Serial Supply Chain

Assumptions and Notation

$$[n] \to [n-1] \to \cdots [1] \to \lambda$$

- The supply chain consists of n levels.
- Level 1 faces the external demand at a constant rate λ.
- Level i receives items from Level $i+1$, $1 \leq i \leq n-1$.
- Level n produces the items.
- K_i = ordering/setup cost at level i.
- Lead time =0.
- h_i' = holding cost per item held at level i per unit time.
- No shortages permitted.

Objective

Find the optimal reordering and production policies at the n levels so as to minimize the long run cost per unit time.

Analysis

- $x_i(t)$ = inventory on hand at level i at time t.
  $x_i(0) = 0$ for all i.
- $n_i(t)$ = number of reorders/setups at level i upto time t.
- Long run average cost

$$\lim_{T\to\infty} \frac{1}{T}[\sum_{i=1}^{n} n_i(T)K_i + \int_0^T h_i' x_i(t)dt]$$

$$= \lim_{T\to\infty} \frac{1}{T}[\sum_{i=1}^{n} n_i(T)K_i + \int_0^T h_i X_i(t)dt]$$

  where $h_i = h_i' - h_{i+1}'$ is the echelon holding cost rate, and $X_i(t) = \sum_{j=1}^{i} x_j(t)$ is the echelon inventory at level i
- A policy is called stationary if the times between consecutive reorder points at level i is a constant, say T_i, i.e., if $n_i(t) = \lceil \frac{t}{T_i} \rceil$, for all t and all i.
- A stationary policy is called *nested* if $T_i = m_i T_{i-1}$ for some positive integer m_i, for $i = 2, 3, ..., n$.
- Result 1: For an n stage serial system it is optimal to follow a stationary, nested policy if all echelon holding costs are positive.
- Result 2: In a stationary nested optimal policy described by $T_1,\ \ T_2,\ \ \cdots, T_n$, $x_i(T_i^-) = 0$.

- Reference: Love, S. F. (1972). A facilities in series inventory model with nested schedules. Management Sci. 18, 327-338.

Special Case: $n = 2$

- Assume $h'_1 > h'_2 > 0$, i.e., $h_i \geq 0$ for $i = 1, 2$.
- $X_i(t), \quad i = 1, 2$ have the usual saw-tooth shaped sample paths.
- Let $T_1 = T$, and $T_2 = mT_1$, for some positive integer m. Also $g_i = \frac{1}{2}\lambda h_i$.
- Long run cost rate

$$\begin{aligned} C(T, m) &= \frac{K_1}{T} + g_1 T + \frac{K_2}{mT} + g_2 mT \\ &= \frac{K_1 + K_2/m}{T} + (g_1 + g_2 m)T. \end{aligned}$$

- For a fixed m, the optimal T value is given by

$$T^*_m = \sqrt{\frac{K_1 + K_2/m}{g_1 + g_2 m}},$$

and the optimal cost is given by

$$C(T^*_m, m) = 2\sqrt{(K_1 + \frac{K_2}{m})(g_1 + g_2 m)}.$$

- Next we choose the optimal m^* as the smallest positive integer for which

$$m(m+1) > \frac{K_2/g_2}{K_1/g_1}.$$

- The optimal cost is given by

$$C^* = C(T^*, m^*) = 2\sqrt{(K_1 + \frac{K_2}{m^*})(g_1 + g_2 m^*)}.$$

  This is the minimum cost rate of operating the 2-station supply chain.

- Optimal reorder interval

$$T_1^* = T^*, \quad T_2^* = m^* T^*.$$

- Optimal order quantity

$$Q_i^* = \lambda T_i^* \quad i = 1, 2.$$

Special Case: $n = 2$

- Assume $h'_2 > h'_1 > 0$. Then reverse nesting is optimal, i.e., $T_1 = mT_2$ for some positive integer m.
- Define $X_2(t) = x_2(t), \;\; X_1(t) = x_1(t) + x_2(t)$. $\{X_i(t), t \geq 0\}$ has the usual saw-tooth shaped sample paths.
- Let $T_2 = T$, and $T_1 = mT_1$, for some positive integer m. Also $g_1 = \frac{1}{2}\lambda h'_1, \;\; g_2 = \frac{1}{2}\lambda(h'_2 - h'_1)$.
- Long run cost rate

$$C(T, m) = \frac{K_1}{mT} + g_1 mT + \frac{K_2}{T} + g_2 T$$
$$= \frac{K_1/m + K_2}{T} + (g_1 m + g_2)T.$$

