# Optimal Cartel Pricing in the Presence of an Antitrust Authority*

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**Abstract**

Price dynamics are characterized when a price-fixing cartel is concerned about creating suspicions of the presence of a cartel. If detection depends only on the price level, the cartel initially raises price and then gradually lowers it. When detection depends only on price changes, the cartel gradually raises price. In that case, the long-run cartel price is decreasing in the damage multiple but is independent of the level of fines. A more stringent standard for calculating damages is shown to induce the cartel to price higher.

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# 1 Introduction

> Since the beginning of FY 1997, the Antitrust Division has prosecuted international cartels affecting over $10 billion in U.S. commerce ... [These cartels] have been bigger, in terms of the volume of affected commerce and the amount of harm caused to American businesses and consumers, than any conspiracies previously encountered by the Antitrust Division. [*Annual Report, Antitrust Division, United States Department of Justice*, 1999: pp. 5-6]
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> International cartels are estimated to represent a drain of hundreds of millions of euros on the European economy. ... Since 1998, the number of cartel cases investigated by the Commission has increased dramatically. [*European Community Competition Policy, XXXth Report on Competition Policy*, 2000: pp. 24-25]

As these quotes from American and European antitrust authorities suggest, price-fixing remains a perennial problem which makes it all the more important that we understand when cartels form and, when they do form, how they behave. Such understanding provides the basis for detecting cartels and developing policies to deter their formation. Though there is a voluminous literature on collusive pricing, an important dimension to price-fixing cartels has received very little attention. In light of the illegality of price-fixing**, a critical goal faced by a cartel is to avoid the appearance that there is a cartel.** Firms want to raise prices but not suspicions that they are coordinating their behavior. If high prices or rapidly increasing prices or, more generally, anomalous price movements may make customers and the antitrust authorities suspicious that a cartel is operating, one would expect this to have implications for how the cartel prices.

This paper is the initial step in a research project whose objective is to explore cartel pricing dynamics in the presence of detection considerations. Some of the questions to be addressed include: What are the properties of the collusive price path when a cartel takes into account the possibility of detection? How does it depend on the way in which detection occurs? How does the decision to form a cartel and the properties of the collusive price path respond to various instruments of antitrust policy? What types of industry traits make detection more difficult and what are the implications of those traits for cartel pricing?

In this paper, the joint profit maximizing price path is characterized under various assumptions about detection. After describing the model in Section II, a general proof of existence of the optimal price path is provided in Section III. Section IV considers when detection is driven only by the price level. If higher prices make detection more likely, the optimal price path is shown to involve initially raising price and then having it fall over time. The model of Section V assumes detection is sensitive to price changes and finds that the cartel steadily raises price. Comparative statics on the steady-state price reveal that it is decreasing in the damage multiple and the probability of detection but, surprisingly, is independent of the level of fines. Furthermore, if fines are the only penalty, the cartel's steady-state price is the same as in the absence of antitrust laws, though fines do affect the path to the steady-state. Another intriguing result is that a more stringent standard for calculating damages actually increases the steady-state price. In Section VI, some directions for future research are provided and it is discussed how this line of work may be able to allow us to distinguish empirically between tacit collusion and explicit collusion.

**Related Work** A few papers have investigated, in a static setting, optimal cartel pricing under the constraint of possible detection. Block, Nold, and Sidak (1981) consider a static oligopoly model in which the probability of detection depends on the price-cost margin and the penalty is a multiple of above-normal profits. They show that the optimal cartel price is below the monopoly price and that the cartel price is decreasing in the penalty multiple and the level of enforcement expenditures (higher levels of which raise the probability of detection). Section III is, in essence, a dynamical extension of their model. Spiller (1986), Salant (1987), and Baker (1988) extend the static formulation to allow buyers to adjust their purchases under the anticipation that they may be able to collect multiple damages if sellers are shown to have been colluding. Also within a static setting, Besanko and Spulber (1989, 1990), LaCasse (1995), Polo (1997), and Saïd (2001) explore a context in which firms have private information, which influences whether or not they collude, and either the government or buyers must decide whether to pursue costly legal action. Two papers consider a dynamic setting. Cyrenne (1999) modifies Green and Porter (1984) by assuming that a price war, and the ensuing raising of price after the war, results in detection for sure and with it a fixed fine. Motta and Polo (2001) consider the effects of leniency programs on the incentives to collude in a repeated game of perfect monitoring. In that model, the probability the cartel is detected is assumed fixed. Though considering collusive behavior in a dynamical setting with antitrust laws, these two papers exclude the sources of dynamics that are the foci of the current analysis; specifically, that the probability of detection and penalties (in the form of damages) are sensitive to firms' pricing behavior. It is that sensitivity that will generate predictions about cartel pricing dynamics.

## 2 Model

The representative firm's profit when all firms charge a price of $P$ is denoted $\pi(P)$. If total market demand is $D(P)$ when the common price is $P$ and a firm's cost function is $C(\cdot)$ then the profit function takes the form $\pi(P) = P(D(P)/n) - C(D(P)/n)$, given $n \geq 2$ firms. $\Omega$ is the set of feasible prices. In the absence of the formation of a cartel, a symmetric equilibrium is assumed to exist which entails a price of $\widehat{P}$ and firm profit of $\widehat{\pi} \geq 0$.

If firms form a cartel, they meet to determine price. Assume these meetings, and any associated documentation, provides the "smoking gun" if an investigation is pursued. The price in period $t$ is denoted $P^t$ and each firm earns profit of $\pi(P^t)$. The cartel is detected with some probability and incurs some level of expected damages in that event. Detection can be viewed as the end of the horizon with a terminal payoff of $[\widehat{\pi}/(1-\delta)] - D^t - F$ where $D^t$ is a firm's (expected) damages in the event the cartel is detected and $F$ is any (expected) fines.[1] If not detected, collusion continues on to the next period. There is an infinite number of periods. Penalties and fines are assumed to be sufficiently bounded from above for all histories so that the expected present value of a firm's income stream is always positive so bankruptcy is avoided.[2]

A cartel member's damages are assumed to evolve in the following manner:

$$D^t = \beta D^{t-1} + \gamma d\left(P^t\right) \text{ where } \beta \in [0,1), \gamma \geq 0. \tag{1}$$

As time progresses, damages incurred in previous periods become increasingly difficult to document and $1-\beta$ measures the rate of the deterioration of the evidence.[3] $d\left(P^t\right)$ is the level of damages incurred in the current period where $\gamma$ is the multiple of damages that a firm can expect to pay if found caught colluding. While U.S. antitrust law specifies treble damages, $\gamma$ could be less than three because a case is settled out-of-court. Single damages are not unusual for an out-of-court settlement.[4] Furthermore, the probability of being found guilty conditional on an investigation is embedded in $\gamma$. In that case, $\gamma$ could be less than one. Current U.S. antitrust practice is $d\left(P^t\right) = \left(P^t - \widehat{P}\right)\left(D\left(P^t\right)/n\right)$.

Detection of a cartel can occur from many sources; some of which are related to price - such as customer complaints - and some of which are unrelated to price - such as internal whistleblowers and incidental detection via an unrelated legal case.[5] Hay and Kelley (1974) find that detection was attributed to a complaint by a customer or a local, state, or federal agency in 13 of 49 price-fixing cases.[6] High prices or anomalous price movements (not easily explained by a non-collusive model) may cause customers to become suspicious and pursue legal action or share their suspicions with the antitrust authorities.[7] Though it isn't important for my model, I do imagine

[1] One could allow for the cartel to be reestablished sometime in the future and I suspect many results would not change. Of the 1300 firms indicted by the Department of Justice over 1962-1980, 14% were recidivists (Bosch and Eckard, 1991).

[2] In this model, damages refers to any penalty that is sensitive to the prices charged while fines refer to penalties that are fixed with respect to the endogenous variables. In practice, fines levied by the antitrust authorities are sensitive to these factors though not as much as damages.

[3] Assuming a depreciation rate to damages is important analytically as it bounds the penalty. An alternative approach is to impose a statute of limitations so that the damage penalty is the sum of damages incurred over a bounded number of periods into the past. I conjecture the same type of insight would emerge under such an assumption. I thank Ted O'Donoghue for making this suggestion. $\beta$ can also capture the fact that the real value of the damages declines over time as defendants are not required to pay foregone interest. Interest is applied only after the judicial determination of an antitrust violation. Blackstone and Bowman (1987) estimate that this reduced the real value of damage penalties by around 50% in 1975 given the average length of a cartel around that time was 8.6 years.

[4] See Connor (2001) and White (2001) for some estimates of damages associated with the lysine cartel. Also see de Roos (1999) for an analysis of the lysine cartel.

[5] Bryant and Eckard (1991) estimate the chances of a price-fixing cartel being indicted to be around 15%.

[6] In the recent graphite electrodes case, the press reported that the investigation began with a complaint from a steel manufacturer which is a purchaser of graphite electrodes (Levenstein and Suslow, 2001).

[7] The Nasdaq case is one in which truly anomalous pricing resulted in suspicions about collusion.

that buyers (in many price-fixing cases, they are industrial buyers) are the first line of defense in detecting cartels.

> As a general rule, the [Antitrust] Division follows leads generated by disgruntled employees, unhappy customers, or witnesses from ongoing investigations. As such, it is very much a reactive agency with respect to the search for criminal antitrust violations. ... Customers, especially federal, state, and local procurement agencies, play a role in identifying suspicious pricing, bid, or shipment patterns. [McAnney, 1991, pp. 529, 530]

To capture these ideas in a tractable manner, I allow the probability of a cartel being detected in period $t$, denoted $\phi\left(P^t, P^{t-1}\right)$, to depend on the current price and the previous period's price. One can interpret $\phi\left(\widehat{P}, \widehat{P}\right)$ as a baseline probability of detection driven by, for example, internal whistleblowers. A central part of the paper is exploring how assumptions on $\phi$ impact pricing dynamics.

This approach to detection has two limitations. First, it is reduced form; specifying a functional relationship between the price series and the likelihood of detection rather than modelling buyers (or whomever is engaging in detection) and deriving the detection function. Second, it presumes buyers are forgetful in that their likelihood of becoming suspicious about collusion depends only on recent prices. The attractiveness of this specification is that it will generate clean pricing dynamics that, as an initial venture into this important problem, are useful for acquiring intuition and insight. Currently, research is in process to develop a more primitive modelling of the detection process.

In period 1, firms have the choice of forming a cartel, and risking detection and damages, or earning non-collusive profit of $\widehat{\pi}$. If they choose the former, they can, at any time, choose to discontinue colluding. In that event, it is assumed they'll never collude again and receive a terminal payoff of $[\widehat{\pi}/(1-\delta)] - \sigma\left(P^{t-1}, D^{t-1}\right)$ where the last period of collusion is period $t-1$. $\sigma\left(P^{t-1}, D^{t-1}\right)$ is to be interpreted as the present value of the expected penalty when collusion is discovered after the dissolution of the cartel (for example, by a whistleblower).

For the purposes of establishing the existence of an optimal cartel price path, the following assumptions are imposed. Additional structure will be required to derive properties of that path.[8]

**A1** $\Omega \subseteq \Re_+$ is compact.

**A2** $\pi : \Omega \rightarrow \Re$ is bounded and continuously differentiable.

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It was scholars rather than market participants who observed that dealers avoided odd-eighth quotes and ultimately explained it as a form of collusive behavior (Christie and Schultz, 1994). Though the market-makers did not admit guilt, they did pay an out-of-court settlement of around $1 billion.

[8] If $d(P) = \left(P - \widehat{P}\right)(D(P)/n)$ then $d$ could be decreasing for sufficiently high prices which contradicts A3. However, it is shown in Harrington (2001) that, under standard assumptions on demand and cost functions, $d$ is increasing for prices on the optimal price path.

**A3** $d : \Omega \to \Re_+$ is bounded, continuously differentiable, and non-decreasing.

**A4** $\phi : \Omega^2 \to [0,1]$ is continuous.

**A5** $\sigma : \Omega \times \Re_+ \to \Re_+$ is bounded, continuous and non-decreasing.

**A6** $\exists P^m > \widehat{P}$ such that $\pi'(P) \gtreqless 0$ as $P \lesseqgtr P^m$.

The cartel chooses an infinite price path so as to maximize the expected sum of discounted income where $\delta \in (0,1)$ is its discount factor. To break indifference, firms are assumed to collude if they are indifferent between colluding and not colluding.

# 3 Existence of an Optimal Price Path

The basic problem is one of the cartel manager choosing a price path to maximize the expected present value of the representative cartel member's income stream. To establish the existence of an optimal price path, dynamic programming is used. The state variables are yesterday's price, $P^{t-1}$, and accumulated damages, $D^{t-1}$. $V\left(P^{t-1}, D^{t-1}\right)$ denotes the value function when the cartel is still functioning as of period $t$. It is defined as the fixed point to:

$$\begin{aligned} V\left(P^{t-1}, D^{t-1}\right) = & \max_{P\in\Omega} \pi(P) \\ & +\delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta D^{t-1} - \gamma d(P) - F\right] \\ & +\delta\left[1-\phi\left(P, P^{t-1}\right)\right] \times \max\{V\left(P, \beta D^{t-1} + \gamma d(P)\right), \\ & \left(\widehat{\pi}/(1-\delta)\right) - \sigma\left(P, \beta D^{t-1} + \gamma d(P)\right)\}. \end{aligned} \quad (2)$$

$(\widehat{\pi}/(1-\delta)) - \beta D^{t-1} - \gamma d(P) - F$ is the terminal payoff associated with the cartel being detected. Also note that firms have the future option of dismantling the cartel and receiving a terminal payoff of $(\widehat{\pi}/(1-\delta)) - \sigma\left(P, \beta D^{t-1} + \gamma d(P)\right)$. All proofs are in Appendix A.

**Theorem 1** *Assume A1-A5. An optimal price path exists.*

A natural specification for the post-cartel penalty function is

$$\sigma\left(P^{t-1}, D^{t-1}\right) = \sum_{\tau=t}^{\infty} \delta^{\tau-t+1}\left[\beta^{\tau-t+1} D^{t-1} + F\right]\omega^{\tau}\left(P^{t-1}\right) \quad (3)$$

where $\omega^{\tau}\left(P^{t-1}\right)$ is the probability of the cartel being discovered in period $\tau$ (which may depend on the initial conditions for price for the post-cartel period). In that case, $\sigma\left(P^{t-1}, D^{t-1}\right)$ is an affine function of $D^{t-1}$. This property is used in the next result which shows that the value function is a decreasing convex function of accumulated damages.

**Theorem 2** *Assume A1-A5 and* $\sigma$ *is a weakly concave function of* $D^{t-1}$*. Then* $V\left(P^{t-1}, D^{t-1}\right)$ *is a decreasing convex function of* $D^{t-1}$*.*

As a higher value for $D^{t-1}$ means a more severe penalty in the event of detection, it is unsurprising that the value of collusion is decreasing in the amount of accumulated damages. It is also easy to explain why the value function is convex. Holding the price path fixed and assuming collusion is infinitely-lived, a firm's payoff is linear and decreasing in $D^{t-1}$ as the expected present value of the penalty associated with $D^{t-1}$ is $D^{t-1}\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t+1}\omega^{\tau}$ where $\omega^{\tau}$ is the probability of detection in period $\tau$.[9] Since the cartel can partially mitigate the effect of increased accumulated damages by adjusting the price path to make detection less likely, the value function, at each value of $D^{t-1}$, is bounded from below by a linear decreasing function of $D^{t-1}$ which is tangent to the value function at that value of $D^{t-1}$. With this lower bound, the value function is then (weakly) convex in $D^{t-1}$. Unfortunately, the convexity of the value function means that the cartel's objective function need not be concave and this will complicate the ensuing analysis.

## 4 When Detection Depends on the Price Level

Previous static analyses of the influence of antitrust law on cartel pricing have assumed that the probability of detection is an increasing function of price. In this section, we consider their dynamical extension by assuming the probability of detection depends only on the current price. The idea is that buyers may have some sense about what is an appropriate non-collusive price (or some idea about the level of cost) and the more that the observed price departs from that competitive benchmark, the more likely they are to become suspicious and pursue a legal investigation. The main conclusion of this section is that this modelling of detection generates nonsensical results from which I conclude that detection must be driven by more than the price level.

