HW 8 Solutions

Due: Thursday, Nov 1, 2007

Ch. 5, Comp. Ex. 10. Let $Y_1 = (C_1 + C_2)X_1 \sim Exp(\lambda_1/(C_1 + C_2))$ and $Y_2 = C_2X_2 \sim Exp(\lambda_2/C_2)$. Using $\mu_1 = \lambda_1/(C_1 + C_2)$ and $\mu_2 = \lambda_2/C_2$, and the results of Section 5.1.7, we get

$$P(C_{12} > x) = P(Y_1 + Y_2 > x) = \frac{\mu_2}{\mu_1 - \mu_2} e^{-\mu_1 x} - \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2 x}.$$

Similarly, using $\nu_1 = \lambda_1/C_1$ and $\nu_2 = \lambda_2/(C_1 + C_2)$, we get

$$P(C_{21} > x) = \frac{\nu_2}{\nu_1 - \nu_2} e^{-\nu_1 x} - \frac{\nu_1}{\nu_1 - \nu_2} e^{-\nu_2 x}.$$

The result follows from brute force calculations, with the sign reversed!

Ch. 5, Comp. Ex. 29. Let T = time pedestrian will have to wait and X_1 = time until the first car crosses the pedestrian crossing.

$$T \sim \begin{cases} 0 & \text{if } X_1 > \frac{x}{u} \\ X_1 + T & \text{if } X_1 \leq \frac{x}{u} \end{cases}$$

$E(T) = E(X_1 + T \mid X_1 \leq \frac{x}{u})(1 - \mathrm{e}^{-\lambda x/u}) \Leftrightarrow \mathrm{e}^{-\lambda x/u} E(T) = E(X_1 \mid X_1 \leq \frac{x}{u})(1 - \mathrm{e}^{-\lambda x/u})$

Hence $E(T) = \mathrm{e}^{\lambda x/u} \int_0^{\frac{x}{u}} \lambda t \mathrm{e}^{-\lambda t}\, dt = \dfrac{e^{\lambda x/u} - 1}{\lambda} - \dfrac{x}{u}.$

Ch. 5, Comp. Ex. 37. Let the number of items produced by time t be $N^I(t)$ $(PP(\lambda))$ and the number of trucks that arrive at the depot by time t be $N^T(t)$ $(PP(\mu))$. Then $Z(t) = N^I(S_{N^T(t)})$ is neither a PP nor an NPP. It is also not a CPP since the batch sizes are not independent of the arrival time.

$$\begin{aligned} E\{Z(t)\} &= E\{N^I(S_{N^T(t)})\} \\ &= E\{E\{N^I(S_{N^T(t)})|S_{N^T(t)}\}\} \\ &= E\{\lambda S_{N^T(t)}\} \\ &= \lambda E\{S_{N^T(t)}\} \end{aligned}$$

where $E\{S_{N^T(t)}\} = t - \dfrac{1}{\mu}(1 - e^{-\mu t})$ from Computational Exercise 21.

Ch. 5, Con. Ex. 14. Let $\{N(t), t \geq 0\}$ be a NPP($\lambda(\cdot)$). Let S_k be the kth event time in the NPP. Define $X_k = 1$ if the kth event is registered, and 0 otherwise. Then, given $S_k = s$, $X_k \sim B(p(s))$. Now

$$\begin{aligned}
P(N_1(t) = k) &= \sum_{n=0}^{\infty} P(N_1(t) = k | N(t) = n) P(N(t) = n) \\
&= \sum_{n=k}^{\infty} P(\sum_{i=1}^{N(t)} X_i = k | N(t) = n) P(N(t) = n) \\
&= \sum_{n=k}^{\infty} P(\sum_{i=1}^{n} B(p(S_i)) = k | N(t) = n) P(N(t) = n) \\
&= \sum_{n=k}^{\infty} P(\sum_{i=1}^{n} B(p(U_i)) = k | N(t) = n) P(N(t) = n),
\end{aligned}$$

where $\{U_i, i \geq 1\}$ are iid random variables with common density $\lambda(u)/\Lambda(t)$ for $0 \leq u \leq t$. Now let $Y_i \sim B(p(U_i))$. Then

$$P(Y_i = 1) = \frac{1}{\Lambda} \int_0^t p(u)\lambda(u) du.$$

Thus $\{Y_i, i \geq 1\}$ are iid B(α) random variables with $\alpha = \frac{1}{\Lambda} \int_0^t p(u)\lambda(u) du$. Hence

$$P(N_1(t) = k) = \sum_{n=k}^{\infty} P(Bin(\alpha, n) = k | N(t) = n) P(N(t) = n).$$

The rest of the calculations follow as in Theorem 5.9.

Ch. 5, Con. Ex. 15. Let the common batch mean be $E(Z_1)$, second moment $E(Z_1^2)$ and LST $A(s)$. From Equation 5.127 we know that $N(t) \sim P(\Lambda(t))$. Then following the proof of Theorem 5.13 we get

$$E(e^{-sZ(t)}) = exp(-\Lambda(t)(1 - A(s))).$$

Similarly, following the calculations in Theorem 5.14 we get

$$E(Z(t)) = \Lambda(t)E(Z_1); \quad Var(Z(t)) = \Lambda(t)E(Z_1^2).$$