HW 5 Solutions

Due: Thursday, Sept 27, 2007

Ch. 3, Comp. Ex. 11. (a). The DTMC is irreducible, positive recurrent and aperiodic. Hence the limiting distribution $[\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4]$ satisfies

$$\begin{aligned} \pi_1 &= .5\pi_1 + .5\pi_2, \\ \pi_2 &= .5\pi_1 + .5\pi_3, \\ \pi_3 &= .5\pi_2 + .5\pi_4, \\ \pi_4 &= .5\pi_3 + .5\pi_4, \end{aligned}$$

The normalizing equation is

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Using MATLAB we get the solution as

$$[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4] = [.25 \;\; .25 \;\; .25 \;\; .25].$$

The long run fraction of the time the DTMC spends in state i is also given by π_i.

(b). The DTMC is irreducible, positive recurrent, but periodic with period 2. Hence the limiting distribution does not exist. The $[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4]$ satisfies

$$\begin{aligned} \pi_1 &= .5\pi_2, \\ \pi_2 &= \pi_1 + .5\pi_3, \\ \pi_3 &= .5\pi_2 + .5\pi_4, \\ \pi_4 &= \pi_3, \end{aligned}$$

The normalizing equation is

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Using MATLAB we get the solution as

$$[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4] = [\frac{1}{6} \;\; \frac{1}{3} \;\; \frac{1}{3} \;\; \frac{1}{6}].$$

The long run fraction of the time the DTMC spends in state i is given by π_i.

(c). The DTMC is irreducible, positive recurrent and aperiodic. Hence the limiting distribution $[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4]$ satisfies

$$\begin{aligned} \pi_1 &= .1\pi_1 + .4\pi_2 + .3\pi_3 + .2\pi_4, \\ \pi_1 &= .2\pi_1 + .1\pi_2 + .4\pi_3 + .3\pi_4, \\ \pi_1 &= .3\pi_1 + .2\pi_2 + .1\pi_3 + .4\pi_4, \\ \pi_1 &= .4\pi_1 + .3\pi_2 + .2\pi_3 + .1\pi_4, \end{aligned}$$

The normalizing equation is

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Using MATLAB we get the solution as

$$[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4] = [.25 \;\; .25 \;\; .25 \;\; .25].$$

The long run fraction of the time the DTMC spends in state i is also given by π_i.

(d). The DTMC is irreducible, positive recurrent and aperiodic. Hence the limiting distribution $[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4]$ satisfies

$$\begin{aligned} \pi_1 &= .1\pi_1 + .1\pi_2 + .1\pi_3 + .1\pi_4, \\ \pi_1 &= .9\pi_1, \\ \pi_1 &= .9\pi_2, \\ \pi_1 &= .9\pi_3 + .9\pi_4, \end{aligned}$$

The normalizing equation is

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Using MATLAB we get the solution as

$$[\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4] = [.1 \;\; .09 \;\; .081 \;\; .729].$$

The long run fraction of the time the DTMC spends in state i is given by π_i.

Ch. 3, Comp. Ex. 14. The DTMC is irreducible and aperiodic. We shall directly solve for π. The steady state equations are

$$\pi_0 = \sum_{j=0}^{\infty} \frac{1}{j+2}\pi_j,$$

$$\pi_i = \sum_{j=i-1}^{\infty} \frac{1}{j+2}\pi_j, \quad i \geq 1.$$

From this we get

$$\pi_0 = \pi_1,$$
$$\pi_i = \frac{1}{i+1}\pi_{i-1} + \pi_{i+1}.$$

Solving recursively we get

$$\pi_i = \frac{1}{i!}\pi_0, \quad i \geq 0.$$

Using the normalization equation we get

$$\pi_0 = [\sum_{i=0}^{\infty} \frac{1}{i!}]^{-1} = e^{-1}.$$

Hence the DTMC is positive recurrent with the limiting distribution

$$pi_i = e^{-1}\frac{1}{i!}, \quad i \geq 0.$$

Ch. 3, Comp. Ex. 16. The balance equations are

$$\pi_j = \pi_0 b_j + \sum_{i=1}^{j+1} \pi_i a_{j-i+1}, \quad j \geq 0.$$

Following the same steps as in Example 3.24, and using the notation given here we get

$$\phi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \pi_0 B(z) + \frac{1}{z}A(z)(\phi(z) - \pi_0).$$

Solving for $\phi(z)$ we get

$$\phi(z) = \pi_0 \frac{A(z) - zB(z)}{A(z) - z}.$$

To compute the unknown π_0, we use the normalization equation $\phi(1) = 1$. Using L'Hopitals rule once we get

$$\pi_0 = \frac{1-a}{1-a+b}.$$

We must have $a < 1$ for this to be positive, hence the condition of stability is

$$a = \sum_{k=0}^{\infty} k a_k < 1.$$

Ch. 3, Comp. Ex. 17. a). Equation (3.201) becomes

$$\rho = \sum_{i=0}^{\infty} a_i \rho^i = (1-c) \sum_{i=0}^{\infty} (\rho c)^i = \frac{1-c}{1-\rho c}.$$

Note that we know that $\rho < 1$ and $.5 < c < 1$, and hence the geometric series converges. Solving the above equation we get

$$\rho^2 c - \rho - c + 1 = 0,$$

which has two solutions

$$\rho = 1, \quad \rho = \frac{1-c}{c}.$$

The second solution is less than 1, and hence is the correct one. Substituting in Equation (3.203) we get the limiting distribution as

$$\pi_j = \frac{2c-1}{c} (\frac{1-c}{c})^j, \quad j \geq 0.$$

(b). Equation (3.201) becomes

$$\rho = \sum_{i=0}^{\infty} a_i \rho^i = \frac{1}{m+1} \frac{1-\rho^{m+1}}{1-\rho}.$$

Solving numerically (run "cmp317b") we get the following table of ρ valus for different ms.

m	ρ
3	0.4142
4	0.2757
5	0.2113
6	0.1727
7	0.1464
8	0.1273
9	0.1127
10	0.1011
11	0.0918
12	0.0840
13	0.0774
14	0.0718
15	0.0670
16	0.0628
17	0.0590
18	0.0557
19	0.0528
20	0.0501

The limiting distribution for a given m is given by Equation (3.203) by using the appropriate value of ρ from the above table.

Ch. 3, Con. Ex. 10. Since $\{X_n, n \geq 0\}$ is a finite state irreducible DTMC, it is positive recurrent (Theorem 3.7). Hence it has a unique limiting distribution (Theorem 3.15). Thus it suffices to check that the soultion $\pi_j = 1/N$ for all $j = 1, 2, ..., N$ satisfies the balance equations 3.120 and 3.121. Clearly, 3.121 is satisfied. Using the definition of doulbly stochastic matrices, we get

$$\sum_{i=1}^{N} \pi_i p_{i,j} = \sum_{i=1}^{N} \frac{1}{N} p_{i,j} = \frac{1}{N} \sum_{i=1}^{N} p_{i,j} = \frac{1}{N} = \pi_j.$$

Thus Equation 3.120 is satisfied. Hence $\pi_j = 1/N$ for all $j = 1, 2, ..., N$ is the liming distribution.