By assumptions B2 and B3, detection depends only on the current price and the probability function is increasing and weakly convex.[10] B4 and B5 imply that the profit and damage functions are twice differentiable and strictly concave. With very mild differences, these assumptions are the same as in Block et al (1981). Note that the only state variable is $D^{t-1}$ since $\phi$ does not depend on $P^{t-1}$.

**B1** $\Omega$ is a compact convex subset of $\Re_+$ and $\left[\widehat{P}, P^m\right] \subseteq \Omega$.

**B2** $\exists\widetilde{\phi} : \Re \to [0,1]$ such that $\phi\left(P^t, P^{t-1}\right) = \widetilde{\phi}\left(P^t\right) \forall \left(P^t, P^{t-1}\right) \in \Omega^2$.

**B3** $\widetilde{\phi}$ is twice continuously differentiable, $\widetilde{\phi}' > 0$, and $\widetilde{\phi}'' \geq 0$.

[9] If collusion is finitely-lived then one has the same type of expression up until the final period of collusion and then $\sigma$ is relevant thereafter.

[10] Since $\widetilde{\phi}$ is bounded above by one, it is restrictive to assume that it is convex over the entire domain. One should then interpret this assumption as saying that $\widetilde{\phi}$ is convex over that part of the domain where optimal cartel pricing occurs.

**B4** $\pi(P) = (P - c)(D(P)/n)$ and $D'' < 0$.

**B5** $d(P) = \max\left\{\left(P - \widehat{P}\right)(D(P)/n), 0\right\}$.

Theorem 3 shows that the optimal price path has a "tendency" to fall over time. To be more precise, prior to possibly reaching a steady-state, in each period price is lower in some future period. Consistent with this characterization is a weakly decreasing price path (and strictly decreasing in some periods).[11]

**Theorem 3** *Assume A6, B1-B5, and* $D^0 = 0$.[12] *If it is optimal to form a cartel and collude forever then* $\left\{D^{t-1}\right\}_{t=1}^{\infty}$ *is non-decreasing over time,* $P^1 > \widehat{P}$*, and: i) if* $D^{t'-1} < D^{t'}$ *then* $\exists t'' > t'$ *such that* $P^{t'} > P^{t''}$*; and ii) if* $D^{t'-1} = D^{t'}$ *then* $P^t = P^{t'} \forall t \geq t'$.

The proof involves presenting the cartel's problem as choosing $D^t$ given $D^{t-1}$ and then showing that the cartel's objective function is strictly supermodular in $D^t$ and $D^{t-1}$. The latter property implies that the optimal value of $D^t$ is non-decreasing in $D^{t-1}$. Showing that $P^1 > \widehat{P}$ and thus $D^1 > 0 = D^0$, it then follows that accumulated damages are non-decreasing over time. With rising damages, a non-decreasing price path is shown to be inconsistent with optimality; price must be lower in some future period. Furthermore, I conjecture that the optimal collusive price path is monotonically (weakly) decreasing over time. In that case, the cartel initially ratchets price above the non-collusive level and reduces it over time with price either becoming constant from some period onward or asymptoting some value. The basis for this conjecture is the following intuition. The cartel begins with no (or low) accumulated damages. As collusion continues, I know by Theorem 3 that accumulated damages grow which means a higher penalty in the event of detection. Since the probability of detection is increasing in price, a natural response to a higher possible penalty is to lower price and thereby reduce the likelihood of detection. The optimal price path is then likely to entail setting an initially high price and reducing it over time.[13]

To provide evidence towards validating this intuition, numerical analysis was conducted. Assume two firms, a market demand function of $D\left(P^t\right) = 1000 - P^t$, and constant marginal cost equalling zero. The simple monopoly price, $P^m$, is then 500 and the non-collusive price, $\widehat{P}$, is specified to be either the Cournot price, which is 333, or the competitive price, which is zero. The damage function is $d\left(P^t\right) = \left(P^t - \widehat{P}\right)\left(D\left(P^t\right)/n\right)$, fines are zero ($F = 0$), and $\sigma\left(D^{t-1}\right) = 0 \forall D^{t-1}$. The probability of detection is specified to be a linear function, $\widetilde{\phi}\left(P^t\right) =$

[11] The result presumes that cartel formation is optimal and the cartel is infinitely-lived. Sufficient conditions for this to be true are that the probability of detection is sufficiently low and the penalty parameters, $\gamma$ and $F$, are sufficiently low.

[12] The assumption of $D^0 = 0$ could be replaced with the condition that, on the optimal path, $D^0 < D^1$.

[13] That I have been unable to prove this result is due to the complications created by the value function being convex (as established in Theorem 2) so that the firm's objective function need not be concave.

$\min\left\{\phi_0 + \phi_1\left(P^t - \widehat{P}\right), 1\right\}$, and I allow three sets of values for $(\phi_0, \phi_1)$ such that $\left(\widetilde{\phi}\left(\widehat{P}\right), \widetilde{\phi}(P^m)\right) \in \{(0, .1), (.1, .4), (.2, .8)\}$. Finally, assume $\delta = .96$, $\gamma \in \{1, 3\}$ and $\beta \in \{.3, .6, .9\}$. The value function was first approximated as a piecewise linear function with 350 breakpoints. With a discretized price space of 100 points (and, in later analysis, 1000), the Bellman equation was solved iteratively starting from an initial guess. To check for the amount of approximation error, the Bellman equation was solved assuming a finer price grid and the value functions were compared. The change in the value function from a finer grid was always extremely small. Having derived the value function, the optimal policy was derived. Specifying an initial level of damages of zero, the optimal price path and path of accumulated damages were then generated. All of this analysis was performed using MatLab.

Out of the 36 parameter configurations considered, 22 resulted in the cartel being infinitely-lived.[14] In all of those cases, the optimal price path had the firm set a price above the non-collusive price in period 1 and then monotonically lower it over time.[15] Representative of results is Figure 1. The price begins around 403, well above the non-collusive price of 333, and then declines; settling down at a price around 367.

My own sense is that a path of raising price to a high level and then lowering it over time is counterfactual. This is certainly contrary to the price paths associated with the citric acid cartel of 1987-97 (Connor, 1998) and the graphite electrodes cartel of 1992-97 (Levenstein and Suslow, 2001) where price was gradually increased. In that a falling price path is the logical implication of having detection depend only on the price level, I infer that detection is not largely driven by the price level. A natural alternative is that detection is driven by price changes. This is considered in the next section.

# 5 When Detection Depends on Price Changes

While it may be difficult for individuals external to a firm, such as buyers, to have reasonably-informed beliefs about the level of cost and demand, it may be quite reasonable for them to receive noisy signals of changes in cost or demand and be able to make assessments about what is a reasonable change in price; "reasonable" in the sense that it is consistent with non-collusive behavior. This view has previously been articulated:

> Simultaneous price increases and output reductions unexplained by an increase in cost may therefore be good evidence of the initiation of a price-fixing scheme, while changes in the opposite direction might be used as evidence that a cartel has just collapsed. Notice that it is not necessary to determine what the firms' marginal costs are or what the competitive price and output would be. One simply observes price and

[14] By this I mean that the cartel is active over the entire length of the policy simulation (which is 100 periods) and it appears a steady-state had been achieved.

[15] The one caveat to that statement is that in some cases the price path, after settling down, entered into a small price cycle. I believe this is an artifact of the price space being discrete.

output changes and asks whether changes in costs or in demand explain them, or whether it is necessary to posit cartelization. [Posner (1976), pp. 66-67.]

In this paper, I've chosen to simplify matters by not explicitly modelling these cost and demand shocks and instead assuming that bigger price changes are more likely to trigger detection. In the context of a stationary environment, bigger price changes should seem more puzzling. The virtue of this simplification is that the optimal price path is deterministic which is conducive to deriving clean results on dynamics. Future research will explicitly allow for cost and demand shocks and, more generally, consider richer models of detection.

To characterize properties of the optimal price path, the following structure is imposed.

**C1** $\Omega$ is a compact convex subset of $\Re_+$ and $\left[\widehat{P}, P^m\right] \subseteq \Omega$ .

**C2** $\exists \widehat{\phi} : \Re \rightarrow [0,1]$ and $g : \Re \rightarrow \Re_{++}$, where $g$ is a strictly positive, non-increasing, continuously differentiable function, such that

$$\phi\left(P^t, P^{t-1}\right) = \widehat{\phi}\left(\left(P^t - P^{t-1}\right) g\left(P^{t-1}\right)\right) \ \forall \left(P^t, P^{t-1}\right) \in \Omega^2.$$

**C3** If $x \geq y \geq 0$ then $\widehat{\phi}(x) \geq \widehat{\phi}(y)$ .

**C4** $\widehat{\phi}(x) \geq \widehat{\phi}(0)\ \forall x \in \Re$ and $\widehat{\phi}(0) \in [0,1)$ .

**C5** $\exists \varepsilon > 0$ such that $\widehat{\phi}$ is continuously differentiable $\forall P'' \in (P' - \varepsilon, P' + \varepsilon), \forall P'$ and $\widehat{\phi}'(0) = 0$.

Assumptions C2-C4 specify that the probability of detection depends only on the change in price, is non-decreasing for price increases, and is minimized by keeping price constant. Note that the case of a constant probability of detection, so that it is uninfluenced by price dynamics, is a special case. C5 requires differentiability around a price change of zero and is a necessary technical condition.[16] It will also ease the analytics by having $\widehat{\phi}'(0) = 0$. The importance of that assumption will be discussed later. Note that if $g$ is a constant then the probability of detection depends only on the size of price movements. If instead $g\left(P^{t-1}\right) = 1/P^{t-1}$ then it depends on the percentage change in price.

Two additional assumptions involving the profit function are required.

**C6** $\pi(P) - \delta\widehat{\phi}(0)\left[\left(\frac{\gamma d(P)}{1-\beta}\right) + F\right] > \widehat{\pi}\ \forall P \in \left(\widehat{P}, P^m\right]$.

[16] I want to acknowledge Ali Khan for the proper statement of C5. By developing an elegant example, he proved to me that a function can be differentiable at a point but not be differentiable in an $\varepsilon$-ball around that point.

**C7** $\exists P^* \in \left(\widehat{P}, P^m\right]$ such that

$$\pi'(P) - \left[\delta\widehat{\phi}(0) / \left(1 - \delta\beta\left(1 - \widehat{\phi}(0)\right)\right)\right]\gamma d'(P) \gtreqless 0 \text{ as } P \lesseqgtr P^*.$$

In Harrington (2001), it is shown that C6 is sufficient to ensure that, at a steady state price of $P$, colluding is preferable to not colluding. C7 requires quasi-concavity of an income function which is defined to be profit less some multiple of damages. An example satisfying these assumptions is provided later.

## 5.1 Properties of the Cartel Price Path

Theorem 4 shows that if collusion is optimal then it is infinitely-lived, involves a non-decreasing price path, and the long-run price is $P^*$ (as defined in C7).[17] These properties for the price path are derived under the assumption that it is optimal to form a cartel.[18]

**Theorem 4** *Assume A2-A6, C1-C7, and* $P^0 \in \left[\widehat{P}, P^*\right)$*. If it is optimal to form a cartel then it is optimal to collude in all periods and if* $\{P^t\}_{t=1}^{\infty}$ *is an optimal price path then: i) it is non-decreasing over time; and ii)* $P^t \to P^*$ *as* $t \to \infty$*.*

In that larger price movements result in a higher probability of detection, the optimal price path has the cartel gradually increase price to its long-run target value of $P^*$ with the hope of not triggering suspicions. If the probability of detection is independent of price then one can show that the cartel immediately increases price to $P^*$ and leaves it there. Thus, dynamics come from detection depending on the change in price.[19] A numerical example in Figure 2 shows a typical price path. Price starts at the non-collusive (Cournot) price of 333 and is gradually raised; asymptoting a value of 470 which is below the simple monopoly price of 500.[20]

The long-run cartel price, $P^*$, is defined as the unique solution to

$$\pi'(P^*) - \left[\delta\widehat{\phi}(0) / \left(1 - \delta\beta\left(1 - \widehat{\phi}(0)\right)\right)\right]\gamma d'(P^*) = 0. \tag{4}$$

[17] Without C5, the proof still establishes that the price path is non-decreasing over time and is bounded from above by $P^*$. C5 serves to show that $lim_{t\to\infty} P^t = P^*$.

[18] Here are two sets of sufficient conditions for cartel formation to occur when $P^0 = \widehat{P}$ and $D^0 = 0$. First, $\gamma$ and $F$ are sufficiently small. Second, $d\left(\widehat{P}\right) = 0$, $F = 0$, and $\sigma\left(P^{t-1}, D^{t-1}\right) = 0 \forall \left(P^{t-1}, D^{t-1}\right)$. The first case is immediate and the second case is shown in Harrington (2001). The latter is robust to small changes in the assumptions.

[19] When the probability of detection is fixed then the expected penalty associated with past damages is independent of what firms do (as long as they continue colluding). Hence, the optimal price doesn't change over time even though damages do grow.

[20] The numerical analysis assumes the demand and cost functions used with the numerical analysis in Section III and that $n = 2$, $\beta = .6$, $\gamma = 1$, $\delta = .96$, and $F = 0$. The probability of detection function is quadratic, $\min\left\{\phi_0 + \phi_1\left(P^t - P^{t-1}\right)^2, 1\right\}$, with $\phi_0 = .05$ and $\phi_1 = .0002592$ so that raising the price 25% of the way from the non-collusive to the simple monopoly price in one period results in a 50% chance of detection. Note that $\widehat{\phi}(0) = .05$ so that the steady-state probability of detection is 5%. More numerical results are in Harrington (2001).

$P^*$ depends on the damage function and multiple, the rate at which damages depreciate, and the probability of detection function. If the profit function is concave ($\pi'' < 0$), the damage function is strictly increasing ($\gamma d' > 0$), and the minimum probability of detection is positive, $\widehat{\phi}(0) > 0$, then $P^* < P^m$ so that the cartel price is bounded below the simple monopoly price in all periods. Thus, antitrust law constrains pricing behavior. However, also note that if $\gamma = 0$, so that the only penalty is fines, then $\pi'(P^*) = 0$ which means $P^* = P^m$. *At the steady-state, fines do not constrain the cartel's price.* It is true, however, that higher fines can be expected to reduce the speed with which price is raised and, if fines are sufficiently high, they can deter cartel formation altogether.

It appears that this result is driven by assumption C5 which states that small price changes have a proportionately small effect on the probability of detection. In the long run, price settles down so that the price change converges to zero. Given that $\widehat{\phi}'(0) = 0$, marginal changes in price have no first-order effect on the probability of detection though continue to have a first-order effect on the potential penalty through the damage function. Thus, factors that influence the relationship between price and the size of the penalty - the discount factor, the rate of depreciation of damages, the damage multiple, and the damage function - all influence the long-run price. As a result, if there are only fines and no damages then, as price changes go to zero, marginal changes in price have no effect on the expected penalty in the long-run so that the cartel price converges to the simple monopoly price. Suppose instead $\widehat{\phi}'(0) > 0$ so that C5 is not satisfied. Inspection of the proof of Theorem 4 reveals that the price path is still non-decreasing and is bounded from above by $P^*$. Hence, price changes still go to zero and, therefore, $\widehat{\phi}'(0) > 0$ implies marginal changes in price do impact the probability of detection. Since a higher fine makes the expected penalty more sensitive to price changes even in the long-run, fines - as well as damages - influence the long-run price. Arguing by continuity, the long-run price is then largely independent of fines when $\widehat{\phi}'(0)$ is positive but small.

The possible independence of the steady-state cartel price with respect to fines is in stark contrast to static models of collusive pricing in the presence of antitrust laws. In those models, there is an equivalence between fines and damages in the sense that any price resulting for some damage multiple could be alternatively generated through an appropriately selected fine. In a dynamical model in which detection depends on price changes, price is bounded below the simple monopoly price when damages are used but that need not be the case when the only penalty is fines.

