HW 1 Solutions

Due: Thursday, August 30, 2007

Ch. 2, Mod. Ex. 1. The state space of $\{X_n, n \geq 0\}$ is $S = \{0, 1, 2, 3, ...\}$. Suppose $X_n = i$. Then the age of the lightbulb in place at time n is i. If this light bulb does not fail at time $n+1$, then $X_{n+1} = i+1$. If it fails at tiem $n+1$, then a new lightbulb is put in at time $n+1$ with age 0, making $X_{n+1} = 0$. Let Z be the lifetime of a lightbulb. We have

$$\begin{aligned} P(X_{n+1} = 0 | X_n = i, X_{n-1}, ..., X_0) &= P(\text{lightbulb of age } i \text{ fails at age } i+1) \\ &= P(Z = i+1 | Z > i) \\ &= \frac{p_{i+1}}{\sum_{j=i+1}^{\infty} p_j} \end{aligned}$$

Similarly

$$\begin{aligned} P(X_{n+1} = 0 | X_n = i, X_{n-1}, ..., X_0) &= P(Z > i+1 | Z > i) \\ &= \frac{\sum_{j=i+2}^{\infty} p_j}{\sum_{j=i+1}^{\infty} p_j} \end{aligned}$$

It follows that $\{X_n, n \geq 0\}$ is a success-runs DTMC with

$$p_i = \frac{\sum_{j=i+2}^{\infty} p_j}{\sum_{j=i+1}^{\infty} p_j},$$

and

$$q_i = \frac{p_{i+1}}{\sum_{j=i+1}^{\infty} p_j},$$

for $i \in S$.

Ch. 2, Mod. Ex. 2. $\{X_n, n \geq 0\}$ is a DTMC with state space $\{0 = \text{dead}, 1 = \text{alive}\}$ because the movements of the cat and the mouse are independent of the past while the mouse is alive. Once the mouse is dead, it stays dead. Tf the mouse is still alive at time n, he dies at time $n+1$ if both the cat and mouse choose the same node to visit at time $n+1$. There are $N-2$ ways for this to happen. In total there are $(N-1)^2$ possible ways for the cat and the mouse to choose the new nodes. Hence

$$P(X_{n+1} = 0 | X_n = 1) = \frac{N-2}{(N-1)^2}.$$

Hence the transition probability matrix is given by

$$P = \begin{bmatrix} 1 & 0 \\ \frac{N-2}{(N-1)^2} & 1 - \frac{N-2}{(N-1)^2} \end{bmatrix}.$$

Ch. 2, Mod. Ex. 4. Suppose $X_n = i$. Then, $X_{n+1} = i+1$ if the first coin shows heads, while the second shows tails, which will happen with probability $p_1(1-p_2)$, independent of the past. Similarly, $X_{n+1} = i-1$ if the first coin shws tails and the second coin shows heads, which will happen with probability $p_2(1-p_1)$, independent of the past. If both coins show heads, or both show tails, $X_{n+1} = i$. Hence, $\{X_n, n \geq 0\}$ is a space heomegeneous random walk on $S = \{..., -2, -1, 0, 1, 2, ...\}$ (see Example 2.5) with

$$p_i = p_1(1-p_2), \quad q_i = p_2(1-p_1), \quad r_i = 1 - p_i - q_i.$$

Ch. 2, Mod. Ex. 6. We define X_n, the state of the weather system on the nth day, as the length of the current sunny or rainy spell. The state is k, $(k = 1, 2, 3, ...)$, if the weather is sunny and this is the kth day of the current sunny spell. The state is $-k$, $(k = 1, 2, 3, ..)$, if the the weather is rainy and this is the kth day of the current rainy spell. Thus the state space is $S = \{\pm 1, \pm 2, \pm 3, ...\}$.

Now suppose $X_n = k$, $(k = 1, 2, 3, ...)$. If the sunny spell continues for one more day, then $X_{n+1} = k+1$, or else a rainy spell starts, and $X_{n+1} = -(k+1)$. Similarly, suppose $X_n = -k$. If the rainy spell continues for one more day, then $X_{n+1} = -(k+1)$, or else a sunny spell starts, and $X_{n+1} = 1$. The MArkov property follows from the fact that the lengths of the sunny and rainy spells are independent. Hence, for $k = 1, 2, 3, ...$,

$$\begin{aligned} P(X_{n+1} = k+1 | X_n = k) &= p_k, \\ P(X_{n+1} = -1 | X_n = k) &= 1 - p_k, \\ P(X_{n+1} = -(k+1) | X_n = -k) &= q_k, \\ P(X_{n+1} = 1 | X_n = -k) &= 1 - q_k. \end{aligned}$$

Ch. 2, Mod. Ex. 10. Let A be the event that the coin 1 is picked for tossing. Let $Y_n = 1$ if the nth toss is a head, and 0 if it is a tail. Let

$$X_n = P(A | Y_n, Y_{n-1}, ... Y_1), \quad n \geq 1.$$

Since the coin is initially at random, we have $X_0 = .5$. Since $0 \leq X_n \leq 1$, the state space is the interval $[0, 1]$. Suppose $X_n = x$. Then

$$P(Y_{n+1} = 1 | X_n = x, Y_n, Y_{n-1}, ... Y_1) = x p_1 + (1-x) p_2$$

$$P(Y_{n+1}=0|X_n=x,Y_n,Y_{n-1},...Y_1)=x(1-p_1)+(1-x)(1-p_2).$$

Consider the case $X_n=x, Y_{n+1}=1$. We have

$$\begin{aligned}
P(A|Y_{n+1}=1,X_n=x,Y_n,...Y_1) &= P(A|Y_{n+1}=1,X_n=x) \\
&= \frac{P(A,Y_{n+1}=1|X_n=x)}{P(Y_{n+1}=1|X_n=x)} \\
&= \frac{P(Y_{n+1}=1|A,X_n=x)P(A|X_n=x)}{P(Y_{n+1}=1|X_n=x)} \\
&= \frac{xp_1}{xp_1+(1-x)p_2}.
\end{aligned}$$

Similarly,

$$P(A|Y_{n+1}=0,X_n=x,Y_n,...Y_1)=\frac{x(1-p_1)}{x(1-p_1)+(1-x)(1-p_2)}.$$

Thus we have

$$X_{n+1}=\begin{cases} \frac{X_np_1}{X_np_1+(1-X_n)p_2} & \text{with prob. } X_np_1+(1-X_n)p_2 \\ \frac{X_n(1-p_1)}{X_n(1-p_1)+(1-X_n)(1-p_2)} & \text{with prob. } X_n(1-p_1)+(1-X_n)(1-p_2). \end{cases}$$

This shows that $\{X_n, n\geq 0\}$ is a DTMC on state space $[0,1]$.