HW 11 Solutions

Due: Thursday, Nov 27, 2007

Ch. 6, Comp. Ex. 15. Using the transition rates given in the solution to Modeling Exercise 13, we get the following balance equations:

$$\begin{aligned}
\lambda p_{0,0} &= \mu_1 p_{1,1} + \mu_2 p_{1,2}, \\
(\lambda + \mu_j) p_{1,j} &= \lambda \alpha_j p_{0,0} + \mu_1 \alpha_j p_{2,1} + \mu_2 \alpha_j p_{2,2}, \quad j = 1, 2, \\
(\lambda + \mu_j) p_{i,j} &= \lambda p_{i-1,j} + \mu_1 \alpha_j p_{i+1,1} + \mu_2 \alpha_j p_{i+1,2}, \quad i \geq 2, j = 1, 2.
\end{aligned}$$

Multiply the second equation by z, the third by z^i and add over all i to get

$$(\lambda+\mu_j)\sum_{i=1}^{\infty} z^i p_{i,j} = z\lambda\alpha_j p_{0,0} + \lambda \sum_{i=2}^{\infty} z^i p_{i-1,j} + \mu_1 \alpha_j \sum_{i=1}^{\infty} z^i p_{i+1,1} + \mu_2 \alpha_j \sum_{i=1}^{\infty} z^i p_{i+1,2}.$$

Simplifying, we get

$$(\lambda+\mu_j)\phi_j(z) = z\lambda\alpha_j p_{0,0} + \lambda z \phi_j(z) + \mu_1 \alpha_j (\phi_1(z) - p_{1,1} z)/z + \mu_2 \alpha_j (\phi_2(z) - p_{1,2} z)/z.$$

Collecting terms,

$$(\lambda(1-z)+\mu_j)\phi_j(z) = z\lambda\alpha_j p_{0,0} + \frac{\alpha_j}{z}(\mu_1\phi_1(z) + \mu_2\phi_2(z)) - \alpha_j(\mu_1 p_{1,1} + \mu_2 p_{1,2}).$$

Using the first balance equation, we get

$$(\lambda(1-z)+\mu_j)\phi_j(z) = \lambda\alpha_j(z-1)p_{0,0} + \frac{\alpha_j}{z}(\mu_1\phi_1(z) + \mu_2\phi_2(z)), \quad j = 1, 2.$$

These are two equations for $\phi_1(z)$ and $\phi_2(z)$. Solving simultaneously, and simplifying, we get

$$\phi_i(z) = \frac{\lambda\alpha_i(\lambda(1-z)+\mu_j)}{\frac{\mu_1\mu_2}{z} - \lambda\mu_1(1-\frac{\alpha_1}{z}) - \lambda\mu_2(1-\frac{\alpha_2}{z}) - \lambda^2(1-z)} p_{0,0}.$$

Using

$$\phi_1(1) + \phi_2(1) + p0, 0 = 1$$

we get

$$p_{0,0} = 1 - \lambda(\alpha_1/\mu_1 + \alpha_2/\mu_2).$$

The condition of stability is $p_{0,0} > 0$.

Ch. 6, Comp. Ex. 19. $X(t)$ = number of working CPUs at time t. $\{X(t), t \geq 0\}$ is a CTMC on $\{0, 1, 2, 3, 4, 5\}$ with rates

$$q_{i,i-1} = i\mu c, \quad q_{i,0} = i\mu(1-c), \quad i = 2, 3, 4, 5,$$

$$q_{1,0} = \mu.$$

Let $T = \min\{t \geq 0 : X(t) = 0\}$, and $m_i = \mathcal{E}(T, |X(0) = i)$. The desired result is m_5. Using Theorem 6.19, we get

$$m_0 = 0,$$

$$m_1 = 1/\mu,$$

$$m_2 = 1/(2\mu) + cm_1,$$

$$m_3 = 1/(3\mu) + cm_2,$$

$$m_4 = 1/(4\mu) + cm_3,$$

$$m_5 = 1/(5\mu) + cm_4.$$

Solving recursively, we get

$$m_5 = \frac{1}{\mu}[\frac{1}{5} + \frac{c}{4} + \frac{c^2}{3} + \frac{c^3}{2} + \frac{c^4}{1}].$$

Ch. 6, Comp. Ex. 21. $X(t)$ = number of working machines at time t. $\{X(t), t \geq 0\}$ is a birth and death process on $\{0, 1, ..., k\}$ with birth parameters

$$\lambda_i = (k-i)\lambda, \quad 0 \leq i \leq k,$$

and death parameters

$$\mu_i = i\mu, \quad 0 \leq i \leq k.$$

Let $T = \min\{t \geq 0 : X(t) = 0\}$, and $m_i = \mathcal{E}(T, |X(0) = i)$. Want m_1. This is a special case of Example 6.36. We have

$$\frac{1}{\lambda_j \rho_j} = \frac{\lambda_1 ... \lambda_{j-1}}{\mu_1 ... \mu_j}$$

$$= \frac{k!}{(k-j)!j!} \frac{1}{k\lambda} (\lambda/\mu)^j, \quad 1 \leq j \leq k.$$