### 5.2 Comparative Statics

Assume the market demand function, $D(\cdot)$, is twice differentiable and each firm has constant marginal cost of $c$. A firm's profit is then $\pi(P) = (P - c)(D(P)/n)$. Assume $D''(P) \leq 0$ so that A6 holds. Next suppose that the damage function is $d(P) = \left(P - \widehat{P}\right)(D(P)/n)$ where $\widehat{P} > c$. To ensure that C7 is satisfied, define

$$\Psi(P) \equiv \pi(P) - \kappa d(P) = (1/n)\left[(P-c)D(P) - \kappa\left(P - \widehat{P}\right)D(P)\right] \tag{5}$$

$$\text{where } \kappa \equiv \delta\widehat{\phi}(0)\gamma \Big/ \left(1-\delta\beta\left(1-\widehat{\phi}(0)\right)\right).$$

Note that if $\Psi''(P) < 0$ then $P^*$ is defined by $\Psi'(P^*) = 0$. Taking the first two derivatives of $\Psi$:

$$\begin{aligned}\Psi'(P) &= (1/n)\left\{(1-\kappa)\left[(P-c)D'(P) + D(P)\right] + \kappa\left(\widehat{P}-c\right)D'(P)\right\}, \quad (6)\\ \Psi''(P) &= (1/n)\left\{(1-\kappa)\left[2D'(P) + (P-c)D''(P)\right] + \kappa\left(\widehat{P}-c\right)D''(P)\right\}.\end{aligned}$$

$\Psi''(P) < 0$ if $\kappa < 1$ and $D'' \leq 0$. For $P^*$ to exceed $\widehat{P}$, one needs:

$$\Psi'\left(\widehat{P}\right) = (1/n)\left\{(\widehat{P}-c)D'\left(\widehat{P}\right) + (1-\kappa)D\left(\widehat{P}\right)\right\} > 0. \quad (7)$$

Since $(\widehat{P}-c)D'\left(\widehat{P}\right) + D\left(\widehat{P}\right) > 0$, as $\widehat{P}$ is associated with the non-collusive outcome, then $\Psi'\left(\widehat{P}\right) > 0$ if $\kappa$ is sufficiently close to zero which holds, for example, if either the probability of detection or the damage multiple is sufficiently small. $P^*$ is then defined by:

$$(1-\kappa)\left[(P^*-c)D'(P^*) + D(P^*)\right] + \kappa\left(\widehat{P}-c\right)D'(P^*) = 0. \quad (8)$$

Taking the total derivative of (8) with respect to $\kappa$,

$$\frac{\partial P^*}{\partial \kappa} = \frac{\left[(P^*-c)D'(P^*) + D(P^*)\right] - \left(\widehat{P}-c\right)D'(P^*)}{(1-\kappa)\left[2D'(P^*) + (P^*-c)D''(P^*)\right] + \kappa\left(\widehat{P}-c\right)D''(P^*)} < 0. \quad (9)$$

It is straightforward to show that $\kappa$ is increasing in $\gamma$, $\widehat{\phi}(0)$, $\beta$, and $\delta$. It is then immediate that the steady-state cartel price is reduced when: i) the damage multiple, $\gamma$, is increased; ii) the probability of detection, $\widehat{\phi}(\cdot)$, is increased; iii) the rate at which damages persist over time, $\beta$, is increased; and iv) the discount factor, $\delta$, is increased.[21] The first three results are quite immediate. To explain the last one, note that the cartel faces an intertemporal trade-off in that a higher price in the current period raises current profit but lowers the future payoff by increasing the likelihood of detection and, in the event of future detection, increasing the penalty. As cartel members become more patient (a higher value for $\delta$), they then prefer lower cartel prices. Note that the standard equilibrium analysis of collusion (where price is driven by incentive compatibility constraints, that we presently ignore), a higher discount factor supports higher collusive prices. An equilibrium analysis of cartel pricing under the constraint of detection will bring to bear these two counteracting forces and could prove insightful.

If $P^*$ is twice differentiable with respect to $\kappa$ then:

$$\frac{\partial^2 P^*}{\partial\gamma\partial\widehat{\phi}(0)} = \left(\frac{\partial^2 P^*}{\partial\kappa^2}\right)\left(\frac{\partial\kappa}{\partial\gamma}\right)\left(\frac{\partial\kappa}{\partial\widehat{\phi}(0)}\right) + \left(\frac{\partial P^*}{\partial\kappa}\right)\left(\frac{\partial^2\kappa}{\partial\gamma\partial\widehat{\phi}(0)}\right). \quad (10)$$

[21] Numerical analysis revealed that when a change in a parameter caused the long-run cartel price to fall (rise), the entire price path declined (rose). These results are in Harrington (2001).

Since

$$\frac{\partial^2 \kappa}{\partial \gamma \partial \widehat{\phi}(0)} = \frac{\delta(1-\delta\beta)}{\left[1-\delta\beta\left(1-\widehat{\phi}(0)\right)\right]^2} > 0, \tag{11}$$

it follows that, if $\partial^2 P^*/\partial\kappa^2 \leq 0$ (which is true when the demand function is linear) then $\partial^2 P^*/\partial\gamma\partial\widehat{\phi}(0) < 0$. The interpretation of the cross partial derivative of the steady-state cartel price with respect to $\gamma$ and $\widehat{\phi}(0)$ is that the damage multiple and enforcement expenditure (assuming that the latter implies a shift up in $\widehat{\phi}$) are complements in their effect on the cartel price; a higher damage multiple makes enforcement expenditure more effective in reducing price. Hence, more severe penalties should probably be complemented with more, not less, enforcement activities.

A final interesting comparative static exercise is to consider the influence of the but-for price, $\widehat{P}$, on the steady-state cartel price. Recall that the but-for price is the price used in calculating damages.[22] It will be useful to generalize the damage function to:

$$d(P) = \left(P - \widehat{P}\right)\left[\alpha\left(D(P)/n\right) + (1-\alpha)\left(D\left(\widehat{P}\right)/n\right)\right] \tag{12}$$

where $\alpha \in [.5, 1]$. U.S. antitrust practice is captured by $\alpha = 1$ while if damages were specified to equal the loss in consumer surplus then $\alpha = .5$, using a linear approximation. Note that as $\alpha$ falls, the cartel's price has less of an influence on the level of demand used for calculating damages. It is straightforward to derive:

$$\frac{\partial P^*}{\partial \widehat{P}} = \frac{\kappa\left[(1-\alpha)D'\left(\widehat{P}\right) - \alpha D'(P^*)\right]}{(1-\kappa\alpha)\left[2D'(P^*) + (P^*-c)D''(P^*)\right] + \kappa\alpha\left(\widehat{P}-c\right)D''(P^*)} < 0. \tag{13}$$

As before, the denominator is negative. Next note that the numerator is increasing in $\alpha$ so it is minimized at $\alpha = .5$. In that case, the numerator is non-negative as long as $D'\left(\widehat{P}\right) \geq D'(P^*)$. Since $P^* > \widehat{P}$ then $D'' \leq 0$ implies $D'\left(\widehat{P}\right) \geq D'(P^*)$. It is concluded that $\partial P^*/\partial\widehat{P} < 0$.[23] To understand this result, first note that lowering $\widehat{P}$ raises the total amount of damages by increasing the amount of damages assigned per unit of damage demand. One response to a lower but-for price is to lower the cartel price so as to bring back down the per unit damage amount. Alternatively, firms could raise the cartel price so as to reduce the number of units upon which damages are assessed. Given $\alpha$ is not too low - so that the number of units used for damage calculation is sufficiently sensitive to the collusive price - the latter effect dominates. Surprisingly, the steady-state cartel price is then decreasing in the but-for price. Thus, if the cartel anticipates that a more competitive standard will be applied in calculating damages, this will result in a *higher* cartel price in the long-run.[24]

[22] $\widehat{P}$ represents two different prices: the non-collusive price and the but-for price. While, in practice, they are intended to be the same, in principle they could be different. The point to make is that it is $\widehat{P}$ as the but-for price which influences the steady-state cartel price.

[23] Except if $\alpha = .5$ and $D'' = 0$, in which case $\partial P^*/\partial\widehat{P} = 0$. Furthermore, $\partial P^*/\partial\widehat{P} > 0$ if one allows $\alpha$ to be close to zero but it is unclear why that is a relevant case.

[24] However, it is important to note that a lower but-for price may dissuade cartel formation altogether.

Two corollaries are worth mentioning. First, if $\alpha = 1$ and the but-for price is the competitive price, $\widehat{P} = c$, then the steady-state cartel price is the same as the price in the absence of antitrust laws, $P^* = P^m$. Just as much as fines can have no effect in the long run, damages can have no effect either. Second, if the but-for price is assumed to be decreasing in $n$ (which is true for most oligopoly models if the but-for price is the non-collusive price) then the steady-state cartel price is *increasing* in the number of firms.

## 6 Concluding Remarks

In choosing a price path, it is natural to expect a price-fixing cartel to try to avoid creating suspicions that there is a cartel. When detection depends only on the price level, I find that the nonsensical result emerges that the cartel initially raises price and then lowers it over time. When instead detection is driven by price movements, the cartel price path is increasing over time and converges to a steady-state. This steady-state price is below the simple monopoly price when penalties include damages but can equal the simple monopoly price when only fines are used. While a higher damage multiple and more enforcement (which serves to raise the probability of detection) results in a lower cartel price path, a lower but-for price actually raises it.

The model and analysis of this paper is an initial attempt to develop a richer dynamical theory of price-fixing cartels by taking account of their illegality and the desire of firms to avoid detection. There are many directions that one can go from here. With this particular model, there is a need to take account of equilibrium conditions so as to ensure that firms do not want to deviate from the cartel price path. Of particular interest is to explore how antitrust policy interacts with incentive compatibility constraints. To what extent do concerns about detection make cheating more or less desirable?[25]

A second set of extensions is to provide a richer model of detection. The reduced form approach of this paper has considerable analytical virtue. However, if one wants to explore the types of industries that are prone to cartel formation and how the properties of the cartel price path vary across industries, it is necessary to endogenize the detection process so that it can depend on these industry traits. One idea is to introduce cost and demand shocks. The degree of cost variability, demand variability, and seasonality of demand varies significantly across industries. Secondly, this will require allowing the detection process to depend on an industry's cost and demand variability. To what extent buyers become suspicious about collusion in response to a series of price increases should depend on the likelihood that cost and demand increases could have generated that price series and whether, historically, price is relatively stable or volatile. The obvious approach is to model this as a game with firms and buyers (or the antitrust authority) as players.[26] Buyers have a prior set of beliefs on a cartel having been formed (presumably driven by some parameter known to firms and unknown to buyers) and hold accurate conjectures of the firms'

[25] Though in a very restrictive model, Cyrenne (1999) provides some initial insight to that issue.

[26] Besanko and Spulber (1989, 1990) use this approach for the static setting.

pricing rules - under both collusion and non-collusion. The probability of collusion is updated using Bayes Rule and the observed prices. **I think this is the wrong way in which to proceed if the objective is to understand cartel pricing.** The entire purpose of this line of research is to engage in genuine discovery as to how cartels price. I do not imagine that the antitrust authority already knows the answer (indeed, in practice, they hire academic economists like myself to help them), nor that the buyers do (why should the buyers know what the collusive pricing rule looks like if they've never previously witnessed a cartel in their industry?). The modelling requirements are then to endogenize detection so that it depends on relevant industry traits like cost variability but to do so in a manner that does not require buyers "knowing" the collusive pricing rule. I currently have an approach to this problem which I am in the process of developing.

A third extension is related to the fact that detection has been assumed to depend only on movements in a common firm price. However, suspicions about collusion are also generated by firms' prices moving in tandem. If buyers may infer, rightly or wrongly, from parallel price movements that a cartel is present, this will also have implications for pricing behavior.

In conclusion, let me say that by taking into account the issue of detection, theory should be able to offer ways in which to empirically distinguish between explicit and tacit collusion. Tacit collusion I define as when firms engage in a pricing arrangement that serves to raise price and is achieved without explicit communication. While it is possible to prosecute tacitly colluding firms, it is very difficult. Explicit collusion is when firms engage in direct communication regarding the setting of prices (or some other form of collusion such as market allocation). Explicit collusion is clearly an antitrust violation. While antitrust law and policy makes a critical distinction between explicit and tacit collusion, existing collusive pricing theories do not.[27] However, if explicit collusion is illegal and tacit collusion is not (or at least it is considerably more difficult to prove illegality) then concerns about detection are much more important when firms have formed a price-fixing cartel (or what is called a "hard-core cartel" in policy circles). In the model of this paper, all pricing dynamics are driven by concerns about detection and punishment. Indeed, if tacit collusion is legal then, in my model, the joint profit-maximizing price path under tacit collusion is to price at the simple monopoly price in all periods. This is strikingly different from how a hard-core cartel behaves as it gradually raises price and price is bounded below the simple monopoly price. The qualitatively different pricing dynamics between explicit and tacit collusion offers some hope to distinguish between the two forms of collusion. This is quite important for policy purposes as it is best if the antitrust authority allocates its resources to prosecuting explicit collusion for there is both more hope of achieving a conviction and in deterring the formation of hard-core cartels.

[27] There are a few exceptions. McCutcheon (1997) models meetings between firms. Athey, Bagwell, and Sanchirico (1998) and Athey and Bagwell (2001) model the exchange of cost information by firms which would seem more appropriate for explicit than tacit collusion (though such exchange could still occur through a trade association).

## Appendix A

**Proof of Theorem 1** The proof is an adaptation of arguments in Stokey and Lucas (1989). Begin by supposing that the cartel has been formed and let $v : \Omega \times \Re_+ \rightarrow \Re$ be a continuous bounded function. Let $T$ be a function with domain $B$ which is the space of continuous bounded functions that map $\Omega \times \Re_+$ into $\Re$. $T$ is defined as follows:

$$\begin{aligned} T\left(v\left(\cdot\right)\right) &= \max_{P \in \Omega} \pi\left(P\right) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta D^{t-1} - \gamma d\left(P\right) - F\right] \\ &\quad + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right]\max\{v\left(P, \beta D^{t-1} + \gamma d\left(P\right)\right), \\ &\quad \left(\widehat{\pi}/\left(1-\delta\right)\right) - \sigma\left(P, \beta D^{t-1} + \gamma d\left(P\right)\right)\}. \end{aligned} \tag{14}$$

By A1-A5 and the presumption that $v$ is a continuous function, the above problem involves maximizing a continuous function over a compact set. Hence, $T\left(v\left(\cdot\right)\right)$ exists by the Theorem of the Maximum (Theorem 3.6, Stokey and Lucas, 1989). As $\pi, \phi, d, \sigma$, and $v$ are bounded functions, $T$ is then a bounded function. Since $\pi, \phi, d, \sigma$, and $v$ are continuous functions and $\Omega$ is compact, $T$ is a continuous function (Theorem 3.6, Stokey and Lucas, 1989). Hence, the range of $T$ is $B$ so that $T : B \rightarrow B$.

