

Homework 0

Purpose: Self Assessment

1. Let U be a $U(0, 1)$ random variable, and let $X = -\ln(U)$. Show that X is an $\text{exp}(1)$ random variable.

Solution:

$$P(X > x) = P(-\ln(U) > x) \quad (1)$$

$$= P(\ln(U) < -x) \quad (2)$$

$$= P(U < e^{-x}) \quad (3)$$

$$= e^{-x}. \quad (4)$$

Hence $X \sim \text{exp}(1)$.

2. Let $X_i \sim P(\lambda_i)$, $i = 1, 2$, be two independent random variables. Show that $X = X_1 + X_2$ is a $P(\lambda_1 + \lambda_2)$ random variable.

Solution: From Appendix C4 we have:

$$P(X = k) = \sum_{i=0}^k p_{X_1}(i)p_{X_2}(k-i) \quad (5)$$

$$= \sum_{i=0}^k e^{-\lambda_1} \frac{\lambda_1^i}{i!} e^{-\lambda_2} \frac{\lambda_2^{k-i}}{(k-i)!} \quad (6)$$

$$= e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!} \quad (7)$$

$$(8)$$

where the last equality follows by using the binomial expansion of $(\lambda_1 + \lambda_2)^k$.

3. A stick of unit length is cut at two points at distances U_1 and U_2 from the left end. Suppose U_1, U_2 are iid $U(0, 1)$ random variables. Compute the probability that the three pieces can form three sides of a triangle.

Solution: A triangle can be formed if the longest piece is less than .5 in length. This can happen if one the following disjoint event occurs: $\{U_1 \leq .5 \leq U_2 \leq U_1 + .5\}$ or $\{U_2 \leq .5 \leq U_1 \leq U_2 + .5\}$. Probability of each of these events is $1/8$, as is most easily seen by looking at the area they occupy in a unit square. Hence the probability of getting a triangle is .25.

4. Suppose X_1, X_2, \dots, X_n are iid $\text{exp}(\lambda)$ random variables. Show that $X = X_1 + X_2 + \dots + X_n$ is an $\text{Erl}(n, \lambda)$ random variable.

Solution: We shall prove this by induction. The density of an $\text{Erl}(n, \lambda)$ random variable is

$$\lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, \quad x \geq 0.$$

Setting $n = 1$ we see that it reduces to

$$\lambda e^{-\lambda x}, \quad x \geq 0.$$

This is the density of $\text{Exp}(\lambda)$ random variable. Hence the result is true for $n = 1$. Now suppose it is true for some $n \geq 1$. Then for

$$f_{X_1+X_2+\dots+X_{n+1}}(x) = \int_0^x f_{X_1+X_2+\dots+X_n}(y) f_{X_{n+1}}(x-y) dy \quad (9)$$

$$= \int_0^x \lambda e^{-\lambda y} \frac{(\lambda y)^{n-1}}{(n-1)!} \lambda e^{-\lambda(x-y)} dy \quad (10)$$

$$= \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!}, \quad x \geq 0. \quad (11)$$

Thus the result is true by induction for all $n \geq 1$.

5. Number votes cast in a county election is a $P(\lambda)$ random variable. Each vote is Democratic with probability p and Republican with probability $1 - p$. All votes are independent of each other. Let D and R be the Democratic and Republican votes counted in the final tally, respectively. Compute the joint pmf of (D, R) .

Solution: Let N be the total votes cast. The $N \sim P(\lambda)$. Thus $(D = i, R = j)$ implies that $N = i + j$. We have

$$P(D = i, R = j) = P(N = i + j, D = i, R = j) \quad (12)$$

$$= P(D = i, R = j | N = i + j) P(N = i + j) \quad (13)$$

$$= \frac{(i+j)!}{i!j!} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} \quad (14)$$

$$= e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}. \quad (15)$$

Thus $D \sim P(\lambda p)$ and $R \sim P(\lambda(1-p))$ are independent random variables.

6. Consider Example 1 on page 575. Let N denote the number of outcomes that never occur in the n trials. Compute $E(N)$.

Solution: Let $X_i = 1$ if outcome i never occurs, and 0 otherwise. Then $N = X_1 + \dots + X_k$ and $E(X_i) = P(X_i = 1) = (1 - p_i)^n$. Hence

$$E(N) = E\left(\sum_{i=1}^k X_i\right) = \sum_{i=1}^k E(X_i) = \sum_{i=1}^k (1 - p_i)^n.$$