Supply Chain Contracts

Reference: “Supply Chain Contracting and Coordination with Stochastic Demand”
by Martin Lariviere

Overview
1. Integrated Firm
2. Price-only contracts
3. Buy-back contracts
4. Quantity flexibility contracts
5. Penalty methods and standard setting methods
Integrated Channel Performance

Definition: Integrated Firm = firm that controls both the manufacturing and the sales to the public

Assumptions and Notation:

- One period setting
- One manufacturer sells to one retailer
- Retailer faces external demand
- Unmet demand is lost
- \( c = \) manufacturer’s cost per unit
- \( r = \) retailer’s price to customer per unit
- \( v = \) salvage value per unit
- \( r > c > v \)
- \( d = \) demand with cdf \( F(d) \) and pdf \( f(d) \)
- Demand distribution, cost, and revenue info is common knowledge
- \( y = \) stocking level
- \( \Pi_I = \) profits of an integrated firm
Analysis for Integrated Firm

Profit Function for Integrated Firm:

\[ \Pi_I(y) = (r - c)y - (r - v) \int_0^y F(d)dd \]
\[ = ry - cy - r \int_0^y (y - d) f(d)dd + v \int_0^y (y - d) f(d)dd \]

Find optimal \( y_I \):

\[ \Pi_I'(y) = (r - c) - (r - v)F(y) = 0 \]

\[ \therefore y_I = F^{-1}(\frac{r-c}{r-v}) \]

Thus, \( \Pi_I(y_I) = \Pi^*_I = \text{Max system profits} \)

Note: system profits are completely determined by the stocking level
Price-Only Contracts

Assumptions and Notation:

- Two players: manufacturer & retailer
- Manufacturer acts as a Stackelberg leader (offer terms of trade as take-it or leave-it; retailer can only accept or reject)
- Retailer accepts if can earn a non-negative return
- \( w \) = wholesale price offered by manufacturer to retailer
- \( w < r \)
- Retailer retains any excess stock but may salvage excess for \( v \) per unit

Analysis of Retailer’s Problem

Profit Function for Retailer:
\[
\Pi_R(y) = (r - w)y - (r - v) \int_0^y F(d)dd \\
= ry - wy - r \int_0^y (y-d) f(d)dd + v \int_0^y (y-d) f(d)dd
\]

Find optimal \( y(w) \) :
\[
\Pi_R'(y) = (r - w) - (r - v)F(y) = 0
\]
\[
\therefore \quad y(w) = F^{-1}\left(\frac{r-w}{r-v}\right)
\]
Analysis of Manufacturer’s Problem

Note: As a Stackelberg leader, the manufacturer correctly anticipates how retailer will order for any $w$
$\implies$ Manufacturer anticipates a demand curve $y(w)$

Profit Function for Manufacturer:

$$\Pi_M(w) = (w - c)y(w)$$

$$= (w - c)F^{-1}(\frac{r-w}{r-v})$$

- Manufacturer’s profits are deterministic (knows $y$ for every $w$ and bears no costs for leftovers)
- All uncertainty is on retailer
- $\Pi_M(w)$ is not convenient to work with $\therefore$ find another expression
Alternative Profit Function for Manufacturer

- Work with inverse demand curve instead of demand curve, \( y(w) \):

\[
w(y) = (r - v) \overline{F}(y) + v
\]

Interpretation:
* Previously, choose \( w \) anticipating selling the most the retailer would take.
* Now manufacturer chooses quantity to sell anticipating the most per unit that the retailer will pay.
* Both are equivalent b/c manufacturer has a monopoly position.

- Alter costing conventions: (markups over salvage value)

let \( \hat{w} = w - v \)

let \( \hat{r} = r - v \)

let \( \hat{c} = c - v \)

\[
\therefore \hat{w}(y) = \hat{r} \overline{F}(y) \quad \text{(Inverse Demand Curve)}
\]

Simpler Profit Function for Manufacturer:
\[
\Pi_M(y) = y(\hat{w}(y) - \hat{c})
\]

\[
= y(\hat{r} \overline{F}(y) - \hat{c})
\]

Now can choose \( y \) to \( \text{Max} \ \Pi_M(y) \).
Characterizing the Optimal Solution for Manufacturer’s Problem

- Manufacturer will never sell more than \( y_I \) because \( w \) cannot be \(< c \)
- Hazard rate of demand distribution \( h(d) = \frac{f(d)}{F(d)} \)
- \( h(d)dd = P(\text{demand will lie in interval } [d,d+dd] \text{ given that demand is at least } d) \)
- Increasing failure rate (IFR): a distribution is IFR if \( h'(d) > 0 \) for all \( d \)
- Generalized failure rate (GFR): \( g(d) = dh(d) \)
- Increasing generalized failure rate (IGFR): a distribution is IGFR if \( g'(d) > 0 \) for all \( d \)

**Theorem 1.** Let \( v(d) \) denote the own-price elasticity of the retailer’s orders to the manufacturer, and let \( y^1 \) be the smallest value of \( d \) s.t. \( g(d)=1 \). If no such \( d \) exists, let \( y^1 = \infty \).