To show that $T$ is a contraction, Blackwell's theorem is used (Theorem 3.3, Stokey and Lucas, 1989). This requires showing that $T$ satisfies monotonicity and discounting. Monotonicity is satisfied when: if $v^o, v^{oo} \in B$ and

$$v^o\left(P^{t-1}, D^{t-1}\right) \leq v^{oo}\left(P^{t-1}, D^{t-1}\right) \forall \left(P^{t-1}, D^{t-1}\right) \in \Omega \times \Re_+$$

then

$$T\left(v^o\left(P^{t-1}, D^{t-1}\right)\right) \leq T\left(v^{oo}\left(P^{t-1}, D^{t-1}\right)\right) \forall \left(P^{t-1}, D^{t-1}\right) \in \Omega \times \Re_+.$$

This is trivially true. Discounting is satisfied when $\exists \theta \in (0,1)$ such that

$$\begin{aligned} T\left(v\left(P^{t-1}, D^{t-1}\right) + a\right) &\leq T\left(v\left(P^{t-1}, D^{t-1}\right)\right) + \theta a \\ \forall v &\in B, a \geq 0, \left(P^{t-1}, D^{t-1}\right) \in \Omega \times \Re_+. \end{aligned}$$

First note that

$$\begin{aligned} & T\left(v\left(P^{t-1}, D^{t-1}\right) + a\right) \\ = & \max_{P} \pi\left(P\right) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta D^{t-1} - \gamma d\left(P\right) - F\right] \\ & + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right]\max\{v^o\left(P, \beta D^{t-1} + \gamma d\left(P\right)\right) + a, \\ & \left(\widehat{\pi}/\left(1-\delta\right)\right) - \sigma\left(P, \beta D^{t-1} + \gamma d\left(P\right)\right)\} \\ \leq & \max_{P} \pi\left(P\right) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta D^{t-1} - \gamma d\left(P\right) - F\right] \\ & + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right]\max\{v^o\left(P, \beta D^{t-1} + \gamma d\left(P\right)\right), \\ & \left(\widehat{\pi}/\left(1-\delta\right)\right) - \sigma\left(P, \beta D^{t-1} + \gamma d\left(P\right)\right)\} + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right]a \\ \leq & \max_{P} \pi\left(P\right) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta D^{t-1} - \gamma d\left(P\right) - F\right] \end{aligned}$$

$$
\begin{aligned}
&+\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{v^{o}\left(P,\beta D^{t-1}+\gamma d\left(P\right)\right), \\
&\left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta D^{t-1}+\gamma d\left(P\right)\right)\}+\delta a \\
=\ & T\left(v\left(P^{t-1},D^{t-1}\right)\right)+\delta a.
\end{aligned}
$$

As $\delta \in (0,1)$, $T$ is then a contraction. By the Contraction Mapping Theorem (Theorem 3.2, Stokey and Lucas, 1989), $T$ has a unique fixed point which is a continuous bounded function. This fixed point is the value function, $V$. Since then

$$
\begin{aligned}
&\pi\left(P\right)+\delta\phi\left(P,P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta D^{t-1}-\gamma d\left(P\right)-F\right] \\
&+\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{V\left(P,\beta D^{t-1}+\gamma d\left(P\right)\right), \\
&\left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta D^{t-1}+\gamma d\left(P\right)\right)\}
\end{aligned}
\tag{15}
$$

is a continuous and bounded function and $\Omega$ is compact, an optimal price path exists.

All of this analysis is for when the cartel has been formed. If $V\left(P^0,D^0\right) \geq \widehat{\pi}/\left(1-\delta\right)$ then it is indeed optimal to form the cartel and the price path is that which maximizes (15). If $V\left(P^0,D^0\right) < \widehat{\pi}/\left(1-\delta\right)$ then it is not optimal to form the cartel and the optimal price path is $\widehat{P}$ in all periods. ■

**Proof of Theorem 2** Define the sequence of value functions $\{v_h\left(\cdot\right)\}_{h=1}^{\infty}$ :

$$
\begin{aligned}
v_{h+1}\left(P^{t-1},D^{t-1}\right) =\ & \max_{P\in\Omega}\pi\left(P\right) \\
&+\delta\phi\left(P,P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta D^{t-1}-\gamma d\left(P\right)-F\right] \\
&+\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{v_h\left(P,\beta D^{t-1}+\gamma d\left(P\right)\right), \\
&\left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta D^{t-1}+\gamma d\left(P\right)\right)\}.
\end{aligned}
\tag{16}
$$

Given A1-A5, it follows from the Contraction Mapping Theorem that $V\left(\cdot\right)=\lim_{h\rightarrow\infty}v_h\left(\cdot\right)$. Any property that holds for the sequence of functions $\{v_h\left(\cdot\right)\}_{h=1}^{\infty}$ then holds for $V\left(\cdot\right)$.

Suppose $v_h\left(\cdot\right)$ is decreasing in $D^{t-1}$. Since $\sigma\left(\cdot\right)$ is nondecreasing in $D^{t-1}$ then $\max\left\{v_h\left(\cdot\right),\left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(\cdot\right)\right\}$ is nondecreasing in $D^{t-1}$. Using this fact along with $\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta D^{t-1}-\gamma d\left(P\right)-F\right]$ being decreasing in $D^{t-1}$, it follows that $v_{h+1}\left(\cdot\right)$ is decreasing in $D^{t-1}$. Therefore, $V\left(\cdot\right)$ is decreasing in $D^{t-1}$.

Suppose $v_h\left(\cdot\right)$ is convex in $D^{t-1}$. As $\sigma\left(\cdot\right)$ is concave in $D^{t-1}$ and, since the maximum of convex functions is convex, then $\max\left\{v_h\left(\cdot\right),\left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(\cdot\right)\right\}$ is convex in $D^{t-1}$.[28] Next note that $\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta D^{t-1}-\gamma d\left(P\right)-F\right]$ is convex in $D^{t-1}$. Thus, the function being maximized on the rhs of (16) is convex in $D^{t-1}$. As $v_{h+1}\left(\cdot\right)$ is simply the maximum of a collection of convex functions - each one parameterized by a different element of $\Omega$ - then $v_{h+1}\left(\cdot\right)$ is convex in $D^{t-1}$. ■

---

[28] Suppose $u_1\left(\cdot\right),u_2\left(\cdot\right),\ldots,u_k\left(\cdot\right)$ are convex in $x$. To show that $U\left(\cdot\right) \equiv \max\left\{u_1\left(\cdot\right),u_2\left(\cdot\right),\ldots,u_k\left(\cdot\right)\right\}$ is convex, suppose to the contrary. This means that $\exists x',x''$ and $\lambda\in(0,1)$ such that $\lambda U\left(x'\right)+\left(1-\lambda\right)U\left(x''\right)<U\left(\lambda x'+\left(1-\lambda\right)x''\right)$. Suppose $U\left(x'\right)=u_i\left(x'\right)$, $U\left(x''\right)=u_j\left(x''\right)$, and $U\left(\lambda x'+\left(1-\lambda\right)x''\right)=u_k\left(\lambda x'+\left(1-\lambda\right)x''\right)$. The condition is then: $\lambda u_i\left(x'\right)+\left(1-\lambda\right)u_j\left(x''\right)<u_k\left(\lambda x'+\left(1-\lambda\right)x''\right)$. Since $u_i\left(x'\right)\geq u_k\left(x'\right)$ and $u_j\left(x''\right)\geq u_k\left(x''\right)$, it follows that: $\lambda u_k\left(x'\right)+\left(1-\lambda\right)u_k\left(x''\right)<u_k\left(\lambda x'+\left(1-\lambda\right)x''\right)$, but this contradicts the assumption that $u_k\left(\cdot\right)$ is convex.

**Proof of Theorem 3** The proof strategy involves first showing that the state variable, $D^{t-1}$, is non-decreasing over time and then showing that the time path for the state variable implies the specified properties of the price path.

To show that $D^1 > D^0 (= 0)$, let me suppose not so that $P^1 \leq \widehat{P}$ and $d(P^1) = 0$. Since $D^1 = D^0$, by stationarity the optimal price in period 2 is as in period 1 and the same is true for all ensuing periods. This results in a payoff no greater than $\widehat{\pi}/(1-\delta)$ which contradicts the opitmality of forming a cartel.[29]

The next step is to show that $\{D^t\}_{t=1}^{\infty}$ is non-decreasing over time. An alternative representation of the problem is to have the cartel choose $D^t$ rather than $P^t$. Note that $D^t \in \left[\beta D^{t-1}, \beta D^{t-1} + \gamma d(P^m)\right] \equiv \Gamma(D^{t-1})$ where wlog we restrict $P^t \leq P^m$. Solving for the value of $P^t$ that generates $D^t$ given $D^{t-1}$,

$$D^t = \beta D^{t-1} + \gamma d(P^t) \Leftrightarrow P^t = \psi\left(\left(D^t - \beta D^{t-1}\right)/\gamma\right),$$

where $\psi \equiv d^{-1}$. Note that $d$ is strictly monotonic $\forall P^t \leq P^m$ so $d^{-1}$ is defined. The cartel problem is then:

$$\begin{aligned} &\max_{D^t \in \Gamma(D^{t-1})} \pi\left(\psi\left(\left(D^t - \beta D^{t-1}\right)/\gamma\right)\right) + V\left(D^t\right) \\ &-\delta\widetilde{\phi}\left(\psi\left(D^t - \beta D^{t-1}\right)/\gamma\right)\left[V\left(D^t\right) - \left(\widehat{\pi}/(1-\delta)\right) + D^t + F\right]. \end{aligned}$$

Denote the maximand to be $f\left(D^t, D^{t-1}\right)$.

Topkis (1978) shows that if $\Gamma\left(D^{t-1}\right)$ is increasing in $D^{t-1}$ and $f\left(D^t, D^{t-1}\right)$ is upper semi-continuous and strictly supermodular in $D^t$ and $D^{t-1}$ then $\arg\max f\left(D^t, D^{t-1}\right)$ is non-decreasing in $D^{t-1}$.[30] If the optimal value for $D^t$ is non-decreasing in $D^{t-1}$, it follows from $D^0 < D^1$ that $\{D^{t-1}\}_{t=1}^{\infty}$ is non-decreasing. $\Gamma\left(D^{t-1}\right)$ is clearly increasing in $D^{t-1}$.[31] As $f\left(D^t, D^{t-1}\right)$ is continuous, it only remains to show that it is strictly supermodular. Given $f$ is defined on a product of ordered sets then strict supermodularity is equivalent to increasing differences where $f$ has increasing differences in $D^t$ and $D^{t-1}$ if $f\left(D^t, \overline{D}^{t-1}\right) - f\left(D^t, \widetilde{D}^{t-1}\right)$ is increasing in $D^t$ when $\overline{D}^{t-1} > \widetilde{D}^{t-1}$.

$$\begin{aligned} & f\left(D^t, \overline{D}^{t-1}\right) - f\left(D^t, \widetilde{D}^{t-1}\right) \\ = & \left[\pi\left(\psi\left(\left(D^t - \beta\overline{D}^{t-1}\right)/\gamma\right)\right) - \pi\left(\psi\left(\left(D^t - \beta\widetilde{D}^{t-1}\right)/\gamma\right)\right)\right] \\ & +\delta\left[\widetilde{\phi}\left(\psi\left(\left(D^t - \beta\widetilde{D}^{t-1}\right)/\gamma\right)\right) - \widetilde{\phi}\left(\psi\left(\left(D^t - \beta\overline{D}^{t-1}\right)/\gamma\right)\right)\right] \times \\ & \left[V\left(D^t\right) - \left(\widehat{\pi}/(1-\delta)\right) + D^t + F\right]. \end{aligned} \tag{17}$$

To begin, let me show that the first bracketed term in (17) is non-decreasing in $D^t$. As $\pi$ and $\psi$ are twice differentiable, a sufficient condition is that

$$\partial^2\pi\left(\psi\left(\left(D^t - \beta D^{t-1}\right)/\gamma\right)\right)/\partial D^t \partial D^{t-1} \geq 0.$$

[29] This statement does presume that cartel formation is strictly preferred so that $V(0) > \widehat{\pi}/(1-\delta)$.

[30] A useful reference for the theory of supermodular functions is Vives (1999).

[31] The set $A$ *is higher than* the set $B$, $A \succeq B$, if for each $a \in A$ and $b \in B$, $\sup\{a,b\} \in A$ and $\inf\{a,b\} \in B$. The correspondence $\phi : A \rightarrow B$ is *increasing* if, when $a \geq a', a \neq a'$, then $\phi(a) \succeq \phi(a')$.

Since

$$\pi\left(\psi\left(\left(D^t-\beta D^{t-1}\right)/\gamma\right)\right)=\left[\psi\left(\left(D^t-\beta D^{t-1}\right)/\gamma\right)-c\right](1/n)\,D\left(\psi\left(\left(D^t-\beta D^{t-1}\right)/\gamma\right)\right)$$

then

$$\begin{aligned}&\partial^2\pi\left(\psi\left(\left(D^t-\beta D^{t-1}\right)/\gamma\right)\right)/\partial D^t\partial D^{t-1}\\ =&\ -\left(\beta/\gamma^2 n\right)\left\{\psi''\left[D(\psi)+(\psi-c)\,D'(\psi)\right]+\left(\psi'\right)^2\left[2D'(\psi)+(\psi-c)\,D''(\psi)\right]\right\}.\end{aligned}$$

Using these substitutions,[32]

$$\begin{aligned}\psi' &= n\Big/\left[D(\psi)+\left(\psi-\widehat{P}\right)D'(\psi)\right]>0\\ \psi'' &= -n\left\{\psi'\left[2D'(\psi)+\left(\psi-\widehat{P}\right)D''(\psi)\right]\right\}\Big/\left[D(\psi)+\left(\psi-\widehat{P}\right)D'(\psi)\right]^2>0,\end{aligned}$$

and multiplying through by $\left[D+\left(\psi-\widehat{P}\right)D'\right]^2/\psi'$, one finds that $\partial^2\pi/\partial D^t\partial D^{t-1}\geq 0$ iff

$$\begin{aligned}0\ \geq\ &\left[2D'+(\psi-c)\,D''\right]\left[D+\left(\psi-\widehat{P}\right)D'\right]\\ &-\left[2D'+\left(\psi-\widehat{P}\right)D''\right]\left[D+(\psi-c)\,D'\right].\end{aligned}\tag{18}$$

Let $\chi\left(\widehat{P}\right)$ denote the expression on the rhs of (18). Since $\chi(c)=0$,

$$\chi'\left(\widehat{P}\right)=-\left[2D'+(\psi-c)\,D''\right]D'+\left[D+(\psi-c)\,D'\right]D''<0,$$

and $\widehat{P}\geq c$ then $\chi\left(\widehat{P}\right)\leq 0$. This proves that the first bracketed term in (17) is non-decreasing in $D^t$.

The next step is to show that the second term in (17),

$$\begin{aligned}&\delta\left[\widetilde{\phi}\left(\psi\left(\left(D^t-\beta\widetilde{D}^{t-1}\right)/\gamma\right)\right)-\widetilde{\phi}\left(\psi\left(\left(D^t-\beta\overline{D}^{t-1}\right)/\gamma\right)\right)\right]\times\\ &\left[V\left(D^t\right)-\left(\widehat{\pi}/(1-\delta)\right)+D^t+F\right],\end{aligned}$$

is increasing in $D^t$. As both of these bracketed terms are positive (the second term is obvious and the first term will be shown), it is sufficient to prove that these terms are increasing in $D^t$. As $\phi$ is twice differentiable then the first term is increasing in $D^t$ if

$$\begin{aligned}&\widetilde{\phi}'\left(\psi\left(\left(D^t-\beta\widetilde{D}^{t-1}\right)/\gamma\right)\right)\psi'\left(\left(D^t-\beta\widetilde{D}^{t-1}\right)/\gamma\right)\\ >\ &\widetilde{\phi}'\left(\psi\left(\left(D^t-\beta\overline{D}^{t-1}\right)/\gamma\right)\right)\psi'\left(\left(D^t-\beta\overline{D}^{t-1}\right)/\gamma\right).\end{aligned}\tag{19}$$

[32] Note that the optimal price path lies below $P^m$. If the cartel prices at $P^m$, by instead marginally lowering its price there is no first-order effect on current profit but there is a first-order effect in reducing the probability of detection and reducing damages (when $\widehat{P}>c$). Given the optimal price is less than than $P^m$ and $\widehat{P}\geq c$ then $D(\psi)+(\psi-c)\,D'(\psi)>0$.

Since $\overline{D}^{t-1} > \widetilde{D}^{t-1}$ then $\left(D^t - \beta\widetilde{D}^{t-1}\right)/\gamma > \left(D^t - \beta\overline{D}^{t-1}\right)/\gamma$ and, given $\psi' > 0$,

$$\psi\left(\left(D^t - \beta\widetilde{D}^{t-1}\right)/\gamma\right) > \psi\left(\left(D^t - \beta\overline{D}^{t-1}\right)/\gamma\right).$$

Since $\widetilde{\phi}'' \geq 0$ then

$$\widetilde{\phi}'\left(\psi\left(\left(D^t - \beta\widetilde{D}^{t-1}\right)/\gamma\right)\right) \geq \widetilde{\phi}'\left(\psi\left(\left(D^t - \beta\overline{D}^{t-1}\right)/\gamma\right)\right).$$

Given $\psi'' > 0$ then

$$\psi'\left(\left(D^t - \beta\widetilde{D}^{t-1}\right)/\gamma\right) > \psi'\left(\left(D^t - \beta\overline{D}^{t-1}\right)/\gamma\right).$$

I conclude that (19) is true.

