

HW 7 Solutions

Due: Thursday, Oct 11, 2007

Ch. 4, Comp. Ex. 7. Let m_i be the expected time to reach the food starting from cell i . First step analysis yields:

$$\begin{aligned}m_1 &= 1 + .5m_2 + .5m_4 \\m_2 &= 1 + (1/3)(m_1 + m_3 + m_5) \\m_3 &= 1 + .5m_2 + .5m_6 \\m_4 &= 1 + (1/3)(m_1 + m_5 + m_7) \\m_5 &= 1 + (1/4)(m_2 + m_4 + m_6 + m_8) \\m_6 &= 1 + (1/3)(m_3 + m_5 + m_9) \\m_7 &= 1 + .5m_4 + .5m_8 \\m_8 &= 1 + (1/3)(m_5 + m_7 + m_9) \\m_9 &= 0.\end{aligned}$$

Solving, we get the desired answer to be $m_1 = 18$.

Ch. 4, Comp. Ex. 10. Equation 4.72 becomes:

$$m_0 = m_N = 0; \quad m_i = 1 + pm_{i+1} + qm_{i-1}, \quad 1 \leq i \leq N - 1.$$

Using $p + q = 1$, we can rewrite the above equation as

$$q(m_i - m_{i-1}) = 1 + p(m_{i+1} - m_i), \quad 1 \leq i \leq N - 1.$$

Let $w_i = m_i - m_{i-1}$, $1 \leq i \leq N - 1$. Then we have

$$w_{i+1} = (q/p)w_i - (1/p), \quad 1 \leq i \leq N - 1.$$

Recursive substitution yields

$$w_i = (q/p)^{i-1}w_1 - (1/p) \frac{1 - (q/p)^{i-1}}{1 - (q/p)}, \quad 1 \leq i \leq N.$$

Now $w_1 + w_2 + \dots + w_N = m_N - m_0 = 0$. Hence, summing the above equation for $i = 1$ to N , we get

$$0 = \sum_{i=1}^N (q/p)^{i-1}w_1 - (1/p) \sum_{i=1}^N \frac{1 - (q/p)^{i-1}}{1 - (q/p)}.$$

Hence,

$$w_1 = m_1 - m_0 = m_1 = \frac{1}{q-p} - \frac{N}{q-p} \cdot \frac{1-q/p}{1-(q/p)^N}.$$

Then routine algebra yields the desired result by using

$$m_i = \sum_{j=1}^i w_j.$$

Ch. 4, Comp. Ex. 13. Let $m_i = E(T|X_0 = i)$. Using the transition probabilities given in Computational problem 14 of Chapter 3, Equations 4.72 yield

$$m_i = 1 + \frac{1}{i+2}(m_1 + m_2 \dots + m_{i+1}), \quad i \geq 1.$$

Rearranging, we get

$$m_2 = 2m_1 - 3, \quad m_{i+1} = (i+2)m_i - (i+1)m_i - 1, \quad i \geq 2.$$

Now, let $u_i = m_i - m_{i-1}$, $i \geq 1$. Then, the above equations can be rearranged as

$$u_2 = m_1 - 3, \quad u_{i+1} = (i+1)u_i - 1, \quad i \geq 2.$$

Solving recursively, we get

$$u_i = \frac{1}{2}i!m_1 - i! \sum_{j=1}^i \frac{1}{j!}, \quad i \geq 2.$$

Finally, using $m_i - m_1 = u_1 + u_2 + \dots + u_i$, we get

$$m_i = \frac{1}{2} \sum_{k=0}^i k!m_1 - \sum_{k=2}^i k! \sum_{j=1}^k \frac{1}{j!}, \quad i \geq 2.$$

Since m_i is supposed to be the smallest non-negative solution, we choose

$$m_1 = \lim_{i \rightarrow \infty} \frac{\sum_{k=2}^i k! \sum_{j=1}^k \frac{1}{j!}}{\frac{1}{2} \sum_{k=0}^i k!}.$$