1. The elasticity of retailer’s orders is given by \( v(d) = \frac{1}{g(d)} \)

2. The manufacturer’s first order conditions may be written as \( \hat{w}(y)(1 - \frac{1}{v(y)}) = c \)

3. If the demand distribution is IGFR, then the manufacturer’s profits are unimodal on \([0,\infty)\), concave on \([0,y^1]\), and strictly decreasing on \([y^1, \infty)\). Any
solution to #2 above is a unique global maximum and must lie in the interval $[0, y^1]$.

Comments:

• Possible for manufacturer’s problem to have multiple local maxima and no solution
  
  – Some distributions fail to yield sensible solutions
  
  – Need limits on demand distribution to assure first order conditions have a unique solution

• The retailer’s ordering policy becomes an induced demand curve to the manufacturer

• $v(d)$ is the elasticity of the orders the retailer places with the manufacturer
  
  – $v(d)$ measures the percent change in the retailer’s orders for a one percent change in $w$
  
  – If $v(d)>1$ (price elastic demand), as $w \downarrow$ total revenue $\uparrow$
  
  – Optimal sales quantity lies on the elastic portion of the demand curve

• #1: Relationship between elasticity of orders and the generalized failure rate
  
  – IGFR $\iff$ elasticity of orders falls monotonically $\iff$ MR falls monotonically and can equal MC at only one point

• #2: Differentiate revenue and set $MR = MC$ for profit maximizing point
- optimal wholesale price

- #3: IGFR is the correct sufficient condition for a well-behaved profit function
  - Limits the distributions that one can be assured are well-behaved
  - Conditions for $\Pi_M$ to be unimodal
  - Maximum occurs between 0 and $y^1$
• IGFR gives some additional attractive properties...

Theorem 2 If the demand distribution is IGFR, then the optimal sales quantity \( y^* \) is increasing in \( \hat{r} \) and decreasing in \( \hat{c} \). Additionally, \( \bar{w}(y^*) \geq \bar{w}(y^1) \).

Comments:
• Provides a lower bound on the manufacturer’s profit maximizing wholesale price
• \( y^* \) must lie in \( [0, y^1] \)
• \( y^1 \) can be found from the generalized failure rate

Special Cases:

• More assumptions regarding the demand distribution allow more analysis

**Definition 1** Scaled Family. Distribution is from a scaled family if the distribution depends on a parameter \( \theta \) and there exists an increasing, positive function \( \tau(\theta) \) s.t. \( F(d|\theta) = F\left(\frac{d}{\tau(\theta)}\right)\).

**Definition 2** Shifted Family. Distribution is from a shifted family if for some parameter \( \theta \geq 0 \), \( F(d|\theta) = F(d-\theta|\theta) \).

Theorem 3 Suppose that the demand distribution \( F(d|\theta) \) is IGFR for all values of \( \theta \).

1. If \( F(d|\theta) \) is from a scaled family, the optimal order quantity is proportional to \( \tau(\theta) \) and
the resulting \( w \) is independent of \( \theta \). That is, 
\[ y^*(\theta) = \tau(\theta)y^*(1) \text{ and } \hat{\omega}(\theta) = \hat{\omega}(1) \]
where \( y^*(\theta) \) and \( \hat{\omega}(\theta) \) are the optimal values of \( y \) and \( \hat{\omega} \) given \( \theta \).

2. If \( F(d|\theta) \) is a shifted family, then for \( \Delta > 0 \) : 
\[ y^*(\theta + \Delta) < y^*(\theta) + \Delta \text{ and } \hat{\omega}^*(\theta) < \hat{\omega}^*(\theta + \Delta). \]

Comments:

• #1: If demand distribution is IGFR and from a scaled family,
  
  – then can find \( y^*(1) \) and plug in \( \tau(\theta) \) to find optimal order quantity
  
  – optimal \( w \) doesn’t depend on distribution parameter

• #2: If demand distribution is IGFR and from a shifted family,

  – then a higher \( \theta \) corresponds to a larger market 
    . leads to increasing \( y \), but the increase is less than the increase in market size
  
  – shifted distribution leads to manufacturer charging a higher price
Theorem 4. If the demand distribution $F(d|\theta)$ is from either a shifted or a scaled family, then the manufacturer’s profits are increasing in $\theta$.