Finally, I need to show that $\left[V\left(D^t\right) - \left(\widehat{\pi}/\left(1-\delta\right)\right) + D^t + F\right]$ is increasing in $D^t$. If $\overline{D}^t > \widetilde{D}^t$, it should be true that

$$\begin{aligned} V\left(\overline{D}^t\right) - \left(\widehat{\pi}/\left(1-\delta\right)\right) + \overline{D}^t + F &> V\left(\widetilde{D}^t\right) - \left(\widehat{\pi}/\left(1-\delta\right)\right) + \widetilde{D}^t + F \Leftrightarrow \\ \overline{D}^t - \widetilde{D}^t &> V\left(\widetilde{D}^t\right) - V\left(\overline{D}^t\right). \end{aligned}$$

This holds because increasing damages by $\overline{D}^t - \widetilde{D}^t$ lowers the cartel's payoff by $\delta\beta\left(\overline{D}^t - \widetilde{D}^t\right)$ when detection occurs for sure in period $t$. Hence, $\delta\beta\left(\overline{D}^t - \widetilde{D}^t\right)$ is an upper bound on the change in the value function.

The final step is to establish the properties of the price path. Since $D^{t-1}$ is non-decreasing, there are two cases: i) $D^{t'-1} < D^{t'}$; and ii) $D^{t'-1} = D^{t'}$. Let $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ denote the optimal price path. Lemma 1 provides a lower and upper bound on the change in the value function.[33]

**Lemma 1**: If $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an optimal price path then

$$\Delta^{t'}\beta\left(D^{t'} - D^{t'-1}\right) \geq V\left(D^{t'-1}\right) - V\left(D^{t'}\right) \geq \Delta^{t'+1}\beta\left(D^{t'} - D^{t'-1}\right)$$

where

$$\Delta^t = \delta\sum_{\tau=t}^{\infty}\left(\delta\beta\right)^{\tau-t}\widetilde{\phi}\left(\overline{P}^{\tau}\right)\prod_{j=t}^{\tau-1}\left[1 - \widetilde{\phi}\left(\overline{P}^j\right)\right].$$

**Proof**: In examining a firm's payoff in period $t$, $D^{t-1}$ enters in the form of the expression $-\Delta^t\beta D^{t-1}$ which is the expected present value of the penalty associated with $D^{t-1}$ (for details see the representation of the payoff in Appendix B). Intuitively, $\Delta^t$ is the sum of the probabilities of being detected in periods $t, t+1, \ldots$ where the penalty (associated with accumulated damages as of period $t-1$) is $\left(\delta\beta\right)^{\tau-t+1}D^{t-1}$ where damages are discounted and depreciated.

[33] Note that this provides an alternative proof that the value function is decreasing in $D^{t-1}$.

Given accumulated damages of $D^{t'}$, suppose the cartel were to use the price path $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$ over periods $t'+1, t'+2, \ldots$ The associated payoff is $V\left(D^{t'-1}\right) - \Delta^{t'}\beta\left(D^{t'} - D^{t'-1}\right)$ where $\Delta^{t'}\beta\left(D^{t'} - D^{t'-1}\right)$ is the rise in the present value of expected damages from increasing accumulated damages from $D^{t'-1}$ to $D^{t'}$ holding the price path fixed at that which occurs when damages are $D^{t'-1}$. Since the optimal price path starting at $t'+1$ may differ from that price path, this is a lower bound on the value function evaluated at $D^{t'}$ :

$$V\left(D^{t'}\right) \geq V\left(D^{t'-1}\right) - \Delta^{t'}\beta\left(D^{t'} - D^{t'-1}\right). \tag{20}$$

By analogous reasoning, if the cartel uses the price path $\left\{\overline{P}^t\right\}_{t=t'+1}^{\infty}$ over periods $t', t'+1, \ldots$ then the associated payoff is $V\left(D^{t'}\right) + \Delta^{t'+1}\beta\left(D^{t'} - D^{t'-1}\right)$. Hence,

$$V\left(D^{t'-1}\right) \geq V\left(D^{t'}\right) + \Delta^{t'+1}\beta\left(D^{t'} - D^{t'-1}\right). \tag{21}$$

Combining (20)-(21) proves Lemma 1. ♦

Consider case (i) so that $D^{t'} - D^{t'-1} > 0$. By Lemma 1, $\Delta^{t'} \geq \Delta^{t'+1}$. Let us next show that $\Delta^t$ is increasing in $P^h \forall h \geq t$.

$$\begin{aligned} \frac{\partial \Delta^t}{\partial P^h} &= \delta\left(\delta\beta\right)^{h-t}\widetilde{\phi}'\left(\overline{P}^h\right)\prod_{j=t}^{h-1}\left[1-\widetilde{\phi}\left(\overline{P}^j\right)\right] \times \\ & \left\{1 - \sum_{\tau=h+1}^{\infty}\left(\delta\beta\right)^{\tau-h}\widetilde{\phi}\left(\overline{P}^\tau\right)\prod_{j=h+1}^{\tau-1}\left[1-\widetilde{\phi}\left(\overline{P}^j\right)\right]\right\}. \end{aligned}$$

Since

$$\sum_{\tau=h+1}^{\infty}\widetilde{\phi}\left(\overline{P}^\tau\right)\prod_{j=h+1}^{\tau-1}\left[1-\widetilde{\phi}\left(\overline{P}^j\right)\right] \leq 1,$$

as it is the probability of detection in period $h+1$ or later (conditional on no detection as of period $h$), it follows that

$$1 - \sum_{\tau=h+1}^{\infty}\left(\delta\beta\right)^{\tau-h}\widetilde{\phi}\left(\overline{P}^\tau\right)\prod_{j=h+1}^{\tau-1}\left[1-\widetilde{\phi}\left(\overline{P}^j\right)\right] > 0$$

and thus $\partial\Delta^t/\partial P^h > 0$. To summarize, $D^{t'} > D^{t'-1}$ implies $\Delta^{t'} \geq \Delta^{t'+1}$ and, in addition, $\Delta^t$ is increasing in $P^h$ for $h \geq t$.

As the next step, it is straightforward to show that

$$\Delta^t = \delta\widetilde{\phi}\left(\overline{P}^t\right) + \left[1-\widetilde{\phi}\left(\overline{P}^t\right)\right]\delta\beta\Delta^{t+1}.$$

This fact combined with $\Delta^{t'} \geq \Delta^{t'+1}$ results in

$$\begin{aligned} \delta\widetilde{\phi}\left(\overline{P}^{t'}\right) + \left[1-\widetilde{\phi}\left(\overline{P}^{t'}\right)\right]\delta\beta\Delta^{t'+1} &\geq \Delta^{t'+1} \Leftrightarrow \\ \delta\widetilde{\phi}\left(\overline{P}^{t'}\right)\Big/\left[1-\left(1-\widetilde{\phi}\left(\overline{P}^{t'}\right)\right)\delta\beta\right] &\geq \Delta^{t'+1}. \end{aligned}$$

Note that the lhs is $\Delta^t$ when the price path is constant at $\overline{P}^{t'}$. Given that property and that $\Delta^{t'+1}$ is increasing in $P^h$, this inequality does not hold if $\overline{P}^t \geq \overline{P}^{t'} \forall t \geq t'+1$ and $\overline{P}^t > \overline{P}^{t'}$ for some $t$. Therefore, either: i) $\overline{P}^t = \overline{P}^{t'} \forall t \geq t'+1$; or ii) $\exists t'' > t'$ such that $\overline{P}^{t'} > \overline{P}^{t''}$. The latter is our desired result. Let us then show that (i) cannot occur.

Suppose the price path is constant at $\overline{P}^{t'}$ over periods $t', t'+1, \ldots$ Using the expression for the payoff function in (34), the derivative of the payoff function at period $t$ with respect to the period $t\,(\geq t')$ price, evaluated at a constant price path of $\overline{P}^{t'} \forall t \geq t'$, is

$$\pi'\left(\overline{P}^t\right) - \Delta^t \gamma d'\left(\overline{P}^t\right) - \left(\frac{\partial \Delta^t}{\partial P^t}\right)\left[\gamma d\left(\overline{P}^t\right) + \beta D^{t-1}\right] \tag{22}$$
$$-\phi'\left(\overline{P}^t\right) \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \prod_{j=t}^{\tau-1}\left[1-\phi\left(\overline{P}^j\right)\right]\left[\pi\left(\overline{P}^\tau\right) - \Delta^\tau \gamma d\left(\overline{P}^\tau\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right].$$

Since the price path is constant, $\Delta^t$ and $\partial \Delta^t / \partial P^t$ take on the same values $\forall t$. Let us argue that $\overline{P}^{t'}$ is in the interior of $\Omega$ so that (22) equals zero. Since it is postulated that $D^{t'-1} < D^{t'}$ then $d\left(\overline{P}^{t'}\right) > 0$ which implies that $\overline{P}^{t'} > \widehat{P}$. If $\overline{P}^{t'} = P^m$ then (22) is negative. Furthermore, it cannot be optimal to price above $P^m$. Hence, $\overline{P}^{t'}$ must be strictly less than $P^m$. Since the optimal price is interior, (22) equals zero $\forall t$. However, the expression is decreasing in $D^{t-1}$ and since $D^{t'-1} < D^{t'}$, if (22) is zero for $t'$ then it must be negative for $t'+1$. This contradiction establishes that the optimal price path cannot be constant and, therefore, $D^{t'-1} < D^{t'}$ implies $\exists t'' > t'$ such that $\overline{P}^{t'} > \overline{P}^{t''}$.

Case (ii) is when $D^{t'-1} = D^{t'}$. By stationarity, $\overline{P}^{t'} = \overline{P}^{t'+1}$. Next note that:

$$D^{t'+1} = \beta D^{t'} + \gamma d\left(\overline{P}^{t'+1}\right) = \beta D^{t'-1} + \gamma d\left(\overline{P}^{t'}\right) = D^{t'}.$$

By induction, $D^{t-1} = D^{t'-1} \forall t \geq t'$ and therefore $\overline{P}^t = \overline{P}^{t'} \forall t \geq t'$. Hence, if $D^{t'-1} = D^{t'}$ then price is constant at $\overline{P}^{t'}$ thereafter. ■

**Proof of Theorem 4** There are several steps in the proof. First, it is shown that if it is optimal to form a cartel then it is optimal to collude forever. Second, the optimal price path is bounded above by $P^*$. Third, the optimal price path is non-decreasing over time. Fourth, the optimal price path converges to $P^*$.

- It is optimal to collude forever.

The strategy is to show that if it is optimal to collude in, say, period $T$ then it must be optimal to collude in period $T+1$. Assume it is optimal to form a cartel. It is sufficient to show that it is optimal to collude forever when $\sigma\left(P^{t-1}, D^{t-1}\right) = 0 \forall \left(P^{t-1}, D^{t-1}\right)$ so that the terminal payoff from stopping collusion is $\widehat{\pi}/(1-\delta)$.

Suppose it is optimal to collude until period $T$ where $T$ is finite. For it to be optimal to collude in $T$, it must be true that:

$$\pi\left(P^T\right)-\delta\phi\left(P^T,P^{T-1}\right)\left[\beta D^{T-1}+\gamma d\left(P^T\right)+F\right]+\frac{\delta\widehat{\pi}}{1-\delta}\geq\frac{\widehat{\pi}}{1-\delta}.$$

The lhs is the payoff from colluding in $T$ and stopping collusion as of $T+1$ and the rhs is the payoff from stopping collusion in $T$. This expression is equivalent to:

$$\pi\left(P^T\right)-\delta\phi\left(P^T,P^{T-1}\right)\left[\beta D^{T-1}+\gamma d\left(P^T\right)+F\right]\geq\widehat{\pi}. \tag{23}$$

For it to be optimal to dismantle the cartel in $T+1$, it is necessary that:

$$\frac{\widehat{\pi}}{1-\delta}>\pi\left(P^T\right)-\delta\widehat{\phi}(0)\left[\beta\left(\beta D^{T-1}+\gamma d\left(P^T\right)\right)+\gamma d\left(P^T\right)+F\right]+\frac{\delta\widehat{\pi}}{1-\delta}\Leftrightarrow$$

$$\widehat{\pi}>\pi\left(P^T\right)-\delta\widehat{\phi}(0)\left[\beta\left(\beta D^{T-1}+\gamma d\left(P^T\right)\right)+\gamma d\left(P^T\right)+F\right]. \tag{24}$$

The rhs of the first line in (24) is the payoff from maintaining a price of $P^T$ in $T+1$ and then stopping collusion as of $T+2$.[34] Note that $\phi\left(P^T,P^T\right)=\widehat{\phi}(0)$. Combining (23)-(24):

$$\begin{aligned}&\pi\left(P^T\right)-\delta\phi\left(P^T,P^{T-1}\right)\left[\beta D^{T-1}+\gamma d\left(P^T\right)+F\right]\\\geq\ &\widehat{\pi}>\pi\left(P^T\right)-\delta\widehat{\phi}(0)\left[\beta\left(\beta D^{T-1}+\gamma d\left(P^T\right)\right)+\gamma d\left(P^T\right)+F\right].\end{aligned}$$

A necessary condition for this to hold is:

$$\begin{aligned}&\pi\left(P^T\right)-\delta\phi\left(P^T,P^{T-1}\right)\left[\beta D^{T-1}+\gamma d\left(P^T\right)+F\right]\\>\ &\pi\left(P^T\right)-\delta\widehat{\phi}(0)\left[\beta\left(\beta D^{T-1}+\gamma d\left(P^T\right)\right)+\gamma d\left(P^T\right)+F\right]\end{aligned}$$

or

$$\widehat{\phi}(0)\left[\beta\left(\beta D^{T-1}+\gamma d\left(P^T\right)\right)+\gamma d\left(P^T\right)+F\right]>\phi\left(P^T,P^{T-1}\right)\left[\beta D^{T-1}+\gamma d\left(P^T\right)+F\right].$$

Since, by C4, $\phi\left(P^T,P^{T-1}\right)\geq\widehat{\phi}(0)$, a necessary condition is:

$$\beta\left(\beta D^{T-1}+\gamma d\left(P^T\right)\right)+\gamma d\left(P^T\right)>\beta D^{T-1}+\gamma d\left(P^T\right)\Leftrightarrow\frac{\gamma d\left(P^T\right)}{(1-\beta)}>D^{T-1}.$$

Intuitively, if it is optimal to collude at a price of $P^T$ in period $T$ but it is not optimal to do so in $T+1$ then damages must be higher in $T+1$. For that to be the case, what is added to damages in $T$, $\gamma d\left(P^T\right)$, must exceed the amount of damages lost through depreciation, $(1-\beta)D^{T-1}$. This produces the above condition.

Next note that it is never optimal for the cartel price to exceed the simple monopoly price of $P^m$. Relative to a price of $P^m$, a higher price yields strictly lower current profit, weakly higher damages, and, as price initially starts below $P^m$, a weakly higher probability of detection. It is straightforward to show that a price

[34] The assumption is used that a firm must strictly prefer not to collude for it to dissolve the cartel.

path with prices above $P^m$ yields a lower payoff to one in which all those prices exceeding $P^m$ are replaced with $P^m$. Since then $P^T \leq P^m$, it follows from C6 that:

$$\pi\left(P^T\right) - \delta\widehat{\phi}(0)\left[\left(\frac{\gamma d\left(P^T\right)}{1-\beta}\right) + F\right] > \widehat{\pi}. \tag{25}$$

Given it has been shown that $D^{T-1}$ is bounded above by $\gamma d\left(P^T\right)/(1-\beta)$, (25) contradicts (24). This contradiction establishes that the claim that collusion stops in finite time is false.

- The optimal price path is bounded above by $P^*$.

The proof strategy is to show that if the price path ever exceeds $P^*$ that a higher payoff is realized by pricing at $P^*$ forever, starting in the period with which price first exceeds $P^*$.

