SOLUTION HW 12.

P4. (a) Yes. The kernel is

$$\begin{aligned} G_{01}(x) &= 1 - e^{-\lambda x} \\ G_{10}(x) &= G(t) , \end{aligned}$$

where G is the service-time cdf.

(b) No. It does not have the Markov property at downward jumps.

(c) No. Same reason as for (b).

P5. First solve $\pi = \pi P$, $\sum_{k \in I} \pi_k = 1$, for the stationary probabilities, π_k, $k \in I$, for the embedded *DTMC*. Then use the *SMP* formula,

$$p_j = \frac{\pi_j \mu_j}{\sum_{k \in I} \pi_k \mu_k} ,$$

to derive the limiting probabilities p_j for the *SMP*:

$$p_1 = 0.0274 , \ p_2 = 0.0454 , \ p_3 = 0.2016 , \ p_4 = 0.7256 .$$

P6. Let $X(t)$ denote the number of down components at time $t \geq 0$. (We could just as easily work with the number of up components.) Since all the random variables have exponential distributions, there are many possible choices for the embedded Markov renewal sequence. For example, each of the following is possible:

1. $S_n =$ time at which repair person returns to the system for the n^{th} time
2. $S_n =$ time at which repair person goes away from the system for the n^{th} time
3. $S_n =$ time of n^{th} failure of a component
4. $S_n =$ time of n^{th} completion of a component repair

In the first and second cases, we can take $Y_n = X(S_n)$, since the definition of S_n includes the information as to whether the repair person is on duty or not just after S_n, which is needed to define the kernel. In cases (3) and (4), we would need a two-dimensional state variable, $Y_n = (X(S_n), m)$, where $m = 1$ if the repair person is on duty just after S_n and $m = 0$ otherwise. The lack-of-memory property of the exponential distribution then ensures that $\{(Y_n, S_n)\}$ is a Markov renewal sequence in each case. In each case, we define the associated semi-Markov process in the usual way: $Z(t) := Y_{N(t)}$. (Various combinations of the four cases are also possible.)

However, if we want to compute the long-run probability that the repair person is on duty in terms of the limiting probabilities for the semi-Markov process $\{Z(t), t \geq 0\}$, then we are further restricted in our definition of the embedded Markov renewal sequence: the process $Z(t)$ must tell us unambiguously whether or not the repair person is on duty at time t, not just at the most recent S_n. This rules out cases (1) and (2) above (with the one-dimensional state variable, $Y_n = X(S_n)$). In case (1), for example, $Z(t) = i$ tells us that there were i down components at the most recent instant when the repair person returned to the system (and became "on duty"), but we don't know if the repair person is still on duty at t. Similarly, in case (2), $Z(t) = i$ tells us that there were i down components at the most recent instant when the repair person left the system (and became "off duty"), but we don't know if the repair person is still off duty at t. But it also rules out cases (3) and (4) (even with the two-dimensional state variable, $Y_n = (X(S_n), m)$), because the repair person might change status (from off duty to on or vice versa) between two observation points S_n.

So, if we adhere to the strict requirement that $Z(t)$ tell us whether or not the repair person is on duty, we need to include among the observation points S_n *both* the arrivals and departures of the repair person. The simplest Markov renewal sequence that will do this is the one that observes the system *only* at these time points. So, let S_n = the n^{th} epoch at which the repair person either arrives or departs. Let $Y_n = (i, 1)$ if there are i down components and the repair person is on duty at S_n+, and let $Y_n = (i, 0)$ if there are i down components and the repair person is off duty at S_n+. In other words, the second element of Y_n is 1 (0) if the repair person arrives (departs) at S_n. (Note that the S_n alternate between arrival and departure points.) Then $Z(t) = Y_{N(t)} = (i, 1)$ if the repair person is on duty at time t and there were i down components in the system when the repair person most recently arrived, and $Z(t) = Y_{N(t)} = (i, 0)$ if the repair person is off duty at time t and there were i down components in the system when the repair person most recently departed. The kernel is given by:

$$\begin{aligned}
G_{(i,1),(i-r,0)}(x) &= \int_0^x \mu e^{-\mu t} \frac{(\mu t)^{(r-1)}}{(r-1)!} dt \ , \ r \leq i \leq n \ ; \\
G_{(i,1),(0,0)}(x) &= \int_0^x \mu e^{-\mu t} \frac{(\mu t)^{(i-1)}}{(i-1)!} dt \ , \ 0 \leq i \leq r \ ; \\
G_{(i,0),(i+k,1)}(x) &= \int_0^x \left(\begin{array}{c} n-i \\ k \end{array} \right) (1-e^{-\lambda t})^k e^{-(n-i-k)\lambda t} dG(t) \ , \ k = 0, 1, \ldots, n-i \ ,
\end{aligned}$$

