OR221 MIDTERM

Due: Thursday March 4, 9:30am

1. The exam is open book/open notes.
2. Attempt all problems. Show all work.
3. All problems carry equal weight.

P1. Customers arrive at a taxi stand according to a Poisson process with rate λ. If a taxi is waiting at the taxi stand, the customer immediately hires it and leaves the taxi stand in the taxi. If there are no taxis available, the customer waits. There is essentially infinite waiting room for the customers. Independently of the customers, taxis arrive at the taxi stand according to a Poisson process with rate μ. There is enough space at the taxi stand to accommodate K (a fixed positive integer) taxis. If a taxi arriving at the taxi stand finds it full, it leaves immediately. Note that customers and taxis cannot be waiting at the taxi stand simultaneously. Model this system as a CTMC as follows:

- What is the state space?
- Show the rate diagram.
- What is the condition of stability?
- Compute the long run probability that there are no taxis waiting at the taxi stand.

P2. A machine can be in two states: up or down. When it is up it fails at rate 1 per day. When it is down, it is repaired at rate 1 per hour, as long as a repair person is on duty. If a repair person is not on duty, a failed machine stays failed. A repair person is on duty from 8am to 5pm every day, and off duty during the remaining time. Suppose the machine is functional at 9am today. What is the probability that it is functional at 9am tomorrow?

P3. Consider a machine shop with two machines and two repair persons. The lifetimes of the machines are iid $\exp(\mu)$. Each repair takes on the average $1/\lambda$ amount of time, and the successive repair times are iid exponential random variables. When both machines are down each machine is repaired by one repair person. However, if only one machine is down, both repair persons work on it together so that the repair occurs at twice the speed. Suppose machine 1 fails at time zero and both repair persons are available at that time. Let T be the time when machine 1 becomes operational again. Compute $E(T)$.

P4. A machine produces items according to a Poisson process with rate 10 per hour. The produced items are stored in a buffer. Demands for these items occur according to a Poisson process with rate 8 per hour. If an item is available in the buffer when a demand occurs, the demand is satisfied immediately. If the buffer is empty when a demand occurs, the demand is lost. It costs 1 dollar to keep an item in the buffer for one hour. Each item sells for ten dollars. The machine is turned off when the number of items in the buffer reaches K, a fixed positive number, and is turned on when the number of items in the buffer falls to 1. It does not cost anything to turn the machine on and off.

- Model this system as a CTMC. State the state space and show the rate diagram.
- Compute the long run net income (revenue-holding cost) per unit time, as a function of K.
- Compute numerically the optimal K that maximizes the net income per unit time.

P5. Let $\{X(t), t \geq 0\}$ be a CTMC on state space $\{0, 1, 2, ...\}$. Suppose the system earns reward at rate r_i per unit time when the system is in state i. Let $g(i)$ be the total expected reward earned by the system until it reaches state 0, given that it starts in state i. Using first passage time derive a set of equations satisfied by $\{g(i), i = 1, 2, 3, ...\}$.