Hence, from Equation 6.220,

$$m_1 = \sum_{j=1}^{k} \frac{k!}{(k-j)!j!}\frac{1}{k\lambda}(\lambda/\mu)^j$$

$$= \frac{1}{k\lambda}[(1+\frac{\lambda}{\mu})^k - 1]$$

where we have used the binomial theorem to compute the sum.

Ch. 6, Comp. Ex. 22. $X(t)$ = number of customers in an $M/M/1/K$ queue at time t. $X(0) = 1$. Let $T = \min\{t \geq 0 : X(t) = 0\}$, and $m_i = \mathcal{E}(T, |X(0) = i)$. The expected time until an arrival to an empty system is thus thus $m_1 + 1/\lambda$. m_1 can be computed using the results of Example 6.36. We have

$$\frac{1}{\lambda_j \rho_j} = \frac{1}{\mu}(\lambda/\mu)^{j-1}, \quad 1 \leq j \leq K.$$

Hence, from Equation 6.220,

$$m_1 = \sum_{j=1}^{K} \frac{1}{\mu}(\lambda/\mu)^{j-1}$$

$$= \frac{1}{\mu}\frac{1-(\lambda/\mu)^K}{1-\lambda/\mu}.$$

$$m_1 + 1/\lambda = \frac{1}{\lambda}\frac{1-(\lambda/\mu)^{K+1}}{1-\lambda/\mu}.$$

Ch. 6, Comp. Ex. 24. Let $X(t)$ be the number of tasks in the system at time t. Suppose $X(t) = i > 0$. An arrival occurs at rate λ and this changes the number of tasks to $i+2$. A task completes at rate 2μ, which reduces the tasks to $i-1$. This shows that $\{X(t), t \geq 0\}$is a CTMC on $\{0, 1, 2, ...\}$ with transition rates:

$$q_{i,i+2} = \lambda, \;\; i \geq 0, \quad q_{i,i-1} = 2\mu, \;\; i \geq 1.$$

The balance equations become

$$\lambda p_0 = 2\mu p_1, \;\; (\lambda + 2\mu)p_1 = 2\mu p_2,$$

$$(\lambda + 2\mu)p_i = \lambda p_{i-2} + 2\mu p_{i+1}, \quad i \geq 2.$$

We compute the generating function $\phi(z) = \sum_{i=0}^{\infty} p_i z^i$. Multiply the equation for p_i by z^i and add to get

$$\lambda p_0 + (\lambda + 2\mu) \sum_{i=1}^{\infty} p_i z^i = 2\mu \sum_{i=0}^{\infty} z^i p_{i+1} + \lambda \sum_{i=2}^{\infty} z^i p_{i-2}.$$

Manipulating the above equation we get

$$(\lambda + 2\mu)\phi(z) = \frac{2\mu}{z}\phi(z) + \lambda z^2 \phi(z) + 2\mu(1 - \frac{1}{z})p_0.$$

This yields

$$\phi(z) = \frac{2\mu(z-1)}{\lambda z(1-z^2) + 2\mu(z-1)} p_0 = \frac{2\mu}{2\mu - \lambda z(1+z)} p_0.$$

We compute p_0 by using $\Phi(1) = 1$. We get

$$p_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho.$$

This shows that condition of stability is $\lambda < \mu$. Next we compute L_T, the expected number of tasks in the system:

$$L_T = \sum_{i=0}^{\infty} i p_i = \phi'(1) = \frac{3\rho}{2(1-\rho)^2} \cdot (1-\rho) = \frac{3\rho}{2(1-\rho)}.$$

To compute L_C, the expected number of customers in the system, let π_i be the limiting probability that there are i customers in the system. Note that $\pi_0 = p_0 = 1 - \rho$. Now, each waiting customer has two tasks with him. However, the customer in service has either one task left or two tasks left with equal probability in steady state. Thus, if there are 0 customers in the system, there are zero tasks in the system. If there are $i > 0$ customers in the system, the expected number of taks is $2(i-1) + 1.5 = 2i - .5$. Hence we can compute the L_T in an alternate way as follows:

$$L_T = \sum_{i=1}^{\infty} (2i - .5)\pi_i = 2L_C - .5(1 - \pi_0) = 2L_C - .5\rho.$$

Hence, we get

$$L_C = L_T/2 + \rho/4.$$