where $G(t)$ is the cdf of the time the repair person spends away. (Here we have $G(t) = 1 - e^{-\eta t}$, but the exponential assumption is not needed for this particular $\{(Y_n, S_n)\}$ to be a Markov renewal sequence.) The transition probabilities for Y_n are as follows:

$$\begin{aligned}
p_{(i,1),((i-r)^+,0)} &= 1 \\
p_{(i,0),(i+k,1)} &= \int_0^\infty \left(\begin{array}{c} n-i \\ k \end{array} \right) (1-e^{-\lambda t})^k e^{-(n-i-k)\lambda t} dG(t) \ , \ k = 0, 1, \ldots, n-i \ ,
\end{aligned}$$

Under the exponential assumption, the above integral "simplifies" to

$$\left(\frac{(n-i)\lambda}{n-i\lambda+\eta} \right) \left(\frac{(n-i-1)\lambda}{(n-i-1)\lambda+\eta} \right) \cdots \left(\frac{(n-i-k+1)\lambda}{(n-i-k+1)\lambda+\eta} \right) \left(\frac{\eta}{(n-i-k)\lambda+\eta} \right) .$$

To derive the limiting probabilities, $p(j, m) := \lim_{t\to\infty} P\{Z(t) = (j, m)\}$, we need the stationary probabilities, $\pi_{(j,m)}$, for the embedded *DTMC*, $\{Y_n\}$, and the mean sojourn times, $\mu_{(j,m)}$. The latter are given by:

$$\begin{aligned}
\mu_{(j,1)} &= \frac{\min\{r, j\}}{\mu} \\
\mu_{(j,0)} &= \frac{1}{\eta}
\end{aligned}$$

The stationary probabilities, $\pi_{(j,m)}$, satisfy the stationary equations,

$$\begin{aligned}
\pi_{(j,0)} &= \pi_{(j+r,1)} \ , \ j=1,\ldots,n \\
&= \sum_{i=0}^{r} \pi_{(i,1)} \ , \ j=0 \\
\pi_{(j,1)} &= \sum_{i=0}^{j} \pi_{(i,0)} p_{(i,0),(j,1)} \ , \ j=0,1,\ldots,n
\end{aligned}$$

The basic limit theorem for semi-Markov processes gives us

$$\begin{aligned}
p(j,1) &= \frac{\pi_{(j,1)}\mu_{(j,1)}}{\sum_{i=0}^{n} \pi_{(i,0)}\mu_{(i,0)} + \sum_{i=0}^{n} \pi_{(i,1)}\mu_{(i,1)}} \\
&= \frac{\pi_{(j,1)} \min\{r,j\}/\mu}{\sum_{i=0}^{n} \pi_{(i,0)}/\eta + \sum_{i=0}^{n} \pi_{(i,1)} \min\{r,i\}/\mu} \ .
\end{aligned}$$

The desired probability is given by

$$\begin{aligned}
\lim_{t\to\infty} P\{ \text{ server is on duty at } t\} &= \sum_{j=0}^{n} p(j,1) \\
&= \frac{\sum_{i=0}^{n} \pi_{(i,1)} \min\{r,i\}/\mu}{\sum_{i=0}^{n} \pi_{(i,0)}/\eta + \sum_{i=0}^{n} \pi_{(i,1)} \min\{r,i\}/\mu} \ .
\end{aligned}$$

Of course, this problem can be solved in a more straightforward way using the machinery of Markov regenerative processes. First we can define a semi-Markov process using Case 1 above (i.e., observing the system only at the arrival points of the repair person, with the state variable Y_n = number of down components at the n^{th} arrival point). Let $p_j = \lim_{t\to\infty} P\{Z(t)=j\}$ for the associated *SMP*. Then the desired probability is given by

$$\lim_{t \ rightarrow\infty} P\{ \text{ server is on duty at } t\} = \sum_{j=0}^{n} p_j \left(\frac{\min\{r,j\}/\mu}{\min\{r,j\}/\mu + 1/\eta} \right) \ .$$

Or, one can use the alternating-renewal-process argument, as follows. Fix a state j (again using Case 1) and consider the renewal process embedded at successive arrivals of the repair person that find j down components. Define the a.r.p. to be "up" if $Z(t)=j$ and the repair person is on duty at time t. The limit theorem for a.r.p.'s says that

$$\begin{aligned}
\lim_{t\to\infty} P\{Z(t)=j \text{ and on duty at } t\} &= \lim_{t\to\infty} P\{ \text{ ``up'' at } t\} \\
&= \frac{\min\{r,j\}/\mu}{\mu_{jj}} \\
&= \min\{r,j\}/\mu \left(\frac{\pi_j}{\sum_{k=0}^{n} \pi_k \mu_k} \right) \\
&= \left(\frac{\pi_j \mu_j}{\sum_{k=0}^{n} \pi_k \mu_k} \right) \left(\frac{\min\{r,j\}/\mu}{\mu_j} \right) \\
&= p_j \left(\frac{\min\{r,j\}/\mu}{\min\{r,j\}/\mu + 1/\eta} \right) \ .
\end{aligned}$$

Summing over j gives

$$\lim_{t\to\infty} P\{\text{ server is on duty at } t\} = \sum_{j=0}^{n} p_j \left(\frac{\min\{r,j\}/\mu}{\min\{r,j\}/\mu + 1/\eta} \right) ,$$

as above.