Assuming firms collude forever and using the representation of the payoff in (34), the payoff starting from period $t'$ for the collusive price path $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right) - \overline{\Delta}^{t'}\gamma d\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \overline{\Delta}^{t'}\beta D^{t'-1} \\
&+ \sum_{t=t'+1}^{\infty}\delta^{t-t'}\left\{\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^t\right) - \overline{\Delta}^t\gamma d\left(\overline{P}^t\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\right\} \\
&+ \left[\left(\widehat{\pi}/(1-\delta)\right) - F\right]
\end{aligned} \tag{26}$$

where

$$\overline{\Delta}^t \equiv \delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\phi\left(\overline{P}^\tau,\overline{P}^{\tau-1}\right)\prod_{j=t}^{\tau-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right].$$

In considering (26), it is as if a colluding firm receives net income in each period equal to $\pi\left(\overline{P}^t\right) - \overline{\Delta}^t\gamma d\left(\overline{P}^t\right)$ where $\pi\left(\overline{P}^t\right)$ is gross profit and $\overline{\Delta}^t\gamma d\left(\overline{P}^t\right)$ is the expected present value of damages associated with colluding in that period.

Suppose it is not true that price is bounded above by $P^*$ so $\exists t'$ such that $\overline{P}^{t'} > P^* \geq \overline{P}^{t'-1}$. If this price path is optimal then, starting from period $t'$, it must yield at least as high a payoff as a price path in which firms collude and price at $P^*$ forever. This is true iff:

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right) - \overline{\Delta}^{t'}\gamma d\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \overline{\Delta}^{t'}\beta D^{t'-1} \\
&+ \sum_{t=t'+1}^{\infty}\delta^{t-t'}\left\{\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^t\right) - \overline{\Delta}^t\gamma d\left(\overline{P}^t\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\right\} \\
\geq\ &\left[\pi\left(P^*\right) - \widetilde{\Delta}^{t'}\gamma d\left(P^*\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \widetilde{\Delta}^{t'}\beta D^{t'-1} \\
&+ \sum_{t=t'+1}^{\infty}\delta^{t-t'}\left\{\left[1-\phi\left(P^*,\overline{P}^{t'-1}\right)\right]\left[1-\widehat{\phi}(0)\right]^{t-t'-1}\left[\pi\left(P^*\right) - \widetilde{\Delta}^t\gamma d\left(P^*\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\right\}
\end{aligned} \tag{27}$$

where

$$\widetilde{\Delta}^{t'} \equiv \delta \left\{ \phi\left(P^*, \overline{P}^{t'-1}\right) + \sum_{\tau=t'+1}^{\infty} (\delta\beta)^{\tau-t'} \left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right] \left[1 - \widehat{\phi}(0)\right]^{\tau-t'-1} \widehat{\phi}(0) \right\},$$

$$\widetilde{\Delta}^{t} \equiv \delta \sum_{\tau=t}^{\infty} (\delta\beta)^{\tau-t} \left[1 - \widehat{\phi}(0)\right]^{\tau-t} \widehat{\phi}(0), \; t \geq t'+1$$

and recall that $\phi(P^*, P^*) = \widehat{\phi}(0)$. To show that $\overline{\Delta}^t \geq \widetilde{\Delta}^t \forall t \geq t'$, first note that these expressions can be represented as:

$$\Delta^t = \delta \sum_{\tau=t}^{\infty} (\delta\beta)^{\tau-t} \omega^{\tau} \prod_{j=t}^{\tau-1} \left(1 - \omega^j\right)$$

where $\omega^{\tau}$ is the probability of detection in period $\tau$ condition on no detection as of $\tau - 1$. Note that:

$$\begin{aligned} \frac{\partial \Delta^t}{\partial \omega^{t^o}} &= \delta \left\{ (\delta\beta)^{t^o-t} \prod_{j=t}^{t^o-1} \left(1-\omega^j\right) - \sum_{\tau=t^o+1}^{\infty} (\delta\beta)^{\tau-t} \omega^{\tau} \prod_{j=t, j\neq t^o}^{\tau-1} \left(1-\omega^j\right) \right\} \\ &= \delta (\delta\beta)^{t^o-t} \prod_{j=t}^{t^o-1} \left(1-\omega^j\right) \left\{ 1 - \sum_{\tau=t^o+1}^{\infty} (\delta\beta)^{\tau-t^o} \omega^{\tau} \prod_{j=t^o+1}^{\tau-1} \left(1-\omega^j\right) \right\}. \end{aligned}$$

$\sum_{\tau=t^o+1}^{\infty} \omega^{\tau} \prod_{j=t^o+1}^{\tau-1} \left(1-\omega^j\right)$ is the probability of detection over periods $t^o+1, \ldots, \infty$. Since it is less than or equal to one, it follows that:

$$1 - \sum_{\tau=t^o+1}^{\infty} (\delta\beta)^{\tau-t^o} \omega^{\tau} \prod_{j=t^o+1}^{\tau-1} \left(1-\omega^j\right) > 0.$$

Thus, $\Delta^t$ is increasing in $\omega^{t^o}$. Since $\overline{P}^{t'} > P^* \geq \overline{P}^{t'-1}$ then, by C3, $\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \geq \phi\left(P^*, \overline{P}^{t'-1}\right)$. By C4, $\phi\left(\overline{P}^{t}, \overline{P}^{t-1}\right) \geq \widehat{\phi}(0)$, $t \geq t'+1$. The probability of detection in period $\tau$ (condition on no detection as of $\tau - 1$) is then weakly higher for $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ than for the alternative price path $\forall t \geq t'$. It is concluded that $\overline{\Delta}^t \geq \widetilde{\Delta}^t \forall t \geq t'$.

Consider the lhs expression in (27). Since it is non-increasing in $\overline{\Delta}^t$ and $\overline{\Delta}^t \geq \widetilde{\Delta}^t \forall t \geq t'$, the expression is weakly increased if $\widetilde{\Delta}^t$ replaces $\overline{\Delta}^t$ $\forall t \geq t'$. It follows that if (27) holds then it must be true that:

$$\begin{aligned} &\left[\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}^{t'} \gamma d\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] - \widetilde{\Delta}^{t'} \beta D^{t'-1} \\ &+ \sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \widetilde{\Delta}^{t} \gamma d\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \right\} \\ \geq\; &\left[\pi(P^*) - \widetilde{\Delta}^{t'} \gamma d(P^*) - \left(\widehat{\pi} - (1-\delta) F\right)\right] - \widetilde{\Delta}^{t'} \beta D^{t'-1} \\ &+ \sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right] \left[1 - \widehat{\phi}(0)\right]^{t-t'-1} \left[\pi(P^*) - \widetilde{\Delta}^{t} \gamma d(P^*) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \right\}. \end{aligned} \tag{28}$$

The objective is to establish that a contradiction follows from (28). The first step is to show that the summation term on the rhs is at least as great as the summation term on the lhs. As $\widetilde{\Delta}^t = \delta\widehat{\phi}(0) / \left[1 - \delta\beta\left(1 - \widehat{\phi}(0)\right)\right]$, it follows from C7 that

$$\pi(P^*) - \widetilde{\Delta}^t \gamma d(P^*) \geq \pi\left(\overline{P}^t\right) - \widetilde{\Delta}^t \gamma d\left(\overline{P}^t\right),\ t \geq t' + 1.$$

Given $\delta\widehat{\phi}(0) / (1 - \beta) \geq \widetilde{\Delta}^t$, C6 implies $\pi(P^*) - \widetilde{\Delta}^t \gamma d(P^*) > \widehat{\pi} + \delta\widehat{\phi}(0) F$ and thus $\pi(P^*) - \widetilde{\Delta}^t \gamma d(P^*) > \widehat{\pi} - (1 - \delta) F$. Finally, note that

$$\left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right]\left[1 - \widehat{\phi}(0)\right]^{t-t'-1} \geq \prod_{j=t'}^{t-1}\left[1 - \phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right],\ t \geq t' + 1.$$

It is concluded that

$$\begin{aligned} &\sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \prod_{j=t'}^{t-1}\left[1 - \phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^t\right) - \widetilde{\Delta}^t \gamma d\left(\overline{P}^t\right) - \left(\widehat{\pi} - (1 - \delta) F\right)\right] \right\} \\ \leq\ &\sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right]\left[1 - \widehat{\phi}(0)\right]^{t-t'-1}\left[\pi(P^*) - \widetilde{\Delta}^t \gamma d(P^*) - \left(\widehat{\pi} - (1 - \delta) F\right)\right] \right\}. \end{aligned}$$

Thus, (28) implies:

$$\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}^{t'} \gamma d\left(\overline{P}^{t'}\right) \geq \pi(P^*) - \widetilde{\Delta}^{t'} \gamma d(P^*). \tag{29}$$

Since $\gamma d\left(\overline{P}^{t'}\right) \geq \gamma d(P^*)$ (so that the lhs is decreasing in $\widetilde{\Delta}^{t'}$ at a faster rate than the rhs), it follows from $\widetilde{\Delta}^{t'} \geq \widetilde{\Delta}^t$ that (29) implies:

$$\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}^t \gamma d\left(\overline{P}^{t'}\right) \geq \pi(P^*) - \widetilde{\Delta}^t \gamma d(P^*).$$

Since $\widetilde{\Delta}^t = \delta\widehat{\phi}(0) / \left[1 - \delta\beta\left(1 - \widehat{\phi}(0)\right)\right]$ and $\overline{P}^{t'} > P^*$, this cannot be true by C7. This proves that the price path is bounded above by $P^*$.

- The optimal price path is non-decreasing over time.

The proof strategy involves two parts. First, suppose that price falls from $t' - 1$ to $t'$ and furthermore that price never exceeds its level prior to the decline, that is, $P^{t'-1} \geq P^t \forall t \geq t'$. It is shown that a higher payoff is realized when price is kept constant at $P^{t'-1} \forall t \geq t'$. Second, suppose that price falls from $t' - 1$ to $t'$ and remains at or below $P^{t'-1}$ over periods $t' + 1, ..., t''$. It is then shown that a higher payoff is realized by skipping the price path over periods $t' + 1, ..., t''$ and jumping to a price of $P^{t''+1}$ in period $t'$, $P^{t''+2}$ in period $t' + 1$, and so forth.

Suppose $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an optimal price path and it is not non-decreasing over time. Hence, $\exists t' > 1$ such that $P^0 < \overline{P}^1 \leq \cdots \leq \overline{P}^{t'-1} > \overline{P}^{t'}$. A necessary condition for optimality is that the payoff, starting in $t'$, from $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$ is at least as great as maintaining price at $\overline{P}^{t'-1}$ forever:

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right)-\overline{\Delta}^{t'}\gamma d\left(\overline{P}^{t'}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]-\overline{\Delta}^{t'}\beta D^{t'-1} \qquad (30)\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]\\
&+\left[\widehat{\pi}/\left(1-\delta\right)-F\right]\\
\geq\ &\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]-\widetilde{\Delta}\beta D^{t'-1}\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\left[1-\widehat{\phi}\left(0\right)\right]^{t-t'}\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]\\
&+\left[\widehat{\pi}/\left(1-\delta\right)-F\right]
\end{aligned}$$

$$\text{where } \widetilde{\Delta}\equiv\delta\sum_{\tau=t}^{T}\left(\delta\beta\right)^{\tau-t}\left[1-\widehat{\phi}\left(0\right)\right]^{\tau-t}\widehat{\phi}\left(0\right).$$

The first step is to show that if $\overline{P}^{t'-1}>\overline{P}^{t'}$ and $\overline{P}^{t'-1}\geq\overline{P}^{t}\forall t\geq t'+1$ then (30) cannot be true; maintaining price at $\overline{P}^{t'-1}$ forever is superior. Recall that price is bounded above by $P^{*}$ so that $\overline{P}^{t'-1}\leq P^{*}$. Since $\widetilde{\Delta}\leq\overline{\Delta}^{t}\forall t$ then the lhs of (30) is less than:

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]-\widetilde{\Delta}\beta D^{t'-1}\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]\\
&+\left[\widehat{\pi}/\left(1-\delta\right)-F\right].
\end{aligned}$$

Hence, a necessary condition for (30) to be true is:

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right] \qquad (31)\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]\\
\geq\ &\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\left[1-\widehat{\phi}\left(0\right)\right]^{t-t'}\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right]
\end{aligned}$$

To show that the summation term on the rhs is at least as great as that on the lhs, first note that C7 implies

$$\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\geq\pi\left(\overline{P}^{t}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right),\ t\geq t'+1.$$

as, by supposition, $\overline{P}^{t'-1} \geq \overline{P}^{t} \forall t \geq t'+1$ and it has already been proven that $P^{*} \geq \overline{P}^{t'-1}$. Next note that C6 implies

$$\begin{aligned} \pi\left(\overline{P}^{t}\right) - \widetilde{\Delta}\gamma d\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right) &> 0 \text{ and} \\ \pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta) F\right) &> 0, \end{aligned}$$

$\forall t \geq t'+1$ because $\overline{P}^{t'-1}, \overline{P}^{t} \leq P^{m}$ and $\widetilde{\Delta} \leq \delta\widehat{\phi}(0) / (1-\beta)$. Finally,

$$\left[1 - \widehat{\phi}(0)\right]^{t-t'} \geq \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right], \ t \geq t'+1.$$

It is concluded that the summation term on the rhs of (31) is at least as great as the summation term on the lhs of (31). Therefore, for (31) (and hence, (30)) to be true, it is necessary that:

$$\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}\gamma d\left(\overline{P}^{t'}\right) \geq \pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right).$$

However, by $\overline{P}^{t'} < \overline{P}^{t'-1} \leq P^{*}$, this contradicts C7. It is concluded that the price path cannot be bounded above by $\overline{P}^{t'-1}$ for $t \geq t'$.

Therefore, if $\overline{P}^{t'-1} > \overline{P}^{t'}$ then $\exists t'' \geq t'$ such that $\overline{P}^{t'-1} \geq \overline{P}^{t'+1}, \ldots, \overline{P}^{t''}$ and $\overline{P}^{t'-1} < \overline{P}^{t''+1}$. Once again compare this price path with one in which price is kept constant at $\overline{P}^{t'-1}$. By the arguments just given, one can show that the income from $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$ is strictly lower at $t'$ and is weakly lower at periods $t'+1, \ldots, t''$. Hence, a necessary condition for optimality is that the sum of the discounted terms for periods $t \geq t''+1$ is strictly higher:

$$\begin{aligned} & \sum_{t=t''+1}^{\infty} \delta^{t-t'} \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \times \\ & \left[\pi\left(\overline{P}^{t}\right) - \overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \\ > & \sum_{t=t''+1}^{\infty} \delta^{t-t'} \left[1 - \widehat{\phi}(0)\right]^{t-t'} \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \end{aligned} \tag{32}$$

or

$$\begin{aligned} & \delta^{t''-t'+1} \prod_{j=t'}^{t''} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \times \\ & \sum_{t=t''+1}^{\infty} \delta^{t-t''-1} \prod_{j=t''+1}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \\ > & \ \delta^{t''-t'+1} \left[1 - \widehat{\phi}(0)\right]^{t''-t'+1} \times \\ & \sum_{t=t''+1}^{\infty} \delta^{t-t''-1} \left[1 - \widehat{\phi}(0)\right]^{t-t''-1} \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right]. \end{aligned}$$

Since

$$\theta \equiv \delta^{t''-t'+1}\left[1-\widehat{\phi}\left(0\right)\right]^{t''-t'+1} \geq \delta^{t''-t'+1}\prod_{j=t'}^{t''}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right] \equiv \xi$$

then a necessary condition for (32) is:

$$\begin{aligned} Y &\equiv \sum_{t=t''+1}^{\infty}\delta^{t-t''-1}\prod_{j=t''+1}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right] \\ &> \sum_{t=t''+1}^{\infty}\delta^{t-t''-1}\left[1-\widehat{\phi}\left(0\right)\right]^{t-t''-1}\times\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right] \equiv X. \end{aligned}$$

From this condition it will be argued that a strictly superior price path to $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ is to set $P^{t}=\overline{P}^{t+t''-t'+1}$, $t\geq t'$. The reason is simple. It has been shown that $\left\{\overline{P}^{t}\right\}_{t=t'}^{t''}$ does worse than a constant price of $\overline{P}^{t'-1}$ over periods $t',\ldots,t''$. The optimality of $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ then requires that a strictly higher payoff be received after $t''$. Beginning from $t'$, a higher payoff to $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ can then be earned by skipping the prices over $t',\ldots,t''$ and start pricing in $t'$ according to the price path as of $t''+1$.