- For a fixed m, the optimal T value is given by

$$T^*_m = \sqrt{\frac{K_1/m + K_2}{g_1 m + g_2}},$$

and the optimal cost is given by

$$C(T^*_m, m) = 2\sqrt{(K_2 + \frac{K_1}{m})(g_2 + g_1 m)}.$$

- Next we choose the optimal m^* as the smallest positive integer for which

$$m(m+1) > \frac{K_1/g_1}{K_2/g_2}.$$

- The optimal cost is given by

$$C^* = C(T^*, m^*) = 2\sqrt{(K_2 + \frac{K_1}{m^*})(g_2 + g_1 m^*)}.$$

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- Optimal reorder interval

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- Optimal order quantity

$$Q_i^* = \lambda T_i^* \quad i = 1, 2.$$

Decentralized Control of 2-Station Supply Chain

- $T_1^* = \sqrt{2K_1/\lambda h_1'}$.
- $C_1^* = \sqrt{2K_1\lambda h_1'}$.
- Let q_2^* be the smallest integer such that

$$q_2^*(q_2^*+1) \geq \frac{K_2}{g_2' T_1^{*2}}.$$

- $T_2^* = q_2^* T_1^*$
- $C_2^* = \frac{K_2}{T_2^*} + g_2'(q_2^*-1)T_1^*$.
- $C^* = C_1^* + C_2^* = \frac{K_1}{T_1^*} + \frac{K_2}{T_2^*} + g_1 T_1^* + g_2 T_2^*$.
- Centralized vs. Decentralized Control: Numerical investigation using eoq2central and eoq2decentral.

Centralized Control: n-Station Chain

- Assume all echelon holding costs are positive.
- The global optimization problem is:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}[\frac{K_i}{T_i} + g_i T_i] \\
\text{Subject to:} \quad & T_i = m_i T_{i-1}, \;\; 2 \leq i \leq n, \\
& m_i \geq 1, \;\; \text{integer} \\
& T_1 \geq 0.
\end{aligned}
$$

- The relaxed problem:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}[\frac{K_i}{T_i} + g_i T_i] \\
\text{Subject to:} \quad & T_i \geq T_{i-1}, \;\; 2 \leq i \leq n, \\
& T_1 \geq 0.
\end{aligned}
$$

Analysis

- An ordered partition of the serial chain $G = \{1, 2, ..., n\}$ is
$$G_1 = \{1, ..., n_1\}, \quad ...G_N = \{n_{N-1}, ..., n\}.$$
- Define
$$K(G) = \sum_{i \in G} K_i, \quad g(G) = \sum_{i \in G} g_i.$$
- Suppose $T_i^*, \quad i \in G$ is an optimal solution to the relaxed problem. Then there is an ordered partition $\{G_1, G_2, ..., G_N\}$ of G such that
$$T_i^* = T^*(k) = \sqrt{\frac{K(G_k)}{g(G_k)}}, \quad i \in G_k.$$
- Conversely, for any ordered partition $\{G_1, G_2, ..., G_N\}$ of G, the solution given above is a feasible solution to the relaxed problem if $T^*(1) \le T^*(2)... \le T^*(N)$.
- Thus the problem reduces to finding an optimal partition.

Main Theorem

Suppose we have an arbitrary collection of N reorder intervals $T*(1), ..., T*(N)$. The following are the necessary and sufficient conditions for these reorder intervals to form an optimal solution to the relaxed problem.

1. There exists an ordered partition $\{G_1, G_2, ..., G_N\}$ of G such that

$$T^*(k) = T^*(G_k) = \sqrt{\frac{K(G_k)}{g(G_k)}}, \quad 1 \leq k \leq N.$$

2. $T^*(1) \leq T^*(2) ... \leq T^*(N)$.

3. For each $k = 1, 2, ..., N$, there does not exist an ordered partition (G_k^-, G_k^+) of G_k for which

$$T^*(G_k^-) < T^*(G_k^+).$$

Reference: Jackson, P. L., W. L. Maxwell, and J. A. Muckstadt (1988). Determining optimal reorder intervals in capacitated production-distribution systems,. Management Sci. 34(8), 938-958.

Algorithm

1. Let $S = \{1, 2, ..., n\}$ be an ordered set. Set $C^i = \{i\}$, for $1 \leq i \leq n\}$. $\{C^i, \imath \in S\}$ is the initial ordered partition of G.
2. Let $\sigma(i)$ be the node that precedes i in S. Thus, $\sigma(i) = i - 1$ for all $2 \leq i \leq n$. set $\sigma(1) = 0$.
3. Set $j = 2$.
4. If $T^*(C^j) \geq T^*(C^{\sigma(j)})$ go to Step 6.
   Else, $C^j = C^j \cup C^{\sigma(j)}, \quad \sigma(j) = \sigma(\sigma(j)), \quad S = S - \{\sigma(j)\}$.
5. If $\sigma(j) > 0$, go to step 4.
6. Set $j = j + 1$. If $j \leq n$, go to Step 4.
7. Re-index the partition $\{C^i, i \in S\}$ such that $S = \{1, 2, ..., N\}$ while maintaining the order.
8. $\{C^i, i \in S\}$ is an optimal partition. Let

   $$T_i^* = T^*(C^k), \quad i \in C^k, \quad 1 \leq k \leq N.$$

   Then $T_1^*, T_2^*, ..., T_n^*$ is an optimal solution to the relaxed problem.

Power of 2 Sub-optimal Solution

- Let $T_L \leq T_1^*$ be the base period. The power of 2 solution is given by

  $$T_i = 2^l T_L, \text{ where } 2^l \geq T^*(C^k)/(\sqrt{2}T_L) > 2^{l-1},$$

  for all $i \in C^k, \quad 1 \leq k \leq N$.

- One can show that the objective function value with this solution is within 6% of the optimal objective function value of the relaxed problem, and hence within 6% of the optimal solution to the non-linear integer program.