HW 6 Solutions

Due: Thursday, Oct 4, 2007

Ch. 3, Comp. Ex. 24. From the Modeling Exercise 1 of Chapter 2, we see that $\{X_n, n \geq 0\}$ is a success-runs DTMC with

$$p_i = \frac{\sum_{j=i+2}^{\infty} \alpha_j}{\sum_{j=i+1}^{\infty} \alpha_j},$$

and

$$q_i = \frac{\alpha_{i+1}}{\sum_{j=i+1}^{\infty} \alpha_j},$$

for $i = 0, 1, 2, ...$ Hence we can use the results of Example 3.22. Using L to represent a generic lifetime os a new item, we heve

$$\rho_0 = 1 = \mathcal{P}(L > 0),$$

$$\rho_n = p_0 p_1 ... p_{n-1} = \sum_{j=n+1}^{\infty} \alpha_j = \mathcal{P}(L > n), \ \ n \geq 1.$$

Now,

$$\sum_{n=0}^{\infty} \rho_n = \sum_{n=0}^{\infty} \mathcal{P}(L > n) = \tau < \infty.$$

Hence, from Equation (3.158), the DTMC is positive recurrent. Using Equation (3.159) we get

$$\pi_n = \frac{\mathcal{P}(L > n)}{\tau}, \quad n \geq 0.$$

Ch. 3, Comp. Ex. 25. To avoid confusing notation, let α_i be the probability that an item produced by machine i is non-defective. Let $Y_{i,n}$, $i = 1, 2$, be the number of non-defective items in the inventory of the ith machine at time n, after all production and any assembly at time n is done. Since the assembly is instantaneous, both $Y_{1,n}$ and $Y_{2,n}$ cannot be positive simultaneously. Now define

$$X_n = M_2 + Y_{1,n} - Y_{2,n}.$$

The state space of $\{X_n, n \geq 0\}$ is $S = \{0, 1, 2, ..., M_1 + M_2 - 1, M_1 + M_2\}$. Now,

$$X_n = k > M_2 \Rightarrow Y_{1,n} = k - M_2, \ Y_{2,n} = 0,$$

$$X_n = k < M_2 \Rightarrow Y_{1,n} = 0, \ Y_{2,n} = M_2 - k,$$

$$X_n = k = M_2 \Rightarrow Y_{1,n} = 0,\ Y_{2,n} = 0.$$

Thus X_n contains complete information about $Y_{1,n}$ and $Y_{2,n}$. $\{X_n, n \geq 0\}$ is a random walk on S as in Example 2.5 with

$$p_{n,n+1} = p_n = \begin{cases} \alpha_1 & \text{if } n = 0, \\ \alpha_1(1-\alpha_2) & \text{if } 0 < n < M_1 + M_2, \end{cases}$$

$$p_{n,n-1} = q_n = \begin{cases} \alpha_2 & \text{if } n = M_1 + M_2, \\ \alpha_2(1-\alpha_1) & \text{if } 0 < n < M_1 + M_2, \end{cases}$$

$$p_{n,n} = r_n = \begin{cases} 1-\alpha_1 & \text{if } n = 0, \\ \alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2) & \text{if } 0 < n < M_1 + M_2, \\ 1-\alpha_2 & \text{if } n = M_1 + M_2. \end{cases}$$

Using Example 3.23, Equation 3.171, we have

$$\rho_0 = 1,$$

$$\rho_n = \frac{\alpha_1}{\alpha_2(1-\alpha_1)} \cdot \left(\frac{\alpha_1(1-\alpha_2)}{\alpha_2(1-\alpha_1)}\right)^{n-1}, \quad \text{if } 1 \leq n \leq M_1 + M_2 - 1,$$

$$\rho_n = \frac{\alpha_1}{\alpha_2} \cdot \left(\frac{\alpha_1(1-\alpha_2)}{\alpha_2(1-\alpha_1)}\right)^{M_1+M_2-1}, \quad \text{if } n = M_1 + M_2 - 1.$$

Ch. 3, Comp. Ex. 26. Let D_n be the number of non-defective items produced on day n. Then, $\{D_n, n \geq 0\}$ is a sequence of iid random variables with common pmf

$$a_0 = (1-p)^2,\ \ a_1 = 2p(1-p),\ \ a_2 = p^2.$$

Since the demand is one per day, we get

$$X_{n+1} = (X_n + D_n - 1)^+.$$

Thus $\{X_n, n \geq 0\}$ is a DTMC. Indeed, it is a random walk on $\{0, 1, 2, ...\}$ with