This can be evaluated to be

$$m_1 = 2(e-1) = 3.4366.$$

Ch. 4, Comp. Ex. 15. Let $X_0 = 0$, and for $n \geq 1$ define $X_n = i$ if the n th toss results in run of i heads in a row, and $X_n = -i$ if it results in a run of i tails in a row. Make states r and $-m$ absorbing. Then $\{X_n, n \geq 0\}$ is a DTMC on state-space $\{-m, -m+1, \dots, -1, 0, 1, \dots, r-1, r\}$, with the non-zero transition probabilities as given below:

$$\begin{aligned} p_{0,1} &= p, \quad p_{0,-1} = q = 1 - p, \\ p_{i,i+1} &= p, p_{i,-1} = q, \quad 1 \leq i \leq r-1, \\ p_{i,i-1} &= q, p_{i,1} = p, \quad -m+1 \leq i \leq -1, \\ p_{r,r} &= 1, p_{-m,-m} = 1. \end{aligned}$$

Define T to be the first passage time into state r , and let $u_i = P(T < \infty | X_0 = i)$. Then the desired answer is given by u_0 . Equations 4.28 yield:

$$\begin{aligned} u_0 &= pu_1 + qu_{-1}, \\ u_i &= pu_{i+1} + qu_{-1}, \quad 1 \leq i \leq r-1, \quad (1) \\ u_i &= qu_{i-1} + pu_1, \quad -m+1 \leq i \leq -1, \quad (2) \\ u_{-m} &= 0, \quad u_r = 1. \end{aligned}$$

Solving equation (1) recursively we get

$$u_1 = p^{i-1}u_i + (1 - p^{i-1})u_{-1}, \quad 1 \leq i \leq r.$$

Using $u_r = 1$ we get

$$u_1 = p^{r-1} + (1 - p^{r-1})u_{-1}. \quad (3)$$

Similarly, solving equation (2) recursively and using $u_{-m} = 0$, we get

$$u_{-1} = (1 - q^{m-1})u_1. \quad (4)$$

Solving equations (3) and (4) simultaneously, we get

$$u_1 = \frac{p^{r-1}}{1 - (1 - p^{r-1})(1 - q^{m-1})}, \quad u_{-1} = \frac{p^{r-1}(1 - q^{m-1})}{1 - (1 - p^{r-1})(1 - q^{m-1})}.$$

Finally, the desired answer is

$$u_0 = pu_1 + qu_{-1} = \frac{p^{r-1}(1 - q^m)}{1 - (1 - p^{r-1})(1 - q^{m-1})}.$$

Ch. 4, Con. Ex. 6. Construct a new DTMC $\{Y_n, n \geq 0\}$ on state-space $\{0, 1, 2, \dots\}$ and with transition probabilities $q_{i,j}$ as follows:

$$q_{0,0} = 1, \quad q_{i,j} = p_{i,j} \quad \text{for } i \geq 1.$$

Suppose it incurs a cost of $d(i)$ every time it visits state i . Let $d(0) = 0$, and $d(i) = c(i)$ for $i \geq 1$. Let $h(i)$ be the total expected discounted cost incurred by the Y process over 0 to ∞ starting from state i . Let T be the first passage time into state 0. Then, recognizing that $X_n = Y_n$ and $d(Y_n) = c(X_n)$ for $0 \leq n < T$ and $d(Y_n) = 0$ for $n \geq T$, we get

$$h(i) = E\left(\sum_{n=0}^{\infty} d(Y_n) | Y_0 = i\right) = E\left(\sum_{n=0}^{T-1} d(Y_n) | Y_0 = i\right) = E\left(\sum_{n=0}^{T-1} c(X_n) | X_0 = i\right) = g(i).$$

However, $h(\cdot)$ satisfies Equation 3.286, which implies that $g(\cdot)$ satisfies the same equation, as desired.