Define $y$ and $x$ as the payoff over $t',\ldots,t''$ from the price path $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ and a constant price of $\overline{P}^{t'-1}$, respectively,

$$y \equiv \sum_{t=t'}^{t''}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right],$$

$$x \equiv \sum_{t=t'}^{t''}\delta^{t-t'}\left[1-\widehat{\phi}\left(0\right)\right]^{t-t'}\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma d\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-\left(1-\delta\right)F\right)\right].$$

Note that $X=x/\left(1-\theta\right)$. In this notation, (30) takes the form:

$$y+\xi Y-\overline{\Delta}^{t'}\beta D^{t'-1}+\left[\widehat{\pi}/\left(1-\delta\right)-F\right] \geq x+\theta X-\widetilde{\Delta}\beta D^{t'-1}+\left[\widehat{\pi}/\left(1-\delta\right)-F\right].$$

Consider:

$$Y-\left(y+\xi Y\right)=\left(1-\xi\right)Y-y>\left(1-\xi\right)Y-x=\left(1-\xi\right)Y-\left(1-\theta\right)X>0.$$

The last inequality follows from $\theta\geq\xi$ and that it has been shown that (30) implies $Y>X$. It is then true that: $Y>y+\xi Y$. Now consider the payoff starting from $t'$ in which $P^{t}=\overline{P}^{t+t''-t'+1}$, $t\geq t'$. It will be shown that it is bounded below by $Y-\overline{\Delta}^{t'}\beta D^{t'-1}+\left[\widehat{\pi}/\left(1-\delta\right)-F\right]$. As defined, $Y$ is the payoff from $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ starting in $t''+1$ and discounting back to $t''+1$ with an initial price of $\overline{P}^{t''}$. It is also the payoff from $P^t = \overline{P}^{t+t''-t'+1}$ for $t \geq t'$, starting in $t'$ and discounting back to $t'$ but with one caveat. The preceding price to $\overline{P}^{t''+1}$ is not $\overline{P}^{t''}$ but rather $\overline{P}^{t'-1}$. Since $\overline{P}^{t''+1} > \overline{P}^{t'-1} \geq \overline{P}^{t''}$ then

$$\left(\overline{P}^{t''+1} - \overline{P}^{t''}\right) g\left(\overline{P}^{t''}\right) \geq \left(\overline{P}^{t''+1} - \overline{P}^{t'-1}\right) g\left(\overline{P}^{t'-1}\right) > 0,$$

so that, by C3, the probability of detection at $t'$ from the price path $P^t = \overline{P}^{t+t''-t'+1}$ is no greater than that at $t''+1$ from $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$.[35] Thus, the associated payoff is weakly higher than $Y - \overline{\Delta}^{t'}\beta D^{t'-1} + [\widehat{\pi}/(1-\delta) - F]$.

To summarize, it has been shown that a price path of $P^t = \overline{P}^{t+t''-t'+1}$ for $t \geq t'$ yields a payoff of at least $Y - \overline{\Delta}^{t'}\beta D^{t'-1} + [\widehat{\pi}/(1-\delta) - F]$ while $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$ yields a payoff of $y + \xi Y - \overline{\Delta}^{t'}\beta D^{t'-1} + [\widehat{\pi}/(1-\delta) - F]$. Since $Y > y + \xi Y$ then the former is larger which contradicts the optimality of $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$. This contradiction shows the falsity of the supposition that $\exists t' > 1$ such that $P^0 < P^1 \leq \cdots \leq \overline{P}^{t'-1} > \overline{P}^{t'}$. It is concluded that the price path is non-decreasing.

- The optimal price path converges to $P^*$.

A variational approach is used to characterize the limiting price. If $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an optimal price path then it is non-decreasing and is bounded above by $P^*$. Therefore, $\lim_{t\to\infty} P^t$ exists and is denoted $\overline{P}$. Consider a price path in which $P^t = \overline{P}^t$ for $t < T$ and $P^t = \overline{P}^t + \varepsilon$ for $t \geq T$. Starting with period $T$, it yields a payoff of

$$\begin{aligned}
&\pi\left(\overline{P}^T + \varepsilon\right) \\
&- \left\{\delta\phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right) + \delta\beta\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right]\overline{\Delta}^{T+1}\right\} \times \\
&\left[\gamma d\left(\overline{P}^T + \varepsilon\right) + \beta D^{T-1}\right] - [\widehat{\pi} - (1-\delta) F] \\
&+ \sum_{t=T+1}^{\infty} \delta^{t-T}\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right] \prod_{j=T+1}^{t-1}\left[1 - \phi\left(\overline{P}^j + \varepsilon, \overline{P}^{j-1} + \varepsilon\right)\right] \times \\
&\left[\pi\left(\overline{P}^t + \varepsilon\right) - \overline{\Delta}^t \gamma d\left(\overline{P}^t + \varepsilon\right) - (\widehat{\pi} - (1-\delta) F)\right] + [\widehat{\pi}/(1-\delta) - F],
\end{aligned}$$

$$\text{where } \overline{\Delta}^t \equiv \delta \sum_{\tau=t}^{\infty} (\delta\beta)^{\tau-t} \phi\left(\overline{P}^\tau + \varepsilon, \overline{P}^{\tau-1} + \varepsilon\right) \prod_{j=t}^{\tau-1}\left[1 - \phi\left(\overline{P}^j + \varepsilon, \overline{P}^{j-1} + \varepsilon\right)\right].$$

This payoff is continuous in $\varepsilon$ and equals the payoff from $\left\{\overline{P}\right\}_{t=T}^{\infty}$ when $\varepsilon = 0$. Optimality requires that if the derivative of the payoff with respect to $\varepsilon$ is defined then it equal 0 at $\varepsilon = 0$. Prior to taking the derivative, recall that

$$\phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right) = \widehat{\phi}\left(\left(\overline{P}^T + \varepsilon - \overline{P}^{T-1}\right) g\left(\overline{P}^{T-1}\right)\right),$$

[35] This is the only step in the proof that requires $g$ to be a non-increasing function.

$$\phi\left(\overline{P}^{t}+\varepsilon,\overline{P}^{t-1}+\varepsilon\right)=\widehat{\phi}\left(\left(\overline{P}^{t}-\overline{P}^{t-1}\right)g\left(\overline{P}^{t-1}+\varepsilon\right)\right),\ t>T$$

When the derivative of $\phi$ is taken, it'll be replaced with its alternative representation of $\widehat{\phi}$ for purposes of the analysis.

Taking the derivative of the payoff with respect to $\varepsilon$ :

$$\begin{aligned}
&\pi'\left(\overline{P}^{T}+\varepsilon\right)-\left(1-\beta\overline{\Delta}^{T+1}\right)\delta\widehat{\phi}'\left(\left(\overline{P}^{T}+\varepsilon-\overline{P}^{T-1}\right)g\left(\overline{P}^{T-1}\right)\right)g\left(\overline{P}^{T-1}\right)\times\\
&\left[\gamma d\left(\overline{P}^{T}+\varepsilon\right)+\beta D^{T-1}\right]\\
&-\delta\beta\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\left[\gamma d\left(\overline{P}^{T}+\varepsilon\right)+\beta D^{T-1}\right]\left(\partial\overline{\Delta}^{T+1}/\partial\varepsilon\right)\\
&-\left\{\delta\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)+\delta\beta\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\overline{\Delta}^{T+1}\right\}\gamma d'\left(\overline{P}^{T}+\varepsilon\right)\\
&-\sum_{t=T+1}^{\infty}\delta^{t-T}\widehat{\phi}'\left(\left(\overline{P}^{T}+\varepsilon-\overline{P}^{T-1}\right)g\left(\overline{P}^{T-1}\right)\right)g\left(\overline{P}^{T-1}\right)\prod_{j=T+1}^{t-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]\times\\
&\left[\pi\left(\overline{P}^{t}+\varepsilon\right)-\overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}+\varepsilon\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]\\
&-\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\times\\
&\sum_{j=T+1}^{t-1}\widehat{\phi}'\left(\left(\overline{P}^{j}-\overline{P}^{j-1}\right)g\left(\overline{P}^{j-1}+\varepsilon\right)\right)\left(\overline{P}^{j}-\overline{P}^{j-1}\right)g'\left(\overline{P}^{j-1}+\varepsilon\right)\times\\
&\prod_{\substack{k=T+1\\k\neq j}}^{t-1}\left[1-\phi\left(\overline{P}^{k}+\varepsilon,\overline{P}^{k-1}+\varepsilon\right)\right]\left[\pi\left(\overline{P}^{t}+\varepsilon\right)-\overline{\Delta}^{t}\gamma d\left(\overline{P}^{t}+\varepsilon\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]\\
&-\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\prod_{j=T+1}^{t-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]\gamma d\left(\overline{P}^{t}+\varepsilon\right)\left(\partial\overline{\Delta}^{t}/\partial\varepsilon\right)\\
&+\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\prod_{j=T+1}^{t-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]\times\\
&\left[\pi'\left(\overline{P}^{t}+\varepsilon\right)-\overline{\Delta}^{t}\gamma d'\left(\overline{P}^{t}+\varepsilon\right)\right].
\end{aligned}$$

$$\begin{aligned}
\text{where }\frac{\partial\overline{\Delta}^{t}}{\partial\varepsilon}\ &=\ \delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\,\widehat{\phi}'\left(\left(\overline{P}^{\tau}-\overline{P}^{\tau-1}\right)g\left(\overline{P}^{\tau-1}+\varepsilon\right)\right)\left(\overline{P}^{\tau}-\overline{P}^{\tau-1}\right)g'\left(\overline{P}^{\tau-1}+\varepsilon\right)\times\\
&\prod_{j=t}^{\tau-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]-\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\,\phi\left(\overline{P}^{\tau}+\varepsilon,\overline{P}^{\tau-1}+\varepsilon\right)\times\\
&\sum_{j=t}^{\tau-1}\widehat{\phi}'\left(\left(\overline{P}^{j}-\overline{P}^{j-1}\right)g\left(\overline{P}^{j-1}+\varepsilon\right)\right)\left(\overline{P}^{\tau}-\overline{P}^{\tau-1}\right)g'\left(\overline{P}^{j-1}+\varepsilon\right)\times
\end{aligned}$$

$$\prod_{\substack{k=t \\ k\neq j}}^{\tau-1} \left[1-\phi\left(\overline{P}^{k}+\varepsilon,\overline{P}^{k-1}+\varepsilon\right)\right].$$

Optimality requires that this derivative (if defined) equal zero at $\varepsilon = 0$, $\forall T$. Consider this derivative, evaluated at $\varepsilon = 0$, as $T \to \infty$. Since $\lim_{T\to\infty} \overline{P}^{T} = \overline{P}$ then

$$\begin{aligned}
\lim_{T\to\infty} \widehat{\phi}'\left(\left(\overline{P}^{T}-\overline{P}^{T-1}\right)g\left(\overline{P}^{T-1}\right)\right)g\left(\overline{P}^{T-1}\right) &= \widehat{\phi}'(0)=0 \\
\lim_{T\to\infty} \widehat{\phi}'\left(\left(\overline{P}^{t}-\overline{P}^{t-1}\right)g\left(\overline{P}^{t-1}\right)\right)g'\left(\overline{P}^{t-1}\right) &= \widehat{\phi}'(0)=0,\ t>T.
\end{aligned}$$

Since $\widehat{\phi}'(0)$ is defined by C5, then the above derivative of the payoff function is defined (also using A2 and A3). Thus, as $T \to \infty$, all of the expressions with $\widehat{\phi}'$ equal zero as do $\partial\overline{\Delta}^{T}/\partial\varepsilon$ and $\partial\overline{\Delta}^{t}/\partial\varepsilon$. This leaves:

$$\begin{aligned}
&\pi'\left(\overline{P}\right)-\left\{\delta\widehat{\phi}(0)+\delta\beta\left[1-\widehat{\phi}(0)\right]\overline{\Delta}\right\}\gamma d'\left(\overline{P}\right) \\
&+\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\widehat{\phi}(0)\right]\prod_{j=T+1}^{t-1}\left[1-\widehat{\phi}(0)\right]\left[\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma d'\left(\overline{P}\right)\right] \\
=\ &\sum_{t=T}^{\infty}\delta^{t-T}\left[1-\widehat{\phi}(0)\right]^{t-T}\left[\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma d'\left(\overline{P}\right)\right] \\
=\ &\frac{\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma d'\left(\overline{P}\right)}{1-\delta\left(1-\widehat{\phi}(0)\right)},
\end{aligned}$$

$$\text{where } \overline{\Delta}\equiv\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\widehat{\phi}(0)\left[1-\widehat{\phi}(0)\right]^{\tau-t}=\frac{\delta\widehat{\phi}(0)}{1-\delta\beta\left(1-\widehat{\phi}(0)\right)}.$$

Optimality then requires that $\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma d'\left(\overline{P}\right)=0$ which, by C7, implies $\overline{P}=P^{*}$. This completes the proof of Theorem 4. ■

## Appendix B

Key to our analysis is a useful representation of a firm's payoff. To save on notation, let $\phi^{t}\equiv\phi\left(P^{t},P^{t-1}\right)$ denote the probability of detection in period $t$, as of the start of period $t$. Suppose collusion is infinitely-lived (subject to detection interrupting it) and the collusive price path is $\left\{P^{t}\right\}_{t=1}^{\infty}$. The payoff as of period $t$ can then be represented as:

$$\begin{aligned}
&\pi\left(P^{t}\right)+\delta\phi^{t}\left[\left(\widehat{\pi}/(1-\delta)\right)-\beta D^{t-1}-\gamma d\left(P^{t}\right)-F\right]+\delta\left(1-\phi^{t}\right)\pi\left(P^{t+1}\right) \\
&+\delta^{2}\left(1-\phi^{t}\right)\phi^{t+1}\left[\left(\widehat{\pi}/(1-\delta)\right)-\beta^{2}D^{t-1}-\beta\gamma d\left(P^{t}\right)-\gamma d\left(P^{t+1}\right)-F\right] \\
&+\delta^{2}\left(1-\phi^{t}\right)\left(1-\phi^{t+1}\right)\pi\left(P^{t+2}\right) \\
&+\delta^{3}\left(1-\phi^{t}\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\times \\
&\left[\left(\widehat{\pi}/(1-\delta)\right)-\beta^{3}D^{t-1}-\beta^{2}\gamma d\left(P^{t}\right)-\beta\gamma d\left(P^{t+1}\right)-\gamma d\left(P^{t+2}\right)-F\right]+\cdots
\end{aligned}$$

A firm earns $\pi\left(P^t\right)$ in the current period. With probability $\phi^t$, detection occurs which results in $\widehat{\pi}$ in all future periods and a penalty of $\beta D^{t-1}+\gamma d\left(P^t\right)+F$. With probability $1-\phi^t$, detection does not occur so $\pi\left(P^{t+1}\right)$ is earned in period $t+1$ and so forth. This expression can be re-arranged to:

$$
\begin{aligned}
&\{\pi\left(P^t\right)-\gamma d\left(P^t\right)\delta\left[\phi^t+\delta\beta\left(1-\phi^t\right)\phi^{t+1}+(\delta\beta)^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}+\cdots\right]\\
&+\delta\left(1-\phi^t\right)\pi\left(P^{t+1}\right)\\
&-\delta\left(1-\phi^t\right)\gamma d\left(P^{t+1}\right)\delta\left[\phi^{t+1}+\delta\beta\left(1-\phi^{t+1}\right)\phi^{t+2}+(\delta\beta)^2\left(1-\phi^{t+1}\right)\left(1-\phi^{t+2}\right)\phi^{t+3}+\cdots\right]\\
&+\cdots\}+\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\left[\delta\phi^t+\delta^2\left(1-\phi^t\right)\phi^{t+1}+\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}+\cdots\right]\\
&-\beta D^{t-1}\delta\left[\phi^t+\delta\beta\left(1-\phi^t\right)\phi^{t+1}+(\delta\beta)^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}+\cdots\right].
\end{aligned}
$$

Let

$$
\Delta^t\equiv\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\phi^\tau\prod_{j=t}^{\tau-1}\left[1-\phi^j\right],
$$

where the convention is adopted that $\prod_{j=t}^{t-1}\left[1-\phi^j\right]=1$. The above expression is then:

$$
\begin{aligned}
&\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)\left[\pi\left(P^\tau\right)-\gamma d\left(P^\tau\right)\Delta^\tau\right] \\
&+\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^\tau\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)-\beta D^{t-1}\Delta^t.
\end{aligned}
\tag{33}
$$

The collusive payoff is represented as the stream of profit net of the expected present value of damages, $\pi\left(P^\tau\right)-\gamma d\left(P^\tau\right)\Delta^\tau$, less the expected present value of the fine, $\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^\tau\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)F$, less the expected present value of inherited damages, $\beta D^{t-1}\Delta^t$, plus the value from not colluding, $\left(\widehat{\pi}/\left(1-\delta\right)\right)\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^\tau\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)$.