Comments:

- for scaled families as $\theta \uparrow$ c.v. stays the same
  - but $y \uparrow \implies \Pi_M \uparrow$
- for shifted families as $\theta \uparrow$ c.v.↓
  - optimal w is higher for lower coefficients of variation $\implies \Pi_M \uparrow$

Theorem 5. Suppose demand is normally distributed. The optimal w is determined by the coefficient of variation and the manufacturer charges more the smaller the coefficient of variation is.

Comments:

- normal is a scaled family and w depends only on c.v.
- w increases as variability decreases $\implies$ c.v.↓
Supply Chain Performance

Summary:

- Determined best for manufacturer, but not best for entire supply chain
  (i.e., sells $y < y_I$ or sets $w > c \implies r > r_I$ . . . $\Pi_D < \Pi_C$)
- Remedies to coordinate system (allowing $\Pi_D = \Pi_C$):

  1. Franchising
  2. Quantity Forcing

- Both allow for arbitrary split of profits

Franchising

- Manufacturer charges $A$ to carry the product (regardless of $y$) and then sells at $w$
- Retailer will pay any $A$ s.t. $A \leq \Pi_R(w) - \kappa$ where $\kappa =$ retailer’s opportunity cost for carrying product
- $\Pi_R(w) \downarrow$ as $w \uparrow$
- Largest $A$ that manufacturer can take is: $A = \Pi_R(c) - \kappa = \Pi_I^* - \kappa$
- Thus, manufacturer captures all channel profits except retailer’s $\kappa$
- Lump sum $A$ serves to redistribute profits

Quantity Forcing
• Manufacturer offers \( w \) and insists that retailer take some quantity \( Q \).
• If \( Q = y_I \), then system profits = \( \Pi^*_I \).
• Retailer’s profit = \( \Pi^*_I - (w - c)y_I \).
• Manufacturer’s profit = \( (w - c)y_I \).
• Retailer accepts contract if \( \kappa \leq \Pi^*_I - (w - c)y_I \).

\[ \therefore \text{accepts if } w \leq \frac{\Pi^*_I - \kappa}{y_I} + c \]
Buy-Back Contracts (a return policy)

Assumptions and Notation:

- Manufacturer posts $w$
- $b = \text{buy back rate (manufacturer buys back any unsold stock)}$
- $b < w$ (no arbitrage)
- If $d < y$, then retailer gets $b(y-d)$
- $v = \text{salvage value to the manufacturer}$

Analysis of Retailer’s Problem

Profit Function for Retailer under Buy-Back Contract:

$$\Pi_R(y) = (r - w) y - (r - b) \int_0^y F(d)dd$$

$$= ry - wy - r \int_0^y (y-d) f(d)dd + b \int_0^y (y-d) f(d)dd$$

Note: as $b \uparrow \Pi_R(y) \uparrow :$ the retailer enjoys a generous return policy

Find optimal $y(w, b)$:

$$\Pi'_R(y) = (r - w) - (r - b)F(y) = 0$$

$$\therefore y(w, b) = F^{-1}\left(\frac{r-w}{r-b}\right)$$

Analysis of Manufacturer’s Problem:
Profit Function for Manufacturer under Buy-Back Contract:

\[ \Pi_M(w, b) = (w - c)y(w, b) - (b - v) \int_0^{y(w, b)} F(d) \, dd \]
Theorem 6: The Hessian of $\Pi_M(w, b)$ is nowhere negative definite. Thus, the second order conditions for a local maximum are never satisfied.

Comments:

- Buy backs are more complicated than price-only contracts
- Unlike price-only contracts, cannot develop sufficient conditions on demand distribution for manufacturer’s problem to be well-behaved
- Cannot identify the optimal contract through standard optimization

Coordinating Contracts:

- No returns has been shown to be suboptimal for entire system
- Full returns has been shown to be suboptimal for entire system
- Need an intermediary policy for channel coordination
Theorem 7  Suppose the manufacturer offers a contract \((w(\varepsilon), b(\varepsilon))\) for \(\varepsilon \in (0, r - c]\) where
\[
w(\varepsilon) = r - \varepsilon \quad \text{and} \quad b(\varepsilon) = r - \frac{\varepsilon(r - \nu)}{r - c}.
\]

1. The retailer orders the integrated channel quantity (i.e., \(y(w(\varepsilon), b(\varepsilon)) = y_I\)) and system profits are equal to the integrated channel profits.

