SOLUTION HW 7.

P1. Let $X(t)$ = number of customers in the system. $\{X(t), t \geq 0\}$ is a birth and death process with birth parameters

$$\lambda_i = \begin{cases} \lambda_1 + \lambda_2 & 0 \leq i < K \\ \lambda_2 & i \geq K \end{cases}$$

and death parameters

$$\mu_i = \mu, i \geq 1.$$

Let

$$\rho_j = \begin{cases} \left(\frac{\lambda_1 + \lambda_2}{\mu}\right)^j & 0 \leq j < K \\ \left(\frac{\lambda_2}{\mu}\right)^{j-K} \left(\frac{\lambda_1 + \lambda_2}{\mu}\right)^K & j \geq K. \end{cases}$$

The system is stable when

$$\sum_{i=0}^{\infty} \rho_i = \sum_{i=0}^{K-1} \left(\frac{\lambda_1 + \lambda_2}{\mu}\right)^i + \sum_{i=K}^{\infty} \left(\frac{\lambda_2}{\mu}\right)^{i-K} \left(\frac{\lambda_1 + \lambda_2}{\mu}\right)^K < \infty \Leftrightarrow \lambda_2 < \mu.$$

Assuming stability,

$$p_0 = 1 \Bigg/ \left(\frac{1 - ((\lambda_1 + \lambda_2)/\mu)^K}{1 - (\lambda_1 + \lambda_2)/\mu} + \left(\frac{\lambda_1 + \lambda_2}{\mu}\right)^K \frac{1}{1 - \lambda_2/\mu} \right),$$

and $p_j = \rho_j p_0$, $j \geq 1$.

P2. Let $X(t)$ be the number of items in the warehouse at time t. $\{X(t), t \geq 0\}$ is a birth and death process with birth rates

$$\lambda_i = \lambda, \quad i \geq 0,$$

and death rates

$$\mu_i = \mu, \quad i \geq 1.$$

Hence, it is an $M/M/1$ queue with traffic intensity

$$\rho = \lambda/\mu.$$

Hence, from Section 3.1, it is stable if $\rho < 1$. When it is stable, its limiting distribution is given by

$$p_j = (1 - \rho)\rho^j, \quad j \geq 0.$$

P6. Note that the server is idle if the system is empty, and busy if there are N or more customers in it. Otherwise, the server may be idle or busy. Hence the state-space is $S = \{0\} \cup \{(i, j) : 1 \leq i \leq N - 1, j = I, B\} \cup \{N, N + 1, N + 2, ...\}$. The positive transition rates are

$$q_{0,(1,I)} = \lambda, \quad q_{(i,I),(i+1,I)} = \lambda, \quad 1 \leq i \leq N - 2, \quad q_{(N-1,I),N} = \lambda,$$

$$q_{(i,B),(i+1,B)} = \lambda, \quad i \geq 1, \quad q_{(i,B),(i-1,B)} = \mu, \quad i \geq 2, \quad q_{(1,B),0} = \mu.$$

The balance equations are

$$\lambda p_0 = \mu p_{1,B},$$

$$\lambda p_{(1,I)} = \lambda p_0,$$
$$\lambda p_{(i,I)} = \lambda p_{(i-1,I)}, \ \ 2 \le i \le N-1,$$
$$(\lambda+\mu)p_{(1,B)} = \mu p_{(2,B)}, \ \ (\lambda+\mu)p_{(i,B)} = \lambda p_{(i-1,B)} + \mu p_{(i+1,B)}, \ \ i \ge 2, \ \ i \ne N,$$
$$(\lambda+\mu)p_{(N,B)} = \lambda p_{(N-1,B)} + \mu p_{(N+1,B)} + \lambda p_{(N-1,I)}.$$

From Equations 2 and 3, we get

$$p_{(i,B)} = p_0, \quad 1 \le i \le N-1.$$

Let $\rho = \lambda/\mu$. Using the fourth set of equations for $i > N$, we get

$$p_i = \rho^{N-i} p_N, \quad i \ge N.$$

Using the fourth set of equations for $1 \le i < N$, we get

$$p_i = \rho \sum_{k=0}^{i-1} \rho^i p_0 = \rho \frac{1-\rho^i}{1-\rho} p_0, \quad 1 \le i \le N.$$

Using the normalization equation, and assuming $\rho < 1$, we get

$$p_0 = \frac{1-\rho}{N}.$$

This yields the solution given in the book.

