Deterministic Non-stationary Demand

Assumptions and Notation

- Periodic review: inventory decisions are done at times $t = 1,2,...,T$.
- d_t = demand at the beginning of period t.
- $h_t(x)$ = cost of holding x items in period t. Concave function of x. Example:

$$h_t(x) = hx.$$

- $c_t(y)$ = cost of ordering cost y items in period t. Concave function of y. Example:

$$c_t(y) = K(1 - \delta_{y,0}) + cy.$$

- No shortages permitted.
- Zero lead time.
- Zero initial inventory.

Objective

Decide how much to order and when to order in order to minimize the total cost.

ANALYSIS

- Sequence of events: Order placed, Procurement cost incurred, Order received, Demand observed, Demand satisfied, Holding cost incurred on the remaining inventory.
- x_t = ending inventory in period t. We start period 1 with $x_0 = 0$.
- y_t = order quantity in period t.
- $x_t = x_{t-1} + y_t - d_t, \quad t = 1, ..., T.$
- Total cost:

$$TC = \sum_{t=1}^{T} [c_t(y_t) + h_t(x_t)].$$

- Math Program:

$$\begin{array}{rl} \text{Minimize} & \sum_{t=1}^{T} [c_t(y_t) + h_t(x_t)] \\ \text{Subject to} & x_t = x_{t-1} + y_t - d_t, \quad t = 1, ..., T, \\ & x_t, y_t \geq 0, \quad t = 1, 2, ..., T, \\ & x_0 = 0. \end{array}$$

Wagner-Whitin Algorithm

- Wagner, H. M., and T. M. Whitin (1958). Dynamic version of the economic lot size model. Management Science, 23, 82-96.
- The optimal value of y_t is in the set

$$\{0, \ \ d_t, \ \ d_t + d_{t+1}, \ \ \cdots, d_t + \cdots + d_T\}.$$

- An optimal policy has the property

$$y_t x_{t-1} = 0.$$

- It is optimal to order only when the starting inventory is zero.
- An optimal policy is obtained by finding the shortest path in a network with $T+1$ nodes labeled $1, 2, ..., T+1$, and with arc length $c_{i,j}$ between nodes $1 \le i < j \le T+1$ given by

$$c_{i,j} = c_i(\sum_{k=i}^{j-1} d_k) + \sum_{k=i}^{j-2} h_k(\sum_{l=k+1}^{j-1} d_l).$$

- Let f_i be the cost of following an optimal policy when i periods remain. Dynamic Programming recursion:

$$\begin{aligned} f_{T+1} &= 0 \\ f_i &= \min_{j=i+1,\ldots T+1} \{c_{i,j} + f_j\}, \quad i = 1, 2, ..., T. \end{aligned}$$

Forward Algorithm

- g_i = minimum policy cost for period $1, 2, ..., i$, given that the inventory level is zero at the end of period i, $0 \le i \le T$.
- g_i can be computed in a forward recursive fashion as follows:

$$\begin{aligned} g_0 &= 0, \\ g_j &= \min_{i=1,2,\ldots,j} \{g_{i-1} + c_{i,j+1}\}, \quad i = 1, 2, ..., T. \end{aligned}$$

- Let i_j be the value that minimizes the RHS for g_j. Then for the j period problem the last production occurs in period i_j in an optimal policy. Thus the optimal policy can be worked out in a backward way.

Example

- $h_t(x) = hx, \quad c_t(y) = K(1 - \delta_{y,0}) + cy.$
- $T = 5, \quad K = 250, \quad c = 2, \quad h = 1,$

  $d_1 = 220, \quad d_2 = 280, \quad d_3 = 360, \quad d_4 = 140, \quad d_5 = 270.$
- The $c_{i,j}$ are as follows:

$i\downarrow \quad j\rightarrow$	2	3	4	5	6
1	690	1530	2970	3670	5290
2	0	810	1890	2450	3800
3	0	0	970	1390	2470
4	0	0	0	530	1340
5	0	0	0	0	790

- Backward recursion:

  $f_6 = 0, \quad f_5 = 790, \quad j_5 = 5, \quad f_4 = 1320, \quad j_4 = 4,$

  $f_3 = 2180, \quad j_3 = 4, \quad f_2 = 2990, \quad j_2 = 2, \quad f_1 = 3680, \quad j_1 = 1.$
- Optimal production:

  $y_1 = 220, \quad y_2 = 280, \quad y_3 = 500, \quad y_4 = 0, \quad y_5 = 270.$
- Optimal cost = \$3680.
- Forward recursion:

  $g_0 = 0, \quad g_1 = 690, \quad i_1 = 1, \quad g_2 = 1500, \quad i_2 = 2,$

  $g_3 = 2470, \quad i_3 = 3, \quad g_4 = 2890, \quad i_4 = 3, \quad g_5 = 3680, \quad i_5 = 5.$
- Produces the same optimal production schedule, as it must.

Planning Horizons

- $l(t)$: the last period when production occurs in an optimal t period problem.
- $l(t)$ is a non-decreasing function of t (consequence of concave costs).
- $l(t) = t$ implies that the optimal policy for periods $1, 2, ..., t-1$ does not depend upon data for periods t onwards.
- If $l(t) = t$, we say that the planning horizon is $1, 2, ..., t-1$.
- In the previous example, $l(1) = 1, \quad l(2) = 2, \quad l(3) = 3, \quad l(4) = 3, \quad l(5) = 5$.

Silver-Meal Heuristics

- Silver, E.A, and H. C. Meal (1973), A heuristics for selecting lot size quantities for the case of a deterministic time-varying demand and discrete opportunities for replenishment, Prod. Invent. Management, 14, 64-74.

- $C(T)$ = average cost of holding and setup per period if the current order satisfies the demand for the next T periods.

  $$C(T) = [K + hd_2 + 2hd_3 + ... + (T-1)hd_T]/T.$$

- Compute $C(T)$ for $T = 1, 2, ...$, and stop as soon as $C(T) > C(T-1)$, and order $d_1 + d_2 + ... + d_{T-1}$ in the first period.

- Example: $K = 250, \quad h = 1, \quad d = [220\ , 280,\ 360,\ 140,\ 270]$.

- Step 1: $C(1) = 250, \quad C(2) = (250 + 280)/2 = 265$. Hence $y_1 = 220$.

- Step 2: $C(1) = 250, \quad C(2) = (250 + 360)/2 = 305$. Hence $y_2 = 280$.

- Step 3: $C(1) = 250, \quad C(2) = (250 + 140)/2 = 195, \quad C(3) = (250 + 140 + 270)/3 = 220$. Hence $y_3 = 360 + 140 = 500$.

- Step 4: $y_4 = 0, y_5 = 270$.

Extensions

- Non-zero lead times.
- Allowing shortages: Zero orders if positive inventory, positive orders only if positive shortages. Reference:
  Zangwill, W. I. (1966). A deterministic multiperiod production scheduling model with backlogging. Management Science 13, 105-119.
- Production Capacities: $y_t \le U_t$, where U_t is the upper bound on the production capacity in period t. Reference:
  Florian, M., and M. Klein (1971). Deterministic production planning with concave costs and capacity constraints. Management Science 18, 12-20.
  Baker, K. R., P. R. Dixon, M. J. Magazine, and E. A. Silver (1978). Algorithm for the dynamic lot-size problem with time-varying production capacity constraints. Management Sci. 16, 1710-1720.
- Convex Costs. Veinott, A. F. (1965). The optimal inventory policy with convex costs: A parametric study. Management Science 10, 441-460.