2. Retailer profits are increasing in \(\varepsilon\). Specifically, \(\Pi_R(w(\varepsilon), b(\varepsilon)) = \frac{\varepsilon}{r - c}\Pi_I^*\).

3. Manufacturer profits are decreasing in \(\varepsilon\). Specifically, \(\Pi_M(w(\varepsilon), b(\varepsilon)) = (1 - \frac{\varepsilon}{r - c})\Pi_I^*\).

Comments:

- The coordinating contract is not unique (a continuum of contracts exist)
- Contracts differ in how to divide channel profits
- Under buy-back contracts, manufacturer can manipulate 2 parameters \((w(\varepsilon), b(\varepsilon))\)
- Manufacturer wants \(\varepsilon\) low, but suboptimal if \(\varepsilon = 0\). (full returns policy)
- No optimal contract exists
- Generous b’s are paired with high w’s, under coordinating contracts
- Coordinating contracts \((w(\varepsilon), b(\varepsilon))\) are independent of the demand distribution
- Manufacturer can contract w/ other retailers w/ different demand distributions, but same cost structure
- Manufacturer can capture same fraction of channel profits across different retailers
- Manufacturer can use one contract to coordinate multiple retailers

- Retailer has opportunity cost $\kappa$ and will not take just any contract the manufacturer offers
  - Manufacturer chooses: $\varepsilon = \frac{\kappa(r-c)}{\Pi_f}$ (manufacturer takes all but $\kappa$)
  - Conversely, if $\varepsilon = r - c$, then manufacturer transfers at cost and $\Pi_M = 0$. (retailer takes all)
  - As $\varepsilon \rightarrow 0, \Pi_M \uparrow$
  - Retailers profits are opposite

- Both players are risk neutral
  - Concerned only with their mean earnings and not the variance of these earnings
  - All coordinating contracts lie on the Pareto frontier (shifting along frontier just reallocates profits) and are all equally effective
  - Buy-backs are not risk-sharing devices
Quantity Flexibility Contracts (a return policy)

Assumptions and Notation:

- Alternative way of implementing returns
- Single period
- Same newsvendor model as above
- Three parameters:
  - \( w \) = wholesale price
  - \( d \) = downward adjustment \( \in [0, 1) \)
  - \( u \) = upward adjustment (\( u \geq 0 \))

Sequence of play:

- Manufacturer offers terms of trade
- Retailer places order of \( y \)
- Manufacturer commits to providing \( y(1+u) \), i.e. upside coverage
- If realized demand, \( x \), is between \( y(1-d) \) and \( y(1+u) \), retailer buys \( x \) at price \( w \), i.e. downside commitment
- Retailer may only cancel \( d \)-percent of his order

\[ \chi = \frac{1-d}{1+u} \] = fraction of channel stock retailer is responsible for

- Measures flexibility to retailer
- Lower \( \chi \), more flexibility to retailer
Analysis of Retailer’s Problem

Profit Function for Retailer under QF contract:
\[ \Pi_R(y) = (r - w)y(1 + u) - (r - w) \int_{y(1-d)}^{y(1+u)} F(x)dx - (r - v) \int_{0}^{y(1-d)} F(x)dx \]

= (profit if sold all)- (adj for demand betw upside & downside)- (adj for demand below downside)

Note: for fixed \( y \) and \( w \), \( \Pi_R(y) \uparrow \) in \( d \) and \( u \)
Theorem 8. Suppose that the manufacturer offers a QF contract with parameters (w,d,u).

1. The retailer’s profits are concave in y.

2. The retailer’s optimal order y* is unique and is implicitly defined by
\[ \Pi'_R(y^*) = (r - w)(1 + u)F(y^*(1 + u)) - (w - v)(1 - d)F(y^*(1 - d)) = 0 \]

3. The retailer’s optimal order y* is decreasing in w and increasing in d.

4. The total amount of stock in the system y*(1 + u) is increasing in u.

Comments:

• #2: If d = u = 0, then retailer’s problem reduces to newsvendor
  – O.w. not generally possible to determine an explicit solution to \( \Pi'_R(y^*) \)
  – QF contract must balance the chance of being over and under 2 distinct points

• #3: Retailer orders less when w↑ (slimmer margin)
  – Retailer orders more when d is increasing (less downside coverage required)

• #4: Retailer orders more as u↑ (more upside coverage offered)
Coordinating QF Contracts