$$r_0 = 1 - p^2,\ \ p_0 = p^2,$$

$$q_i = (1-p)^2,\ \ r_i = 2p(1-p),\ \ p_i = p^2,\ \ i \geq 1.$$

Use the results of Example 3.23. We get

$$\rho_i = (p/(1-p))^{2i},\ \ i \geq 0.$$

Using Eq. 3.174, we see that the DTMC is positive recurrent iff $\sum \rho_n < \infty$. This is the case iff, $p/(1-p) < 1$, i.e., $p < .5$. From Eq. 3.175, we see that the limiting distribution is given by

$$\pi_i = (p/(1-p))^{2i}(1-(p/(1-p))^2), \ \ i \geq 0.$$

Ch. 3, Comp. Ex. 28. Let X_n be the inventory at the beginning of period n, and let Y_n be the state of the machine (1 if up, 0 if down), at the beginning of period n. Note that $0 \leq X_n \leq k \Rightarrow Y_n = 1$. Thus the state-space of the bivariate process $\{(X_n, Y_n), n \geq 0\}$ is $\{i = (i,1) : 0 \leq i \leq K-1\} \cup \{(i,0) : k < i \leq K\}$. It is a DTMC since the production in each time period is iid. The transition probabilities are given by

$$p_{i,i+1} = p^2, \ 0 \leq i < K-1,$$

$$p_{i,i-1} = q^2, \ 1 \leq i \leq K-1,$$

$$p_{i,i} = 2pq, \ 0 \leq i \leq K-1,$$

$$p_{K-1,(K,0)} = p^2, \ p_{(i,0),(i-1,0)} = 1, \ k+1 < i \leq K,$$

$$p_{(k+1,0),k} = 1.$$

The balance equations, using judicious cuts, are

$$p^2\pi_i = q^2\pi_{i+1}, \ 0 \leq i \leq k-1, \quad (1)$$

$$p^2\pi_i = q^2\pi_{i+1} + \pi_K, \ k \leq i \leq K-2, \quad (2)$$

$$p^2\pi_{K-1} = \pi_{(K,0)}, \quad (3)$$

$$\pi_{(i,0)} = \pi_{(K,0)}, \ k+1 \leq i \leq K. \quad (4)$$

Solving equation (1) recursively yields

$$\pi_i = \pi_0(p^2/q^2)^i, \ 0 \leq i \leq k. \quad (5)$$

Solving equation (2) recursively, and simplifying yields

$$\pi_i = \pi_k(p^2/q^2)^{i-k} + \pi_{(K,0)}\frac{1-(p^2/q^2)^{i-k}}{q^2-p^2}, \ k \leq i \leq K-1. \quad (6)$$

Substituting for π_k from equation (5) in equation (6) yields

$$\pi_i = \pi_0(p^2/q^2)^i + \pi_{(K,0)}\frac{1-(p^2/q^2)^{i-k}}{q^2-p^2}, \ k \leq i \leq K-1. \quad (7)$$

Using equation (4) and (7) in the normalizing equation

$$\sum_{i=0}^{K-1} \pi_i + \sum_{i=k+1}^{K} \pi_{(i,0)} = 1$$

yields

$$\pi_0 \sum_{i=0}^{K-1} (p^2/q^2)^i + \pi_{(K,0)} \sum_{i=k+1}^{K-1} \frac{1-(p^2/q^2)^{i-k}}{q^2-p^2} + (K-k)pi_{(K,0)} = 1.(8)$$

Substitute equation (3) in equation (7) (with $i = K-1$) to get

$$\pi_{(K,0)}/p^2 = \pi_{K-1} = \pi_0(p^2/q^2)^{(K-1)} + \pi_{(K,0)} \frac{1-(p^2/q^2)^{K-1-k}}{q^2-p^2}. \quad (9)$$

Solve equations (8) and (9) simultaneously to obtain π_0 and $\pi_{(K,0)}$.
(a) Steady state probability that the machine is off $= \sum_{i=k+1}^{K} \pi_{(i,0)} = (K-k)\pi_{(K,0)}$.
(b) Probability of i items in the inventory $= \pi_i$ if $0 \le i \le k$, $\pi_i + \pi_{(i,0)}$ if $k+1 \le i \le K-1$, and $\pi_{(K,0)}$ if $i = K$.

Ch. 3, Comp. Ex. 29.

$$(a). \quad \begin{bmatrix} 4/11 & 7/11 & 0 & 0 \\ 4/11 & 7/11 & 0 & 0 \\ 4/11 & 7/11 & 0 & 0 \\ 4/11 & 7/11 & 0 & 0 \end{bmatrix}.$$

$$(b). \quad \begin{bmatrix} 90/172 & 27/172 & 55/172 & 0 & 0 & 0 \\ 90/172 & 27/172 & 55/172 & 0 & 0 & 0 \\ 90/172 & 27/172 & 55/172 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1170/3956 & 351/3956 & 715/3956 & 10/23 & 0 & 0 \\ 540/3956 & 162/3956 & 330/3956 & 17/23 & 0 & 0 \end{bmatrix}.$$