SOLUTION HW 9.

P17. $\tilde{G}(s) = \frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu}$.

$$\tilde{M}(s) = \frac{\tilde{G}(s)}{1-\tilde{G}(s)} = \frac{\lambda\mu}{s(s+\lambda+\mu)} = \frac{\lambda\mu}{\lambda+\mu}\frac{1}{s} - \frac{\lambda\mu}{\lambda+\mu}\frac{1}{s+\lambda+\mu}.$$

$$M(t) = \frac{\lambda\mu}{\lambda+\mu}t - \frac{\lambda\mu}{(\lambda+\mu)^2}(1-\mathrm{e}^{-(\lambda+\mu)t}).$$

P21. If $t \le x$, $H(t) = 1$. This can be written as

$$H(t) = 1 - G(t) + \int_0^t H(t-u)\,dG(u).$$

If $t > x$,

$$\Pr\{A(t) \le x \mid X_1 = u\} = \begin{cases} H(t-u) & 0 \le u < t-x \\ 1 & t-x \le u < t \\ 0 & u \ge t \end{cases}.$$

Unconditioning gives

$$\begin{aligned} H(t) &= \int_{t-x}^t dG(u) + \int_0^{t-x} H(t-u)\,dG(u) \\ &= \int_{t-x}^t dG(u) + \int_0^t H(t-u)\,dG(u) - \int_{t-x}^t H(t-u)\,dG(u) = \int_0^t H(t-u)\,dG(u). \end{aligned}$$

Thus H satisfies the following renewal type equation

$$H(t) = D(t) + \int_0^t H(t-u)\,dG(u)$$

where

$$D(t) = \begin{cases} 1-G(t) & t \le x \\ 0 & \text{otherwise} \end{cases}.$$

Then D is monotone and bounded. Assuming $G(\cdot)$ is aperiodic, we get

$$\lim_{t\to\infty} H(t) = \frac{1}{\tau}\int_0^x (1-G(t))\,dt.$$

P23. We have

$$P(N(t) \text{ is odd}|X_1 = x) = \begin{cases} 1-p(t-x) & \text{if } x \le t \\ 1 & \text{if } x \ge t \end{cases}.$$

Hence, unconditioning yields

$$p(t) = \int_0^t (1-p(t-x))dG(x) + \int_t^\infty dG(x) = 1 - \int_0^t p(t-x)dG(x).$$

This is not a renewal equation due to the minus sign on the right hand side.

P24. Let U_1 = first up time, D_1 = first down time and $S_1 = U_1 + D_1$. Use renewal argument by conditioning on S_1.

$$E(W(t) \mid S_1 = x) = \begin{cases} E(U_1 \mid S_1 = x) + H(t-x) & \text{if } x < t \\ E(\min(U_1, t) \mid S_1 = x) & \text{if } x \geq t \end{cases}$$

Unconditioning

$$\begin{aligned} H(t) &= \int_0^t E(U_1 \mid S_1 = x)\, dG(x) + \int_t^\infty E(\min(U_1, t) \mid S_1 = x)\, dG(x) + \int_0^t H(t-x)\, dG(x) \\ &= E(\min(U_1, t)) + \int_0^t H(t-x)\, dG(x) \\ &= \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda t}) + H * G(t). \end{aligned}$$

Taking LSTs, we get

$$\tilde{H}(s) = \frac{1}{s+\lambda} + \tilde{H}(s)\frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu}.$$

This yields,

$$\tilde{H}(s) = \frac{\mu + s}{s(s+\lambda+\mu)} = \frac{\mu}{\lambda+\mu}\frac{1}{s} + \frac{\lambda}{\lambda+\mu}\frac{1}{s+\lambda+\mu}.$$

Inverting, we get

$$H(t) = \frac{\mu}{\lambda+\mu}t + \frac{\lambda}{(\lambda+\mu)^2}(1 - \mathrm{e}^{-(\lambda+\mu)t}).$$

P26. Let $H(t) = E(A(t)^k)$. $E(A(t)^k \mid X_1 = u) = \begin{cases} H(t-u) & u < t \\ t^k & u \geq t. \end{cases}$

$$H(t) = \int_t^\infty t^k\, dG(u) + \int_0^t H(t-u)\, dG(u) \text{ and } D(t) = \int_t^\infty t^k\, dG(u) = t^k(1 - G(t)).$$

$$\lim_{t\to\infty} E(A(t)^k) = \frac{1}{\tau}\int_0^\infty \int_t^\infty t^k\, dG(u)\, dt = \frac{1}{\tau}\int_0^\infty \int_0^u t^k\, dt\, dG(u) = \frac{E(X^{k+1})}{(k+1)\tau}.$$