

OR 351 Presentation

On (R, nQ) Policies in Serial Inventory Systems

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Introduction

- A multi-stage, serial inventory system with stochastic demands
 - material is processed sequentially before being used to satisfy uncertain customer demands;
 - consists of multiple stages representing different stocking points in a production-distribution process;
 - material flow from one stage to the next requires a leadtime and incurs a fixed cost in addition to a variable cost;
 - inventory becomes more expensive to carry as it moves downstream (closer to customers);

- demand unsatisfied from on-hand inventory is backlogged, incurring a penalty cost;
- the entire supply chain is controlled by a central planner whose goal is to satisfy customer demands with minimum long-run average system-wide costs.

- For our serial model with random demand and setup costs, Clark and Scarf (1962) have shown that the optimal policy does not have a simple structure. This means that once managerial effort of implementation is taken into account, the “optimal” policy is no longer optimal or even attractive.
- Therefore, simple and practical heuristic policies are considered. An (R, nQ) policy operates as follows: whenever the inventory position at a stage is at or below R , order nQ units where n is the minimum integer required to increase the inventory position to above R .
- We call R the *reorder point* and Q the *base order quantity*.

- Two variations of (R, nQ) policies
 - echelon-stock (R, nQ) policy, which requires centralized demand information;
 - installation-stock (R, nQ) policy, which only requires local inventory information.
- An integer-ratio constraint: the base order quantity at an upstream stage is an integer multiple of the base order quantity at a downstream stage.

(R, nQ) policies in Single-Location Systems

- A single-location inventory system
 - periodic demands for a single item;
 - demands are i.i.d. nonnegative integer-valued random variables;
 - linear holding cost h and linear backorder cost p (per unit per period);
 - inventory is replenished from an outside source with a constant leadtime of L periods.

- (R, nQ) policy: at beginning of each period, if inventory position y (on-hand inventory + outstanding orders – backorders) is at or below R , an order of size nQ is placed where $n = \min\{n : y + nQ > R\}$; otherwise, no order is placed.
- Fixed setup cost K is incurred for each Q units of order.
- The objective is to minimize the long-run average total cost with decision variables R and Q .
- Assume that ordering decision is made at beginning of each period, demands occur during the period, and holding and backorder costs are assessed at end of the period.

- $IP(t)$ = inventory position at beginning of period t after placing an order (if any) and before demand occurs. Thus

$$R + 1 \leq IP(t) \leq R + Q$$

- $IL(t)$ = inventory level (on-hand inventory – backorders) at end of period t .
- $D[t_1, t_2]$ = total demand in periods $t_1, t_1 + 1, \dots, t_2$.
- Inventory balance equation

$$IL(t + L) = IP(t) - D[t, t + L]$$

- Note that $IP(t)$ and $D[t, t + L]$ are independent. Given $IP(t) = y$, the expected holding and backorder costs in period $t + L$ is

$$G(y) \stackrel{\text{def}}{=} \mathbb{E}[h(y - D[t, t + L])^+ + p(D[t, t + L] - y)^+]$$

Note $G(\cdot)$ is convex.

- We assume that DTMC $\{IP(t)\}$ is irreducible. Hadlay and Whitin (1961) showed that under this mild condition its steady state distribution is uniform over $R + 1, \dots, R + Q$.

- The long-run average holding and backorder cost is $\frac{1}{Q} \sum_{y=R+1}^{R+Q} G(y)$ and long-run average setup cost is $\mu K/Q$ where μ is the mean of one-period demand. Thus the long-run average total cost associated with the (R, nQ) policy is

$$C(R, Q) \stackrel{\text{def}}{=} \frac{\mu K + \sum_{y=R+1}^{R+Q} G(y)}{Q} \quad (1)$$

- **Optimization** $C(R, Q)$ is jointly convex in the decision variables (?). We can minimize it as follows:
 - Fix Q and minimize $C(R, Q)$ over R . Since $G(\cdot)$ is convex, the Q smallest values of $G(\cdot)$ are achieved at Q contiguous points.
 - The optimal R is the one such that the summation in (1) is over those Q points.
 - The optimal Q is obtained by minimizing $C(Q) \stackrel{\text{def}}{=} \min_R C(R, Q)$, a unimodal function.

- **Sensitivity Analysis** The cost performance of (R, nQ) policies is insensitive to the choice of Q . Zhang (1992) has shown that

$$\frac{C(\alpha Q^*)}{C(Q^*)} \leq \frac{1}{2} \left(\alpha + \frac{1}{\alpha} \right) \text{ for all } \alpha > 0.$$

For example, if $Q = \sqrt{2}Q^*$ is used instead of the optimal Q^* then the relative cost increase is no more than 6%.

