SOLUTION HW 10.

P31. Let $Z_i(t) = 1$ if the particle is at site i and zero otherwise. Suppose the particle starts in site i at time 0. Then $\{Z_i(t), t \geq 0\}$ is an ARP with $E(U) = \mu_i$, and $E(U + D) = \mu$. Using aperiodicity of $U + D$, we get

$$\lim_{t\to\infty} P(Z_i(t) = 1) = \mu_i/\mu.$$

P33. ILet $0 \leq T \leq a$. Then, from Equation 8.269, the long run cost rate is given by

$$g(T) = \frac{C_2 + C_1 T/a}{T(1 - T/(2a))}.$$

Setting $g'(T) = 0$ we get

$$C_1 T^2 + 2aC_2 T - 2a^2 C_2 = 0.$$

The positive root is given by

$$r = a\frac{C_2}{C_1}[\sqrt{1 + \frac{2C_1}{C_2}} - 1].$$

Thus, $g(T)$ attains its minimum at $T = r$ if $r \leq a$, else it achieves it at $T = a$.. The optimal value of T is given by $T^* = \min\{r, a\}$.

P34. The state space of $\{X(t), t \geq 0\}$ is $\{s + 1, s + 2, ..., S\}$. Suppose $X(0) = S$. Then the process regenerates whenever it enters state S, due to the Poisson demands. We see that the first regeneration epoch S_1 is a sum of $S - s$ Exp(λ) random variables, and is thus aperiodic. Also, $U_j \sim Exp(\lambda)$ for $s + 1 \leq j \leq S$. Hence, from Theorem 8.26, we see that

$$p_j = \frac{E(U_j)}{E(S_1)} = \frac{1/\lambda}{(S - s)/\lambda} = \frac{1}{S - s}, \quad s + 1 \leq j \leq S.$$

P39. Consider the $M|G|1|1$ queue with arrival rate λ and mean service time τ. Suppose a service starts at time 0, and let S_n be the starting time of the nth service, and $X_n = S_n - S_{n-1}$. Let $Z(t)$ be the total number of customers rejected upto time t. Let R_n be the total number of customers rejected during $(S_{n-1}, S_n]$. Then $\{Z(t), t \geq 0\}$ is a renewal reward process generated by $\{(X_n, R_n), n \geq 1\}$. Now, $E(S_1) = \tau + 1/\lambda$, and $E(R_1) = \lambda\tau$. Hence, from Theorem 8.24, the long run rate at which customers are rejected is given by

$$\lim_{t\to\infty} \frac{Z(t)}{t} = \frac{E(R_1)}{E(S_1)} = \frac{\lambda\tau}{1 + \lambda\tau}.$$

Dividing by the arrival rate λ, we get the long run fraction of customers lost as $\frac{\tau}{1+\lambda\tau}$.

P42. Let $Z(t)$ be the amount of inventory on hand at time t. Then $\{Z(t), t \geq 0\}$ is regenerative process with state space $\{0, 1, 2, ..., S\}$, that regenerates every time it enters state S. Let $Z(0) = S$ and let S_1 be the first regeneration epoch. Then, assuming the arrival rate to be $\lambda = 2$ per day, and mean lead time $\tau = 3$ days, we get

$$E(S_1) = S/\lambda + \tau = .5S + 3, \quad E(U_j) = 1/\lambda = .5 \quad 1 \leq j \leq S, \quad E(U_0) = \tau = 3.$$

Since S_1 is aperiodic, we get from Theorem 8.26,

$$p_j = \frac{.5}{.5S+3} = \frac{1}{S+6}, \quad 1 \le j \le S, \quad p_0 = \frac{3}{.5S+3} = \frac{6}{S+6}.$$

Using an argument as in Example 8.41, we see that the long run ordering cost is $50/E(S_1) = 100/(S+6)$. In state j we make profit at rate $10\lambda = 20$ dollars per day, while we pay holding cost at rate j. Hence the net cost rate in state j is $f(j) = j - 20$. The net cost rate is

$$g(S) = 100/(S+6) + \sum_{j=1}^{S}(j-20)p_j =$$

$$\frac{100}{S+6} + \frac{S(S+1)}{2S+12} - 20(1-p_0) = -20 + \frac{S(S+1)+440}{2S+12}.$$

Evaluating $g(S)$ numerically, we see that $g(S)$ is minimized at $S = 10$. The minimium cost is -9.0625 dollars per day.