## SOLUTION HW 5.

P17. Let $X(t)$ be the state of the system at time $t$. State 0 indicates that the system has crashed at time $t$. State $i$, $1 \leq i \leq 5$, indicates that the system is functioning with $i$ CPUs working, and $5 - i$ CPUs down. The iid exponenitial lifetimes of the CPUs and the instantaneous recovery mechanism implies that $\{X(t), t \geq 0\}$ is a CTMC with the generator matrix given below:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & -\mu & 0 & 0 & 0 & 0 \\ 2\mu(1-c) & 2\mu c & -2\mu & 0 & 0 & 0 \\ 3\mu(1-c) & 0 & 3\mu c & -3\mu & 0 & 0 \\ 4\mu(1-c) & 0 & 0 & 4\mu c & -4\mu & 0 \\ 5\mu(1-c) & 0 & 0 & 0 & 5\mu c & -5\mu \end{bmatrix}.$$

P21. Let $Y(t)$ be the number of packets in the buffer and $Z(t)$ be the number of tokens in the token pool at time $t$. Define $X(t) = M - Z(t) + Y(t)$. Since $0 \leq Z(t) \leq M$ and $Y(t) \geq 0$, we get $0 \leq X(t) < \infty$. Also, $0 \leq X(t) < M \Rightarrow Y(t) = 0, Z(t) = M - X(t)$, and $M < X(t) \Rightarrow Y(t) = X(t) - M, Z(t) = 0$. Thus $X(t)$ has complete information about $(Y(t), Z(t))$. Now, if a token arrives, $X(t)$ reduces by one (unless it is at 0, in which case the token is lost, and $X(t)$ remains unchanged.) If a packet arrives, $X(t)$ increases by 1. Thus $\{X(t), t \geq 0\}$is birth and death process with birth rates $\lambda_i = \lambda, \ \ i \geq 0$ and death rates $\mu_0 = 0, \mu_i = \mu, \ \ i \geq 1$.

P15. Using the transition rates given in the solution to Modeling Exercise 13, we get the following balance equations:

$$\begin{aligned} \lambda p_{0,0} &= \mu_1 p_{1,1} + \mu_2 p_{1,2}, \\ (\lambda + \mu_j) p_{1,j} &= \lambda \alpha_j p_{0,0} + \mu_1 \alpha_j p_{2,1} + \mu_2 \alpha_j p_{2,2}, \quad j = 1, 2, \\ (\lambda + \mu_j) p_{i,j} &= \lambda p_{i-1,j} + \mu_1 \alpha_j p_{i+1,1} + \mu_2 \alpha_j p_{i+1,2}, \quad i \geq 2, j = 1, 2. \end{aligned}$$

Multiply the second equation by $z$, the third by $z^i$ and add over all $i$ to get

$$(\lambda + \mu_j) \sum_{i=1}^{\infty} z^i p_{i,j} = z\lambda\alpha_j p_{0,0} + \lambda \sum_{i=2}^{\infty} z^i p_{i-1,j} + \mu_1 \alpha_j \sum_{i=1}^{\infty} z^i p_{i+1,1} + \mu_2 \alpha_j \sum_{i=1}^{\infty} z^i p_{i+1,2}.$$

Simplifying, we get

$$(\lambda + \mu_j)\phi_j(z) = z\lambda\alpha_j p_{0,0} + \lambda z \phi_j(z) + \mu_1 \alpha_j (\phi_1(z) - p_{1,1} z)/z + \mu_2 \alpha_j (\phi_2(z) - p_{1,2} z)/z.$$

Collecting terms,

$$(\lambda(1-z) + \mu_j)\phi_j(z) = z\lambda\alpha_j p_{0,0} + \frac{\alpha_j}{z}(\mu_1 \phi_1(z) + \mu_2 \phi_2(z)) - \alpha_j(\mu_1 p_{1,1} + \mu_2 p_{1,2}).$$

Using the first balance equation, we get

$$(\lambda(1-z) + \mu_j)\phi_j(z) = \lambda\alpha_j (z-1) p_{0,0} + \frac{\alpha_j}{z}(\mu_1 \phi_1(z) + \mu_2 \phi_2(z)), \quad j = 1, 2.$$

These are two equations for $\phi_1(z)$ and $\phi_2(z)$. Solving simultaneously, and simplifying, we get

$$\phi_i(z) = \frac{\lambda\alpha_i(\lambda(1-z) + \mu_j)}{\frac{\mu_1\mu_2}{z} - \lambda\mu_1(1 - \frac{\alpha_1}{z}) - \lambda\mu_2(1 - \frac{\alpha_2}{z}) - \lambda^2(1-z)}.$$

P16. Using the birth and death rates given in the solution to Modeling Exercise 14, we get the following

$$\rho_0 = 1, \quad \rho_i = \Pi_{j=1}^{i} \frac{\lambda}{\mu + (j-1)\theta}, \quad i \geq 1.$$

This birth and death process is always stable, and the limiting distribution is given by

$$p_i = \frac{\rho_i}{\sum_{j=0}^{\infty} \rho_j}, \quad i \geq 0.$$

P17. If $Q$ is irreducible, it has a unique limiting distribution $p = [p_1, ..., p_N]$. Since $Q$ is doubly stochastic, we can verify that $p_i = 1/N, \ \ 1 \leq i \leq N$ satisfies the balance equations:

$$\sum_{i=1}^{N} p_i q_{ij} = (1/N) \sum_{i=1}^{N} q_{ij} = 0.$$

Now let $\{X(t), t \geq 0\}$be a $PP(\lambda)$, and define $Y(t) = X(t) mod(21)$. Then $\{Y(t), t \geq 0\}$ is a CTMC on $\{0, 1, ..., 20\}$ with transition rates

$$q_{i,i+1} = \lambda, \ \ 0 \leq i \leq 19, \ \ q_{20,0} = \lambda.$$

Thus the $Q$ matrix is doubly stochastic. Hence the limiting distribution of $Y$ is uniform over the state space. Now $X(t)$ is divisible by 3 or 7 if and only if $Y(t) \in \{0, 3, 6, 7, 9, 12, 14, 15, 18\}$. Hence the desired probability is $9/21 = 3/7$.