- **Remark** Alternatively, a fixed setup cost is incurred for each order, independent of its size. The resulting cost function is no longer jointly convex in the decision variables.

(R, nQ) Policies in Serial Systems

- A serial inventory system with N stages
 - stage 1 orders from stage 2, 2 from 3, \dots , and stage N orders from an outside supplier with unlimited stock. For convenience the outside supplier is referred to as stage $N + 1$.
 - demands are i.i.d. nonnegative integer-valued random variables. When stage 1 runs out of stock, the excess demand is backlogged.

- Each stage controls a stage-specific inventory position according to a stage-specific (R, nQ) policy.
 - when the inventory position falls to or below a reorder point R , the stage orders a minimum integer multiple of Q (base quantity) from its upstream stage to increase the inventory position to above R ;
 - In case the upstream stage does not have sufficient on-hand inventory, the unfilled quantity is backlogged at the upstream stage;
 - leadtimes from one stage to the next are constant;
 - other costs, decision variables and objective are similar to single-location systems.

- To simplify the analysis, we assume the integer-ratio constraints

$$Q_{i+1} = n_i Q_i, \quad i = 1, \dots, N - 1 \quad (2)$$

where Q_i is the base quantity at stage $i, i = 1, \dots, N$ and n_i is a positive integer.

- We assume that replenishment activities—ordering, shipping and receiving—occur at beginning of a period. Demand occurs during the period. Holding and backorder costs are assessed at the end of the period.

- Two variations of (R, nQ) policies
 - one is based on *echelon-stock*. The echelon-stock at a stage is the inventory position of the subsystem consisting of the stage and the downstream stages. Each stage replenishes its echelon-stock with an echelon reorder point, R_i ($i = 1, \dots, N$);
 - the other is based on *installation-stock*. The installation-stock at a stage is just the inventory position of the stage itself. Each stage controls its installation-stock with an installation reorder point, r_i ($i = 1, \dots, N$).

- System parameters
 - μ = mean of the demand in one period
 - L_i = leadtime from stage $i+1$ to stage i , a given nonnegative integer
 - K_i = fixed cost of each Q_i ordered at stage i
 - H_i = installation holding cost per unit per period at stage i
 - h_i = echelon holding cost per unit per period at stage i
 $i = H_i - H_{i+1} > 0$ with $H_{N+1} = 0$
 - p = backorder cost per unit per period (at stage 1 only)
 - $D[t_1, t_2]$ = total demand in period t_1, \dots, t_2
 - $D[t_1, t_2) =$ total demand in period $t_1, \dots, t_2 - 1$
 - $D(t_1, t_2) =$ total demand in period $t_1 + 1, \dots, t_2 - 1$

- Let t^- be an epoch after replenishment activities but before demand in period t , and t^+ be an epoch after demand in period t . Define
 - $I_i(t) =$ echelon inventory at stage i at $t^+ =$ on-hand inventory at stage i plus inventories on hand at, and in transit to, stage $1, \dots, i - 1$
 - $B(t) =$ backorder level at stage 1 at t^+
 - $IL_i(t) =$ echelon inventory level at stage i at $t^+ = I_i(t) - B(t)$
 - $IL_i^-(t) =$ echelon inventory level at stage i at $t^- =$ on-hand inventory at stage i plus inventories in transit to or on hand at stage $1, \dots, i - 1$ minus backorders at stage 1 at t^-

- $IP_i(t)$ = echelon inventory position at stage i at $t^- = IL_i^-(t)$ plus orders in transit to stage i
- $ES_i(t)$ = echelon stock at stage i at $t^- = IP_i(t)$ plus outstanding orders of stage i that are backlogged at stage $i + 1$
- $IS_i(t)$ = installation stock at stage i at $t^- =$ outstanding orders at stage i (in transit to stage i or backlogged at stage $i + 1$) plus on-hand inventory at stage i minus backlogged orders from stage $i - 1 = ES_i(t) - ES_{i-1}(t)$, $i \geq 2$, and $IS_1(t) = ES_1(t)$

Note $ES_N(t) = IP_N(t)$ for all t .

- It is reasonable to assume that the initial on-hand inventory at stage i is an integer multiple of Q_{i-1} , $i = 2, \dots, N$.
- Axsater and Rosling (1993) has shown that any installation-stock (R, nQ) policy, $(r_i, Q_i)_{i=1}^N$, is equivalent to an echelon-stock (R, nQ) policy, $(R_i, Q_i)_{i=1}^N$, as long as its echelon reorder points satisfy

$$R_1 = r_1, \text{ and}$$

$$R_i = R_{i-1} + Q_{i-1} + r_i, \quad i = 2, \dots, N \quad (3)$$

In other words, installation-stock (R, nQ) policies are special cases of echelon-stock (R, nQ) policies.

