HW 10 Solutions

Due: Thursday, Nov 20, 2007

Ch. 6, Comp. Ex. 1. Define $Z(t) = 0$ if $X(t)$ is even, and $Z(t) = 1$ if $X(t)$ is odd. Then $\{Z(t), t \geq 0\}$ is a CTMC on state-space $\{0, 1\}$ with generator matrix

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$

Using results of Example 6.12, equation 6.38, we get

$$P(X(t) \text{ odd}|X(0) = 0) = P(Z(t) = 1|Z(0) = 0) = p_{0,1}(t) = \frac{\alpha}{\alpha+\beta}(1-e^{-(\alpha+\beta)t}).$$

Ch. 6, Comp. Ex. 6. Using the transition rates from the modeling Exercise 15, the forward equations 6.29 become

$$p'_{i,j}(t) = .3(j+1)\mu p_{i,j+1}(t) - j\mu p_{i,j} + .4(j-1)\mu p_{i,j-1} + .3(j-2)\mu p_{i,j-2}(t), \;\; j \geq 0,$$

where we interpret $p_{i,j}(t) = 0$ for $j < 0$. Following the steps in the solution to Computational Exercise 2, we get

$$m'(t) = .7\mu m(t), \quad m(0) = i.$$

Solution is given by

$$m(t) = ie^{.7\mu t}.$$

Thus the size of the amoeba colony explodes exponentially with time.

Ch. 6, Comp. Ex. 10. The system is stable if $\lambda < \mu = \mu_1 + \mu_2$. Using the transition rates developed in the solution to Modeling Exercise 7, we get the following balance equations (use $\lambda_1 = \lambda\alpha, \;\; \lambda_2 = \lambda(1-\alpha)$):

$$\begin{aligned} \lambda p_0 &= \mu_1 p_{1a} + \mu_2 p_{1b}, \\ (\lambda+\mu_1)p_{1a} &= \lambda_1 p_0 + \mu_2 p_2, \\ (\lambda+\mu_2)p_{1b} &= \lambda_2 p_0 + \mu_1 p_2, \\ (\lambda+\mu)p_2 &= \lambda p_{1a} + \lambda p_{1b} + \mu p_3, \\ (\lambda+\mu)p_i &= \lambda p_{i-1} + \mu p_{i+1}, \;\; i \geq 3. \end{aligned}$$

The last set of equations is identical to the birth and death equations, and hence yields the solution

$$p_i = p_2\rho^{i-2}, \;\; i \geq 2,$$

where $\rho = \lambda/\mu$. Next we solve the first three equations to obtain p_0, p_{1a} and p_{1b} in terms of p_2. We get (using $\beta = 1 - \alpha$):

$$\begin{aligned} p_0 &= \frac{\mu_1\mu_2}{\lambda^2} \cdot \frac{2\lambda + \mu}{\lambda + \alpha\mu_2 + \beta\mu_1} p_2, \\ p_{1a} &= \frac{\mu_2}{\lambda} \cdot \frac{\lambda + \alpha\mu}{\lambda + \alpha\mu_2 + \beta\mu_1} p_2, \\ p_{1b} &= \frac{\mu_1}{\lambda} \cdot \frac{\lambda + \beta\mu}{\lambda + \alpha\mu_2 + \beta\mu_1} p_2. \end{aligned}$$

Finally we compute p_2 by using the normalizing equation:

$$p_0 + p_{1a} + p_{1b} + \sum_{i=2}^{\infty} p_i = p_0 + p_{1a} + p_{1b} + p_2/(1-\rho) = 1.$$

Ch. 6, Comp. Ex. 13. This system is always stable. Let $a_k = (1-p)p^{k-1}, k \geq 1$. Using the transition rates given in the solution to Modeling Exercise 10, we get the following balance equations:

$$(\lambda + j\mu)p_j = \lambda \sum_{i=0}^{j-1} a_{j-i} p_i + (j+1)\mu p_{j+1}, \quad j \geq 0.$$

We have

$$G(z) = \sum_{j=0}^{\infty} z^j p_j, \quad G'(z) = \sum_{j=0}^{\infty} j z^{j-1} p_j,$$

$$A(z) = \sum_{j=1}^{\infty} z^j a_j = \frac{(1-p)z}{1-pz}.$$

Multiply the jth balance equation by z^j and sum over all $j = 0, 1, 2, \ldots$. We get

$$\sum_{j=0}^{\infty} z^j (\lambda + j\mu) p_j = \sum_{j=0}^{\infty} z^j \lambda \sum_{i=0}^{j-1} a_{j-i} p_i + \sum_{j=0}^{\infty} z^j (j+1)\mu p_{j+1}.$$

Simplifying, we get

$$\lambda G(z) + \mu z G'(z) = \lambda \sum_{i=0}^{\infty} z^i p_i \sum_{j=i+1}^{\infty} z^{j-i} a_{j-i} + \mu G'(z),$$

which yields

$$\lambda G(z) + \mu(z-1)G'(z) = \lambda G(z) A(z).$$

This can be simplified to get

$$G'(z) = \frac{\lambda}{\mu(1-pz)} G(z).$$

Integrating, and using the fact that $G(1) = 1$, we get

$$G(z) = \left(\frac{1-p}{1-pz}\right)^{\frac{\lambda}{\mu p}}.$$

Ch. 6, Comp. Ex. 14. Using the birth and death parameters from the solution to Modeling Exercise 11 in Equation 6.170, we get

$$\rho_i = \frac{(\lambda_1 + \lambda_2)^i}{i!\mu^i}, \quad 0 \le i \le s,$$

$$\rho_i = \frac{(\lambda_1 + \lambda_2)^s \lambda_1^{i-s}}{s! s^{i-s} \mu^i}, \quad i \ge s.$$

Now if $\rho = \lambda_1/(s\mu) < 1$,

$$S = \sum_{i=0}^{\infty} \rho_i = \sum_{i=0}^{s-1} \rho_i + \frac{(\lambda_1 + \lambda_2)^s}{s!\mu^s} \frac{1}{1-\rho},$$

else it is infinity. Thus the system is stable if $\rho < 1$, i.e., $\lambda_1 < s\mu$. If it is stable, the limiting distribution is given by

$$p_0 = 1/S, \quad p_i = \rho_i/S, \quad i \ge 1.$$