**SOLUTION HW 3.**

P14. Suppose $X(t) = i$. Then the next arrival occurs with rate $\lambda$. The next service completion occurs with rate $\mu$. Each of the $i-1$ customers in the queue waiting for servcie leaves due to impatience with rate $\theta$. Hence $\{X(t), t \geq 0\}$is birth and death process with birth parameters $\lambda_i = \lambda, \ \ i \geq 0$ and death parameters $\mu_0 = 0, \ \ \mu_i = \mu + (i-1)\theta, \ \ i \geq 1$.

P15. $\{X(t), t \geq 0\}$ is a CTMC on state-space $\{0, 1, 2, ...\}$ with positive elements of the generator matrix as given below:

$$q_{i,i+1} = .4i\lambda, \ \ q_{i,i+2} = .3i\lambda, \ \ q_{i,i-1} = .3i\mu, \ \ i \geq 0.$$

P16. Let $X(t)$ be the number of customers in the system at time $t$, and $Y(t) = 1$ if the server is busy at time $t$, and 0 otherwise. The operating policy implies that the server cannot be idle if there are $N$ or more customers in the system. Thus $X(t) \geq N \Rightarrow Y(t) = 1$. Hence the state space is $\{1, 2, ...\} \cup \{(0,0), (1,0), ..., (N-1, 0)\}$. Here state $i$ indicates that $X(t) = i$ and $Y(t) = 1$, and state $(i, 0)$ indicates that $X(t) = i$ and $Y(t) = 0$. The triggering event analysis yiled tha following rates:

$$q_{(i,0),(i+1,0)} = \lambda, \ \ 0 \leq i \leq N-1,$$

$$q_{(N-1,0),N} = \lambda, q_{(1,(0,0)} = \mu,$$

$$q_{i,i+1} = \lambda, \ \ i \geq 1, \ \ q_{i,i-1} = \mu, \ \ i \geq 2.$$

P1. Define $Z(t) = 0$ if $X(t)$ is even, and $Z(t) = 1$ if $X(t)$ is odd. Then $\{Z(t), t \geq 0\}$ is a CTMC on state-space $\{0, 1\}$ with generator matrix

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$

Using results of Example 6.12, equation 6.38, we get

$$P(X(t) \text{ odd}|X(0) = 0) = P(Z(t) = 1|Z(0) = 0) = p_{0,1}(t) = \frac{\alpha}{\alpha + \beta}(1 - e^{-(\alpha+\beta)t}).$$

P6. Using the transition rates from the modeling Exercise 15, the forward equations 6.29 become

$$p'_{i,j}(t) = .3(j+1)\mu p_{i,j+1}(t) - j\mu p_{i,j} + .4(j-1)\mu p_{i,j-1} + .3(j-2)\mu p_{i,j-2}(t), \ \ j \geq 0,$$

where we interpret $p_{i,j}(t) = 0$ for $j < 0$. Following the steps in the solution to Computational Exercise 2, we get

$$m'(t) = .7\mu m(t), \quad m(0) = i.$$

Solution is given by

$$m(t) = ie^{.7\mu t}.$$

Thus the size of the amoeba colony explodes exponentially with time.