SOLUTION HW 6.

P22. $X(t)$ = number of customers in an $M/M/1/K$ queue at time t. $X(0) = 1$. Let $T = \min\{t \geq 0 : X(t) = 0\}$, and $m_i = \mathcal{E}(T, |X(0) = i)$. The expected time until an arrival to an empty system is thus thus $m_1 + 1/\lambda$. m_1 can be computed using the results of Example 6.36. We have

$$\frac{1}{\lambda_j \rho_j} = \frac{1}{\mu}(\lambda/\mu)^{j-1}, \quad 1 \leq j \leq K.$$

Hence, from Equation 6.220,

$$\begin{aligned} m_1 &= \sum_{j=1}^{K} \frac{1}{\mu}(\lambda/\mu)^{j-1} \\ &= \frac{1}{\mu}\frac{1-(\lambda/\mu)^K}{1-\lambda/\mu}. \\ m_1 + 1/\lambda &= \frac{1}{\lambda}\frac{1-(\lambda/\mu)^{K+1}}{1-\lambda/\mu}. \end{aligned}$$

P23. Using the result of Example 6.38 with $c = 0$, we get R, the total expected discounted revenue from a single machine (operating at time 0) over the infinite time horizon as

$$R = \frac{r(\alpha+\lambda)}{\alpha(\alpha+\lambda+\mu)}.$$

Hence the total discounted revenue from k machines is kR.

P25. We describe a space as E if it is empty, B if it is occupied by a car in service, and W if it is occupied by a car that is waiting to begin service or has finished service. The state space is $S = \{1 = EEE, 2 = BEE, 3 = BBE, 4 = EBE, 5 = BBW, 6 = BWE, 7 = EBW, 8 = BWW\}$. Thus state is EBW if the space 1 is empty, space two has a car that is pumping gas, and space 3 is occupied by a car that is waiting for service. The triggering event analysis yields the following rate matrix:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\lambda+2\mu) & \mu & \lambda & \mu & 0 & 0 \\ \mu & 0 & 0 & -(\lambda+\mu) & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -2\mu & 0 & \mu & \mu \\ \mu & 0 & 0 & 0 & 0 & -(\lambda+\mu) & 0 & \lambda \\ 0 & \mu & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & -\mu \end{bmatrix}.$$

The balance equations are $pQ = 0, \sum_1^8 p_i = 1$. These can be solved to obtain p

The long run fraction of the customers who enter =

$$1 - p_5 - p_7 - p_8 = \frac{2*(2\rho+1)(2+\rho)(1+\rho)}{3\rho^4 + 11\rho^3 + 14\rho^2 + 14\rho + 2}.$$

P26. Let $X(t)$ be the number of machines in use at time t, and $Y(t)$ be the status of the standby machine at time t (0 if there is none, U if it is up, and D if it is down with an undetected failure). The state of the system is $(X(t), Y(t))$. The state-space is $\{1 = (2, U), 2 = (2, 0), 3 = (2, D), 4 = (1, 0), 5 = (0, 0)\}$. The triggerning event nalysis yields the following rate matrix

$$Q = \begin{bmatrix} -(2\lambda+\theta) & 2\lambda & \theta & 0 & 0 \\ \mu & -(2\lambda+\mu) & 0 & 2\lambda & 0 \\ 0 & 0 & -2\lambda & 2\lambda & 0 \\ 0 & \mu & 0 & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 & \mu & -\mu \end{bmatrix}.$$

The forward equations are:

$$\begin{aligned} p'_{i1}(t) &= -(2\lambda+\theta)p_{i1}(t) + \mu p_{i2}(t), \\ p'_{i2}(t) &= -(2\lambda+\mu)p_{i2}(t) + 2\lambda p_{i1}(t) + \mu p_{i4}(t), \\ p'_{i3}(t) &= -2\lambda p_{i3}(t) + \theta p_{i1}(t), \\ p'_{i4}(t) &= -(\lambda+\mu)p_{i4}(t) + 2\lambda p_{i2}(t) + 2\lambda p_{i3}(t) + \mu p_{i5}(t), \\ p'_{i5}(t) &= -\mu p_{i5}(t) + \lambda p_{i4}(t), \end{aligned}$$

with boundary conditions: $p_{ij}(0) = \delta_{ij}$. The balance equations are:

$$(2\lambda+\theta)p_1 = \mu p_2,$$

$$(2\lambda+\mu)p_2 = 2\lambda p_1 + \mu p_4,$$

$$2\lambda p_3 = \theta p_1,$$

$$(\lambda+\mu)p_4 = 2\lambda p_2 + 2\lambda p_3 + \mu p_5,$$

$$\mu p_5 = \lambda p_4,$$

$$\sum_1^5 p_i = 1.$$

The repair person is idle in states 1 and 2. Hence the desired probability is given by

$$p_1 + p_2 = \frac{2\lambda\mu^2(\mu+\theta+2\lambda)}{8\lambda^4 + 4\lambda^3\theta + 8\lambda^3\mu + 6\lambda^2\mu\theta + 4\lambda\mu^2\theta + 2\lambda\mu^3 + \mu^3\theta + 4\lambda^2\mu^2}.$$

P28. By using the rates in the solution to Modeling problem 22, we get the following balance equations:

$$\begin{aligned} \lambda p_{R+K} &= \theta p_R \\ \lambda p_{R+K-i} &= \theta p_{R-i} + \lambda p_{R+K+1-i}, \quad i = 1, 2, .., R, \\ \lambda p_i &= \lambda p_{i+1}, \quad i = R+1, ..., K-1, \end{aligned}$$

$$(\lambda+\theta)p_i = \lambda p_{i+1}, \quad i = 1, 2, ..., R,$$

$$\theta p_0 = \lambda p_1.$$

From the third set of equations, we get

$$p_i = A(\text{ a constant}), \quad i = R+1, ..., K.$$

Using this in the fourth set of equations, and solving recursively, we get

$$p_i = (\frac{\lambda}{\lambda+\theta})^{R-i+1}A, \quad i = 1, 2, ..., R.$$

The last balance equation yields

$$p_0 = \frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R A.$$

Using these in the first two sets of the balance equations, and solving recursively, we get

$$p_i = (1 - (\frac{\lambda}{\lambda+\theta})^{R+K+1-i})A, \quad i = K+1, K+2, ..., K+R.$$

Summing the above four solutions, and using th normalizing equation, we get

$$A = \frac{1}{K + \frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R}.$$

This gives the limiting distribution of the $\{X(t), t \geq 0\}$ process.
Demands are lost in state 0. Hence the desired answer is

$$p_0 = \frac{\frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R}{K + \frac{\lambda}{\theta}(\frac{\lambda}{\lambda+\theta})^R}.$$