Comp, P1. From Equation 7.63, we get the following generating function of X, the steady state number in the $M|M|1$ system:

$$\phi(z) = \sum_{j=0}^{\infty} p_j z^j = (1-\rho)\sum_{j=0}^{\infty} (\rho z)^j = \frac{1-\rho}{1-\rho z}.$$

Hence,

$$L = E(X) = \phi'(z)|_{z=1} = -\frac{1-\rho}{(1-\rho z)^2}(-\rho)|_{z=1} = \frac{\rho}{1-\rho}.$$

$$L^{(2)} = E(X(X-1)) = \phi''(z)|_{z=1} = 2\frac{1-\rho}{(1-\rho z)^3}(\rho^2)|_{z=1} = 2(\frac{\rho}{1-\rho})^2.$$

Hence,

$$\sigma^2 = Var(X) = L^{(2)} + L - L^2 = 2(\frac{\rho}{1-\rho})^2 + \frac{\rho}{1-\rho} - (\frac{\rho}{1-\rho})^2 = \frac{\rho}{(1-\rho)^2}.$$

P5. By PASTA,

$$\hat{\pi}_j = p_j, \quad j \ge 0.$$

Then

$$\begin{aligned}
\mathcal{P}(\text{An arriving customer joins}) &= \sum_{j=0}^{\infty} \mathcal{P}(\text{An arriving customer joins} \mid \hat{X}_n = j)\mathcal{P}(\hat{X}_n = j) \\
&= \sum_{j=0}^{\infty} \alpha_j \hat{\pi}_j \\
&= \sum_{j=0}^{\infty} \alpha_j p_j.
\end{aligned}$$

P8. When there are i customers in the system, $\min(i, s)$ servers are busy. Hence, the expected number of busy servers is given by

$$\begin{aligned}
\sum_{i=0}^{s} ip_i + s\sum_{i=s+1}^{\infty} p_i &= p_0[\sum_{i=0}^{s} i(\lambda/\mu)^i/i! + s(s^s/s!)\sum_{i=s+1}^{\infty}(\lambda/s\mu)^i] \\
&= (\lambda/\mu)p_0[\sum_{i=0}^{s-1}(\lambda/\mu)^i/i! + (s^s/s!)(\lambda/s\mu)^s/(1-\lambda/s\mu))] \\
&= \lambda/\mu,
\end{aligned}$$

since

$$p_0 = [\sum_{i=0}^{s-1}(\lambda/\mu)^i/i! + (s^s/s!)(\lambda/s\mu)^s/(1-\lambda/s\mu))]^{-1}.$$

The expected number in the system is given by

$$\begin{aligned}
L = &= \sum_{i=1}^{\infty} ip_i \\
&= \sum_{i=0}^{s} ip_i + s\sum_{i=s+1}^{\infty} p_i + \sum_{i=s+1}^{\infty}(i-s)p_i \\
&= \mathcal{E}(\text{Number of Busy servers}) + \sum_{i=s+1}^{\infty}(i-s)p_i \\
&= \lambda/\mu + p_0(s^s/s!)\sum_{i=s+1}^{\infty}(i-s)(\lambda/s\mu)^i \\
&= (\lambda/\mu) + \rho p_0(s^s/s!)(\lambda/s\mu)^s/(1-\rho)^2 \\
&= \frac{\lambda}{\mu} + p_s\frac{\rho}{(1-\rho)^2}.
\end{aligned}$$