Performance Evaluation

- We want to determine the long-run average system-wide cost of an (R, nQ) policy in the serial system. It suffices to consider echelon-stock (R, nQ) policies with control parameters $(R_i, Q_i)_{i=1}^N$.
- On-hand inventory at stage $i + 1$ is always a nonnegative integer multiple of Q_i , $i = 1, \dots, N - 1$. Therefore,

$$IL_{i+1}^-(t) - IP_i(t) = mQ_i, \quad m \geq 0 \text{ integer} \quad (4)$$

- Echelon-stock policy implies that $IP_i(t)$ is above R_i whenever stage $i + 1$ has positive on-hand inventory. Therefore, following from (4)
 - For $IL_{i+1}^-(t) \leq R_i$, $IP_i(t) \leq R_i$ since $m \geq 0$. Thus the on-hand stock at stage $i + 1$ cannot be positive, i.e., $m = 0$ or $IP_i(t) = IL_{i+1}^-(t)$;
 - For $IL_{i+1}^-(t) > R_i$, if $IP_i(t) \leq R_i$ then $m > 0$ implying that stage $i + 1$ has positive on-hand inventory, which in turn implies that $IP_i(t) > R_i$, a contradiction. Thus $IP_i(t)$ must belong to $\{R_i + 1, \dots, R_i + Q_i\}$.

- To summarize,

$$IP_i(t) = O_i[IL_{i+1}^-(t)] \quad \text{for } i = 1, \dots, N - 1 \quad (5)$$

where

$$O_i[x] = \begin{cases} x & \text{if } x \leq R_i \\ x - nQ_i & \text{otherwise} \end{cases}$$

with n being the largest integer such that $x - nQ_i > R_i$.

- Note

$$IL_i(t + L_i) = IP_i(t) - D[t, t + L_i] \quad (6)$$

$$IL_i^-(t + L_i) = IP_i(t) - D[t, t + L_i] \quad (7)$$

- Viewing the whole serial system as a single location system following (R_N, nQ_N) policy, IP_N , the steady state echelon N inventory position is uniformly distributed over $R_N + 1, \dots, R_N + Q_N$.
- The system-wide holding and backorder costs incurred in period t is

$$\sum_{i=1}^N h_i I_i(t) + pB(t)$$

- Since $I_i(t) = IL_i(t) + B(t)$ by definition,

$$\sum_{i=1}^N h_i IL_i(t) + (p + H_1)B(t)$$

Let $l_i = \sum_{j=i}^N L_j$ for $i = 1, \dots, N$ with $l_{N+1} \equiv 0$, i.e., the total leadtime from the outside supplier to stage i .

$$\sum_{i=1}^N h_i IL_i(t + l_i) + (p + H_1)B(t + l_1) \quad (8)$$

Note the above merely shifts costs in time and thus does not affect the long-run average cost.

- Take any period t in steady state. For $i = 1, \dots, N$, define

$$IP_i = IP_i(t + l_{i+1})$$

$$IL_i^- = IL_i^-(t + l_i)$$

$$IL_i = IL_i(t + l_i)$$

$$B = B(t + l_i) = (IL_1)^-$$

$$D_i = D_i[t + l_{i+1}, t + l_i]$$

$$D_i^- = D_i[t + l_{i+1}, t + l_i]$$

- From (5),

$$IP_i = O_i[IL_{i+1}^-], \quad i = 1, \dots, N - 1 \quad (9)$$

From (6) and (7),

$$IL_i = IP_i - D_i \quad (10)$$

$$IL_i^- = IP_i - D_i^- \quad (11)$$

where $i = 1, \dots, N$.

- Using (9)-(11), we can recursively determine the distributions of IP_i and IL_i , $i = 1, \dots, N$.
- Expression (8) is thus equal to

$$\mathbf{E} \left[\sum_{i=1}^N h_i IL_i + (p + H_1)B \right].$$

- Thus we have the long-run average system-wide cost

$$\begin{aligned}
 & \sum_{i=1}^N \frac{\mu K_i}{Q_i} + \mathbf{E} \left[\sum_{i=1}^N h_i I L_i + (p + H_1) B \right] \\
 &= \sum_{i=1}^N \frac{\mu K_i}{Q_i} + \sum_{i=2}^N h_i \mathbf{E}[I L_i] + \mathbf{E} G_1(I P_1) \quad (12)
 \end{aligned}$$

where

$$\begin{aligned}
 G_1(y) &= \mathbf{E}[h_1 I L_1 + (p + H_1) B | I P_1 = y] \\
 &= \mathbf{E}[h(y - D_1) + (p + H_1)(y - D_1)^-].
 \end{aligned}$$