HW 12 Solutions

Due: Thursday, Dec 4, 2007

Ch. 6, Comp. Ex. 25. We describe a space as E if it is empty, B if it is occupied by a car in service, and W if it is occupied by a car that is waiting to begin service or has finished service. The state space is $S = \{1 = EEE, 2 = BEE, 3 = BBE, 4 = EBE, 5 = BBW, 6 = BWE, 7 = EBW, 8 = BWW\}$. Thus state is EBW if the space 1 is empty, space two has a car that is pumping gas, and space 3 is occupied by a car that is waiting for service. The triggering event analysis yields the following rate matrix:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\lambda+2\mu) & \mu & \lambda & \mu & 0 & 0 \\ \mu & 0 & 0 & -(\lambda+\mu) & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -2\mu & 0 & \mu & \mu \\ \mu & 0 & 0 & 0 & 0 & -(\lambda+\mu) & 0 & \lambda \\ 0 & \mu & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & -\mu \end{bmatrix}.$$

The balance equations are $pQ = 0, \sum_1^8 p_i = 1$. These can be solved to obtain p

The long run fraction of the customers who enter =

$$1 - p_5 - p_7 - p_8 = \frac{2 * (2\rho + 1)(2 + \rho)(1 + \rho)}{3\rho^4 + 11\rho^3 + 14\rho^2 + 14\rho + 2}.$$

Ch. 6, Comp. Ex. 26. Let $X(t)$ be the number of machines in use at time t, and $Y(t)$ be the status of the standby machine at time t (0 if there is none, U if it is up, and D if it is down with an undetected failure). The state of the system is $(X(t), Y(t))$. The state-space is $\{1 = (2, U), 2 = (2, 0), 3 = (2, D), 4 = (1, 0), 5 = (0, 0)\}$. The triggerning event nalysis yields the following rate matrix

$$Q = \begin{bmatrix} -(2\lambda+\theta) & 2\lambda & \theta & 0 & 0 \\ \mu & -(2\lambda+\mu) & 0 & 2\lambda & 0 \\ 0 & 0 & -2\lambda & 2\lambda & 0 \\ 0 & \mu & 0 & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 & \mu & -\mu \end{bmatrix}.$$

The forward equations are:

$$\begin{aligned}
p'_{i1}(t) &= -(2\lambda+\theta)p_{i1}(t)+\mu p_{i2}(t),\\
p'_{i2}(t) &= -(2\lambda+\mu)p_{i2}(t)+2\lambda p_{i1}(t)+\mu p_{i4}(t),\\
p'_{i3}(t) &= -2\lambda p_{i3}(t)+\theta p_{i1}(t),\\
p'_{i4}(t) &= -(\lambda+\mu)p_{i4}(t)+2\lambda p_{i2}(t)+2\lambda p_{i3}(t)+\mu p_{i5}(t),\\
p'_{i5}(t) &= -\mu p_{i5}(t)+\lambda p_{i4}(t),
\end{aligned}$$

with boundary conditions: $p_{ij}(0)=\delta_{ij}$. The balance equations are:

$$(2\lambda+\theta)p_1=\mu p_2,$$

$$(2\lambda+\mu)p_2=2\lambda p_1+\mu p_4,$$

$$2\lambda p_3=\theta p_1,$$

$$(\lambda+\mu)p_4=2\lambda p_2+2\lambda p_3+\mu p_5,$$

$$\mu p_5=\lambda p_4,$$

$$\sum_1^5 p_i=1.$$

The repair person is idle in states 1 and 2. Hence the desired probability is given by

$$p_1+p_2=\frac{2\lambda\mu^2(\mu+\theta+2\lambda)}{8\lambda^4+4\lambda^3\theta+8\lambda^3\mu+6\lambda^2\mu\theta+4\lambda\mu^2\theta+2\lambda\mu^3+\mu^3\theta+4\lambda^2\mu^2}.$$

Ch. 6, Comp. Ex. 28. By using the rates in the solution to Modeling problem 22, we get the following balance equations:

$$\begin{aligned}
\lambda p_{R+K} &= \theta p_R\\
\lambda p_{R+K-i} &= \theta p_{R-i}+\lambda p_{R+K+1-i},\quad i=1,2,..,R,\\
\lambda p_i &= \lambda p_{i+1},\quad i=R+1,...,K-1,
\end{aligned}$$

$$(\lambda+\theta)p_i=\lambda p_{i+1},\quad i=1,2,...,R,$$

$$\theta p_0=\lambda p_1.$$

From the third set of equations, we get

$$p_i=A(\text{ a constant}),\quad i=R+1,...,K.$$

Using this in the fourth set of equations, and solving recursively, we get

$$p_i = (\frac{\lambda}{\lambda+\theta})^{R-i+1}A, \quad i = 1, 2, ..., R.$$

The last balance equation yields

$$p_0 = \frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^{R}A.$$

Using these in the first two sets of the balance equations, and solving recursively, we get

$$p_i = (1 - (\frac{\lambda}{\lambda+\theta})^{R+K+1-i})A, \quad i = K+1, K+2, ..., K+R.$$

Summing the above four solutions, and using th normalizing equation, we get

$$A = \frac{1}{K + \frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R}.$$

This gives the limiting distribution of the $\{X(t), t \geq 0\}$ process.
Demands are lost in state 0. Hence the desired answer is

$$p_0 = \frac{\frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R}{K + \frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R}.$$

Ch. 6, Conc. Ex. 12. Let Z be the time time spent in state j during $(0, T)$, and $M_{ij} = E(Z|X(0) = i)$. Clearly $M_{Nj} = 0$.We have

$$E(Z|X(0) = i, X(S_1) = k) = \begin{cases} E(S_1|X(0) = i) & \text{if } i = j \text{ and } k = N, \\ E(S_1|X(0) = i) + E(Z|X(0) = k) & \text{if } i = j \text{ and } k < N \\ 0 & \text{if } i \neq j \text{ and } k = N \\ E(Z|X(0) = k) & \text{if } i \neq j \text{ and } k < N. \end{cases}$$

Hence, for $0 \leq i \leq N-1$, we get

$$\begin{aligned} M_{ij} &= E(Z|X(0) = i) \\ &= \sum_{j=0}^{N} E(Z|X(0) = i, X(S_1) = j)P(X(S_1) = j|X(0) = i) \\ &= \frac{\delta_{ij}}{q_i} + \sum_{j=0, j\neq i}^{N-1} M_{kj}\frac{q_{i,j}}{q_i}. \end{aligned}$$

Ch. 6, Conc. Ex. 14. Suppose $X(0) = i$. The expected discounted reward incurred during the first sojourn time is (following the same argument leading to 6.233)

$$\frac{r_i}{q_i + \alpha} + \frac{q_i}{q_i + \alpha} \sum_{j \neq i} \frac{q_{ij}}{q_i} r_{ij} = \frac{r(i)}{q_i + \alpha},$$

where

$$r(i) = r_i + \sum_{j \neq i} q_{ij} r_{ij}.$$

The result follows from this.