Let us manipulate the term $\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^\tau\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)$:

$$
\begin{aligned}
&\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^\tau\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)\\
=\;&\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\left\{\delta\phi^t+\delta^2\left(1-\phi^t\right)\phi^{t+1}+\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}+\cdots\right\}\\
=\;&\left[\widehat{\pi}-\left(1-\delta\right)F\right]\left\{\delta\phi^t\sum_{\tau=0}^{\infty}\delta^\tau+\delta^2\left(1-\phi^t\right)\phi^{t+1}\sum_{\tau=0}^{\infty}\delta^\tau\right.\\
&\left.+\,\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\sum_{\tau=0}^{\infty}\delta^\tau+\cdots\right\}\\
=\;&\left[\widehat{\pi}-\left(1-\delta\right)F\right]\left\{\delta\phi^t+\delta^2\left[\phi^t+\left(1-\phi^t\right)\phi^{t+1}\right]+\right.\\
&\left.\delta^3\left[\phi^t+\left(1-\phi^t\right)\phi^{t+1}+\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\right]+\cdots\right\}\\
=\;&\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]-\left[\widehat{\pi}-\left(1-\delta\right)F\right]\left[\left(1+\delta+\delta^2+\cdots\right)-\delta\phi^t\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\delta^2 \left(\phi^t + \left(1-\phi^t\right)\phi^{t+1}\right) \\
& -\delta^3 \left(\phi^t + \left(1-\phi^t\right)\phi^{t+1} + \left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\right) - \cdots] \\
= \; & [(\widehat{\pi}/(1-\delta)) - F] - [\widehat{\pi} - (1-\delta)F] \times \\
& \{1 + \delta\left(1-\phi^t\right) + \delta^2\left[1-\phi^t - \left(1-\phi^t\right)\phi^{t+1}\right] \\
& +\delta^3\left[1-\phi^t - \left(1-\phi^t\right)\phi^{t+1} - \left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\right] + \cdots\} \\
= \; & [(\widehat{\pi}/(1-\delta)) - F] - [\widehat{\pi} - (1-\delta)F] \times \\
& \{1 + \delta\left(1-\phi^t\right) + \delta^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right) \\
& +\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\left(1-\phi^{t+2}\right) + \cdots\} \\
= \; & [(\widehat{\pi}/(1-\delta)) - F] - [\widehat{\pi} - (1-\delta)F] \sum_{\tau=t}^{\infty} \delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right).
\end{aligned}
$$

Substituting this expression into (33):

$$
\begin{aligned}
& \sum_{\tau=t}^{\infty} \delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)\{[\pi(P^\tau) - \gamma d(P^\tau)\Delta^\tau] - [\widehat{\pi} - (1-\delta)F]\} \\
& -\beta D^{t-1}\Delta^t + [(\widehat{\pi}/(1-\delta)) - F].
\end{aligned}
\tag{34}
$$

## References


[1] Athey, Susan, Kyle Bagwell, and Chris Sanchirico, "Collusion and Price Rigidity," MIT, Working Paper No. 98-23, November 1998.

[2] Athey, Susan and Kyle Bagwell, "Optimal Collusion with Private Information," *RAND Journal of Economics*, 32 (2001), 428-465.

[3] Baker, Jonathan B., "Private Information and the Deterrent Effect of Antitrust Damage Remedies," *Journal of Law, Economics, and Organization*, 4 (1988), 385-408.

[4] Besanko, David and Daniel F. Spulber, "Antitrust Enforcement under Asymmetric Information," *Economic Journal*, 99 (1989), 408-425.

[5] Besanko, David and Daniel F. Spulber, "Are Treble Damages Neutral? Sequential Equilibrium and Private Antitrust Enforcement," *American Economic Review*, 80 (1990), 870-887.

[6] Blackstone, Erwin A. and Gary W. Bowman, "Antitrust Damages: The Loss from Delay," *Antitrust Bulletin*, Spring 1987, pp. 93-100.

[7] Block, Michael K., Frederick C. Nold, and Joseph G. Sidak, "The Deterrent Effect of Antitrust Enforcement," *Journal of Political Economy*, 89 (1981), 429-445.

[8] Bosch, Jean-Claude and E. Woodrow Eckard, Jr., "The Profitability of Price Fixing: Evidence from Stock Market Reaction to Federal Indictments," *Review of Economics and Statistics*, 73 (1991), 309-317.

[9] Bryant, Peter G. and E. Woodrow Eckard, "Price Fixing: the Probability of Getting Caught," *Review of Economics and Statistics*, 73 (1991), 531-536.

[10] Christie, William G. and Paul H. Schultz, "Why Do Nasdaq Market Makers Avoid Odd-Eighth Quotes?," *Journal of Finance*, 49 (1994), 1813-1840.

[11] Connor, John M., "What Can We Learn from the ADM Global Price Conspiracies?." Staff Paper #98-14, Department of Agricultural Economics, Purdue University, August 1998.

[12] Connor, John M., "'Our Customers Are Our Enemies': The Lysine Cartel of 1992-1995," *Review of Industrial Organization*, 18 (2001), 5-21.

[13] Cyrenne, Philippe, "On Antitrust Enforcement and the Deterrence of Collusive Behavior," *Review of Industrial Organization*, 14 (1999), 257-272.

[14] de Roos, Nicolas, "Examining Models of Collusion: The Market for Lysine," Yale University, photocopy, October 1999.

[15] Green, Edward and Robert H. Porter, "Noncooperative Collusion under Imperfect Price Information," *Econometrica*, 52 (1984), 87-100.

[16] Harrington, Joseph E. Jr., "Optimal Cartel Pricing in the Presence of an Antitrust Authority," Johns Hopkins University Working Paper #460, November 2001.

[17] Hay, George and Daniel Kelly, "An Empirical Survey of Price Fixing Conspiracies," *Journal of Law and Economics*, 17 (1974), 13-38.

[18] LaCasse, Chantale, "Bid Rigging and the Threat of Government Prosecution," *RAND Journal of Economics*, 26 (1995), 398-417.

[19] Levenstein, Margaret and Valerie Suslow, "Private International Cartesl and Their Effect on Developing Countries," University of Massachusetts, pdf copy, 2001.

[20] McAnney, Joseph W., "The Justice Department's Crusade Against Price-Fixing: Initiative or Reaction?," *Antitrust Bulletin*, Fall 1991, 521-542.

[21] McCutcheon, Barbara, "Do Meetings in Smoke-Filled Rooms Facilitate Collusion?," *Journal of Political Economy*, 105 (1997), 330-350.

[22] Motta, Massimo and Michele Polo, "Leniency Programs and Cartel Prosecution," European University Institute, pdf copy, February 2001.

[23] Polo, Michele, "The Optimal Prudential Deterrence of Price Fixing Agreements," Bocconi University, pdf copy, September 1997.

[24] Posner, Richard A., *Antitrust Law: An Economic Perspective,* Chicago: The University of Chicago Press, 1976.

[25] Saïd, Souam, "Optimal Antitrust Policy under Different Regimes of Fines," *International Journal of Industrial Organization*, 19 (2001), 1-26.

[26] Salant, Stephen W., "Treble Damage Awards in Private Lawsuits for Price Fixing," *Journal of Political Economy*, 95 (1987), 1326-1336.

[27] Spiller, Pablo T., "Treble Damages and Optimal Suing Time," *Research in Law and Economics*, 9 (1986), 45-56.

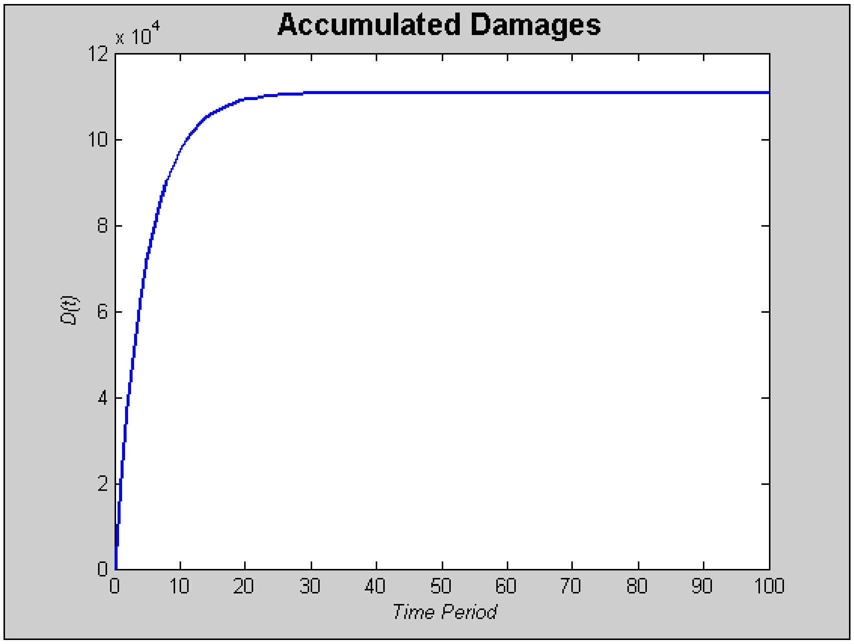
[28] Stokey, Nancy L. and Robert E. Lucas, Jr. with Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Cambridge, Mass.: Harvard University Press, 1989.

[29] Topkis, Donald M., "Minimizing a Submodular Function on a Lattice," *Operations Research*, 26 (1978) 305-321, 1978

[30] Vives, Xavier, *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge, Mass.: MIT Press, 1999.

[31] White, Lawrence J., "Lysine and Price Fixing: How Long? How Severe?," *Review of Industrial Organization*, 18 (2001), 23-31.

**Figure 1**


Accumulated Damages
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12
10
8
6
4
2
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D(t)
0
10
20
30
40
50
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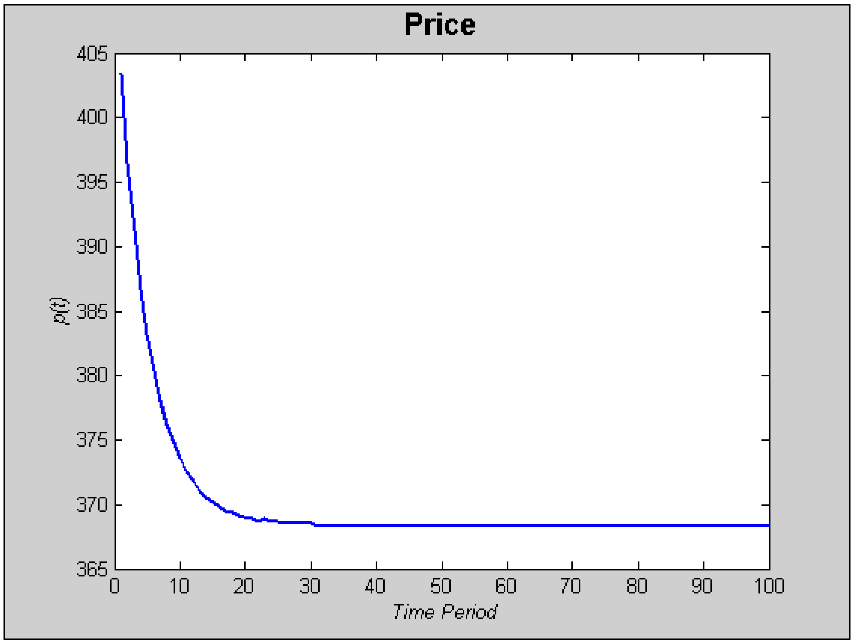
70
80
90
100
Time Period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395
390
385
380
375
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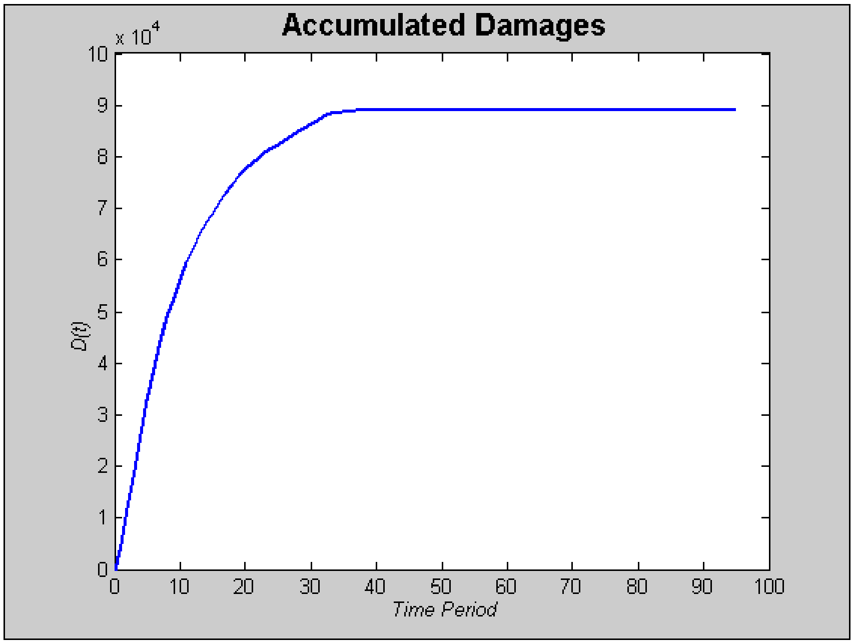
p(t)
0
10
20
30
40
50
60
70
80
90
100
Time Period

# Figure 2



Accumulated Damages
x 10^4
D(t)
Time Period
0
10
20
30
40
50
60
70
80
90
100
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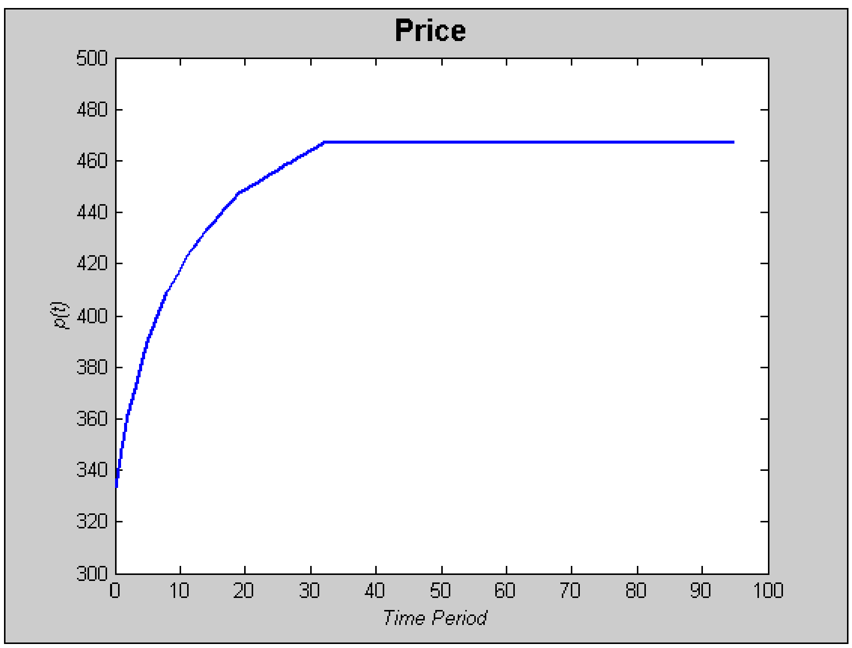
3
4
5
6
7
8
9

Price
p(t)
Time Period
300
320
340
360
380
400
420
440
460
480
500
0
10
20
30
40
50
60
70
80
90
100