The conditional cdf's of the sojourn times are given by:

$$\begin{aligned} Q_{(j,1),(j+r,0)}(x) &= \int_0^x \mu e^{-\mu t} \frac{(\mu t)^{(}r-1)}{(r-1)!} dt \ , \ 0 \le j \le n-r \ ; \\ Q_{(j,1),(n,0)}(x) &= \int_0^x \mu e^{-\mu t} \frac{(\mu t)^{(}n-j-1)}{(n-j-1)!} dt \ , \ n-r \le j \le n \ ; \\ Q_{(j,0),(j-k,1)}(x) &= 1 - e^{-\eta x} \ , \ k = 0, 1, \ldots, j \ . \end{aligned}$$

P7. i) Promotion and tolerance times in different grades must be mutually independent and also not depend on the time at which the grade is entered, in order that the n^{th} sojourn time, X_n, as well as the next state, Y_n, should depend only on the current state, Y_{n-1}, and not on past history nor the time point, S_{n-1}, at which the current state was entered. Note that the promotion and tolerance times within a grade may be dependent. That is, for each i, A_i and B_i could be dependent random variables. For simplicity, however, assume that A_i and B_i are independent, with c.d.f.'s $F_i(t)$ and $G_i(t)$, respectively.

The kernel, $G_{ij}(t)$ for the Markov renewal sequence is given by:

$$\begin{aligned} G_{i,i+1}(t) &= P\{Y_1 = i+1, X_1 \le t | Y_0 = i\} \\ &= P\{A_i \le B_i, A_i \le t\} \\ &= \int_0^t (1 - G_i(x)) dF_i(x) \ , \end{aligned}$$

and

$$\begin{aligned} G_{i,1}(t) &= P\{Y_1 = 1, X_1 \le t | Y_0 = i\} \\ &= P\{A_i > B_i, B_i \le t\} \\ &= \int_0^t (1 - F_i(x)) dG_i(x) \ , \end{aligned}$$

with $G_{ij}(t) \equiv 0$ for all other i, j. (Note that $F_4(t) = 0$ for all $0 \le t < \infty$, so that $G_{4,5}(t) = 0$ and $G_{4,1}(t) = G_4(t)$, for all $t \ge 0$.) Since $p_{ij} = G_{ij}(\infty)$ and $Q_{ij}(t) = G_{ij}(t)/G_{ij}(\infty)$, it follows that

$$\begin{aligned} p_{i,i+1} &= \int_0^\infty (1 - G_i(x)) dF_i(x) = P\{A_i \le B_i\} =: \alpha_i \\ p_{i,1} &= \int_0^\infty (1 - F_i(x)) dG_i(x) = P\{B_i < A_i\} = 1 - \alpha_i \end{aligned}$$

with $p_{ij} = 0$ for all other i, j, and

$$\begin{aligned} Q_{i,i+1}(t) &= \frac{\int_0^t (1 - G_i(x)) dF_i(x)}{\int_0^\infty (1 - G_i(x)) dF_i(x)} \\ Q_{i,1}(t) &= \frac{\int_0^t (1 - F_i(x)) dG_i(x)}{\int_0^\infty (1 - F_i(x)) dG_i(x)} \ , \end{aligned}$$

with $Q_{ij}(t) \equiv 0$ for all other i, j.

ii) The stationary equations for the embedded $DTMC$ are

$$\begin{aligned}
\pi_1 &= \pi_1(1-\alpha_1) + \pi_2(1-\alpha_2) + \pi_3(1-\alpha_3) + \pi_4 \\
\pi_2 &= \pi_1\alpha_1 \\
\pi_3 &= \pi_2\alpha_2 \\
\pi_4 &= \pi_3\alpha_3 \\
1 &= \pi_1 + \pi_2 + \pi_3 + \pi_4
\end{aligned}$$