- Want system to stock $y_I$ units
- Induce retailer to order $\frac{y_I}{1+u}$
- The optimal wholesale price is:

$$w^*(d, u) = v + \frac{(r-v)(1+u)(\frac{c-v}{c-d})}{(1+u)(\frac{c-v}{c-d})+(1-d)F(y_I, \frac{c-v}{c-d})} = v + \frac{c-v}{\frac{c-v}{c-d}+\chi F(\chi y_I)}$$

- Coordinating contract only has 2 parameters: $w$ and $\chi$. Write $w^*(\chi)$
- A continuum of coordinating contracts exist, since $0 < \chi \leq 1$

Theorem 9 If the manufacturer offers a coordinating contract $(w^*(\chi), \chi)$ the retailer’s profits $\Pi_R(w^*(\chi), \chi)$ are increasing in $\chi$.

Comments

- The more flexibility (i.e. $\chi \downarrow$) offered by manufacturer, the higher $w$ and $\Pi_M$ (i.e. $\Pi_R \downarrow$)
  - conversely, $\chi \uparrow \implies \Pi_R \uparrow$

- Under QF contracts, $w$ depends on the demand distribution (unlike buy-backs)
  - Manufacturer cannot use one contract to coordinate all markets

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Theorem 10 If \( F(d|\theta) \) is from a scaled family, the coordinating wholesale price \( w^*(\chi|\theta) \) is independent of \( \theta \) for all values of \( \chi \). That is, \( w^*(\chi|\theta) = w^*(\chi|1) \).

Comments

- The same coordinating contract can be used in multiple markets if the demand distribution for each market is from the same scaled family.
Alternative Contracts (a penalty scheme)

- Return policies work by manipulating consequence of excess stock
  Versus...
- Penalty schemes work by manipulating consequence of being short
  - Difficult because must observe lost sales
  - Suited for intrafirm coordination

Assumptions and Notation:
- Manufacturer offers \( w \)
- No return policy
- \( p = \) penalty per unit for any missed sales (placed on retailer)

Profit Function for Retailer under Penalty contract:
\[
\Pi_R(y) = (r - w)y - (r - v) \int_0^y F(d) \, dd - p \int_y^\infty \overline{F}(d) \, dd
\]
\[
= ry - wy - r \int_0^y (y - d) f(d) \, dd + v \int_0^y (y - d) f(d) \, dd
\]
\[
- p \int_y^\infty (d - y) f(d) \, dd
\]

Find optimal \( y(w, p) \):
\[
\Pi'_R(y) = r - w - rF(y) + vF(y) + p(1 - F(y)) = 0
\]
\[
= r - w + p - (r - v + p)F(y)
\]
\[
\therefore \ y(w, p) = F^{-1} \left( \frac{r + p - w}{r + p - v} \right)
\]
• If manufacturer imposes a penalty of \( p(w) = \frac{(w-c)(r-v)}{e-v} \), then retailer orders integrated channel quantity, \( y_I \)

Alternative penalty scheme:

• Instead of linear payment \( p \), manufacturer charges a lump sum in the event of a stock out
  
  – Now manufacturer only has to observe that a stockout occurs, not how many lost sales

• Manufacturer offers a menu of contracts depending on initial order, \( y \)

• Penalty payment = \( P(y|w) = p(w) \frac{\int_y^\infty F(d)dd}{F(y)} = p(w)*E[\#\text{short given } d >y] \)

• Implementation difficulty; retailer can make audit process hard, better suited for intrafirm
Alternative Contracts (a standard-setting scheme)

Assumptions and Notation:

- Suited for intrafirm coordination
- Classical principal-agent format
  - Principal is risk-neutral
  - Agent is risk-averse
- Principal hires agent to set stocking level in newsvendor problem
  - Typically, agent will set below what principal would choose when have same evaluation of demand distribution
- Manager throughout day has more info available than principal before stocking decision must be made
  - Manager revises estimate of demand distribution
  - New stock level always lies between the standard and what the risk-neutral principal would have chosen given revised information
- Standards-based contract: principal offers a fixed wage $w$ and a gains sharing $\phi$ (where $0 < \phi < 1$) while posting a standard $y_0$
- $\pi(y) = \text{system profits when the initial stocking level is } y$
- Retailer’s total compensation $= w + \phi(\pi(y) - \pi(y_0))$
- Receives w for certain; shares in fraction of gain or loss from deviating from principal’s standard

- Since updating in Bayesian fashion, better for principal to implement standard based scheme

- Not necessarily optimal

- Risk-sharing
  - induces manager to act on his revised information