HW 2 Solutions

Due: Thursday, Sept 6, 2007

Ch. 2, Mod. Ex. 13. Let X_n be the number of passengers on the bus when it leaves the nth stop. Let D_{n+1} be the number of passengers that alight at the $(n+1)$st stop. Since each person on board the bus gets off with probability p in an independent fashion, D_{n+1} is $Bin(X_n, p)$ random variable. Also, $X_n - D_{n+1}$ is a $Bin(X_n, 1-p)$ random variable. Y_{n+1} is the number of people that get on the bus at the $(n+1)$st busstop. Hence

$$X_{n+1} = X_n - D_{n+1} + Y_{n+1}.$$

Since $\{Y_n, n \geq 0\}$ is a sequence of iid random variables, it follows from the above recursive relationship, that $\{X_n, n \geq 0\}$ is a DTMC. The state space is $\{0, 1, ...\}$. The transition probabilities are

$$\begin{aligned}
p_{i,j} &= P(X_{n+1} = j | X_n = i) \\
&= P(X_n - D_{n+1} + Y_{n+1} = j | X_n = i) \\
&= P(Y_{n+1} - Bin(i,p) = j - i) \\
&= \sum_{k=0}^{i} P(Y_{n+1} - Bin(i,p) = j - i | Bin(i,p) = k) P(Bin(i,p) = k) \\
&= \sum_{k=0}^{i} P(Y_{n+1} = k + j - i | Bin(i,p) = k) \binom{i}{k} p^k (1-p)^{i-k} \\
&= \sum_{k=0}^{i} \binom{i}{k} p^k (1-p)^{i-k} p_{k+j-i},
\end{aligned}$$

where we use the convention that $p_k = 0$ if $k < 0$.

Ch. 2, Mod. Ex. 21. Let X_n be the number of packets ready for transmission at time n. Let Y_n be the number of packets that arrive during time $(n, n+1]$. If $X_n = 0$, no packets are transmitted during the nth slot and we have

$$X_{n+1} = Y_n.$$

If $X_n > 0$, exactly one packet is transmitted during the nth time slot. Hence,

$$X_{n+1} = X_n - 1 + Y_n.$$

Since $\{Y_n, n \geq 0\}$ are iid, we see that $\{X_n, n \geq 0\}$ is identical to the DTMC given in Example 2.8.

Ch. 2, Con. Ex. 3. (b). True. We have

$$\begin{aligned}
& P(X_n = j_0 | X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k) \\
= & \frac{P(X_n = j_0, X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)}{P(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= & \frac{P(X_{n+2} = j_2, ..., X_{n+k} = j_k | X_n = j_0, X_{n+1} = j_1) P(X_n = j_0, X_{n+1} = j_1)}{P(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= & \frac{P(X_{n+2} = j_2, ..., X_{n+k} = j_k | X_{n+1} = j_1) P(X_n = j_0, X_{n+1} = j_1)}{P(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= & \frac{P(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k) P(X_n = j_0, X_{n+1} = j_1)/P(X_{n+1} = j_1)}{P(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= & P(X_n = j_0, X_{n+1} = j_1)/P(X_{n+1} = j_1) \\
= & P(X_n = j_0 | X_{n+1} = j_1).
\end{aligned}$$

(c). False. Time shifting is allowed only in conditional probabilities, not in joint probabilities. Consider the special case of $k = 0$. Then the equation reduces to

$$P(X_n = j_0) = P(X_0 = j_0).$$

This is clearly not vaid in general. For the DTMC in part (e) starting in state 1, for example, $P(X_0 = 1) = 1$, but $P(X_1 = 1) = .8$

(e). False. (True only if f is one-to-one function, in which case $f(X_n)$ is a relabeled version of X_n.) As a counterexample, consider the DTMC with state space $\{1, 2, 3\}$ and transition probability matrix

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & .5 & .5 \\ .75 & .25 & 0 \end{bmatrix}.$$

Let

$$f(1) = f(2) = 1, f(3) = 2.$$

Then $Y_n = 1$ if $X_n \in \{1, 2\}$, and $Y_n = 2$ if $X_n = 3$. The numerical calculations in part (a) show that $\{Y_n, n \geq 0\}$ is not a DTMC.

Ch. 2, Comp. Ex. 4. Let X_n be as defined in Example 2.1b. Then $\{X_n, n \geq 0\}$ is a DTMC with transition matrix given below:

$$P = \begin{bmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{bmatrix}.$$

Using the results of Computational exercise 3 above, we get

$$P^n = \frac{1}{2-p_1-p_2}\begin{bmatrix} 1-p_2 & 1-p_1 \\ 1-p_2 & 1-p_1 \end{bmatrix} + \frac{(p_1+p_2-1)^n}{2-p_1-p_2}\begin{bmatrix} 1-p_1 & p_1-1 \\ p_2-1 & 1-p_2 \end{bmatrix}.$$

Using the fact that the first patient is given a drug at random, we have

$$P(X_1 = 1) = P(X_1 = 2) = .5.$$

Hence, for $n \geq 1$, we have

$$\begin{aligned} P(X_n = 1) &= P(X_n = 1 | X_1 = 1) * .5 + P(X_n = 1 | X_1 = 2) * .5 \\ &= \frac{1}{2} \cdot ([P^{n-1}]_{1,1} + [P^{n-1}]_{2,1}) \\ &= 1 - \frac{(p_1 - p_2) * ((p_1 + p_2 - 1)^{(n-1)} - 1)}{2 - a - b}. \end{aligned}$$

Now, let $Y_r = 1$ if the rth patient gets drug 1, and 0 otherwise. Then

$$Z_n = \sum_{r=1}^{n} Y_r$$

is the number of patients among the first n who receive drug 1. Hence

$$\begin{aligned} E(Z_n) &= E(\sum_{r=1}^{n} Y_r) \\ &= \sum_{r=1}^{n} E(Y_r) \\ &= \sum_{r=1}^{n} P(Y_r = 1) \\ &= \sum_{r=1}^{n} P(X_r = 1) \\ &= \sum_{r=1}^{n} \left[1 + \frac{(p_2 - p_1) * ((p_1 + p_2 - 1)^{(n-1)} - 1)}{2 - a - b}\right] \\ &= n\frac{2(1-p_2)}{2-p_1-p_2} - \frac{(p_1 - p_2)}{(2-p_1-p_2)^2} \cdot ((p_1 + p_2 - 1)^n - 1). \end{aligned}$$