Solving for π_j in terms of π_1, we obtain

$$\begin{aligned}
\pi_2 &= \pi_1\alpha_1 \\
\pi_3 &= \pi_1\alpha_1\alpha_2 \\
\pi_4 &= \pi_1\alpha_1\alpha_2\alpha_3 \ ,
\end{aligned}$$

so that

$$\begin{aligned}
1 &= \pi_1 + \pi_2 + \pi_3 + \pi_4 \\
&= \pi_1(1 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3) \ ,
\end{aligned}$$

which yields the solution

$$\begin{aligned}
\pi_1 &= (1 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3)^{-1} \\
\pi_2 &= \alpha_1(1 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3)^{-1} \\
\pi_3 &= \alpha_1\alpha_2(1 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3)^{-1} \\
\pi_4 &= \alpha_1\alpha_2\alpha_3(1 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3)^{-1} \ ,
\end{aligned}$$

or, equivalently,

$$\pi_j = \rho_j / \sum_{i=1}^{4} \rho_i \ ,$$

where $\rho_j := \prod_{i=1}^{j-1} \alpha_i$, $j = 1, 2, 3, 4$. The mean sojourn time in state i is given by

$$\begin{aligned}
\tau_i &= E[X_1 | Y_0 = i] \\
&= \int_0^\infty P\{\min\{A_i, B_i\} > t\} dt \\
&= \int_0^\infty (1 - F_i(t))(1 - G_i(t)) dt \ .
\end{aligned}$$

The limiting probabilities for the SMP are given by

$$\begin{aligned}
p_j &= \lim_{t\to\infty} P\{X(t) = j | X(0) = i\} \\
&= \frac{\pi_j\tau_j}{\sum_k \pi_k\tau_k} \\
&= \frac{\rho_j\tau_j}{\sum_k \rho_k\tau_k} \ .
\end{aligned}$$

Now assume $A_i \sim \exp(\lambda)$, $B_i \sim \text{gamma}(2, \mu_i)$. In this case

$$\begin{aligned} \alpha_i &= P\{A_i < B_i\} \\ &= \int_0^\infty P\{B_i > x\}\lambda_i e^{-\lambda_i x} dx \\ &= \int_0^\infty (1+\mu_i x)e^{-\mu_i x}\lambda_i e^{-\lambda_i x} dx \\ &= \frac{\lambda_i^2 + 2\lambda_i \mu_i}{(\lambda_i+\mu_i)^2}, \end{aligned}$$

and

$$\begin{aligned} \tau_i &= \int_0^\infty (1+\mu_i x)e^{-\mu_i x}e^{-\lambda_i x} dx \\ &= \frac{\lambda_i + 2\mu_i}{(\lambda_i+\mu_i)^2}. \end{aligned}$$

p12. Let $X(t)$ be the bunker occupied by the despot at time t. $\{X(t), t \geq 0\}$ is given to be an SMP. The embedded DTMC $\{X_n, n \geq 0\}$ has state space $\{1, 2, ..., 16\}$. Define

$$Y_n = \begin{cases} 1 & \text{if } X_n = 1, 4, 13, \text{ or } 16, \\ 2 & \text{if } X_n = 2, 3, 5, 8, 9, 12, 14, \text{ or } 15, \\ 3 & \text{if } X_n = 6, 7, 10, \text{ or } 11. \end{cases}$$

From symmetry, we see that $\{Y_n, n \geq 0\}$ is also a DTMC with transition probabilities

$$p_{1,2} = 1, \ \ p_{2,1} = p_{2,2} = p_{2,3} = \frac{1}{3}, \ \ p_{3,2} = p_{3,3} = \frac{1}{2}.$$

Then, the limiting probabilities for the Y process are given by

$$a_1 = \frac{1}{6}, \ \ a_2 = \frac{1}{2}, \ \ a_3 = \frac{1}{3}.$$

Now, again from symmetry, we see that the limiting probabilities π_i of the $\{X_n, n \geq 0\}$ process must satisfy

$$\pi_1 = \pi_4 = \pi_{13} = \pi_{16} = a_1/4 = \frac{1}{24},$$

$$\pi_2 = \pi_3 = \pi_5 = \pi_8 = \pi_9 = \pi_{12} = \pi_{14} = \pi_{15} = a_2/8 = \frac{1}{16},$$

$$\pi_6 = \pi_7 = \pi_{10} = \pi_{11} = a_3/4 = \frac{1}{12}.$$

Now, let

$$C = \sum_{i=1}^{16} \tau_i \pi_i.$$

From Theorem 9.27, we get the long run fraction of the time the despot spends in bunker i as

$$p_i = \begin{cases} \frac{\tau_i/24}{C} & \text{if } i = 1, 4, 13, \text{ or } 16, \\ \frac{\tau_i/16}{C} & \text{if } i = 2, 3, 5, 8, 9, 12, 14, \text{ or } 15, \\ \frac{\tau_i/12}{C} & \text{if } i = 6, 7, 10, \text{ or } 11. \end{cases}$$