

Inflation, Exchange Rates and PPP in a Multivariate Panel Cointegration Model*

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Abstract

New multivariate panel cointegration methods are used to analyze nominal exchange rates and prices in four major economies in Europe; France, Germany, Italy and the United Kingdom for the post-Bretton Woods period. We test for purchasing power parity between these four countries and find that the theoretical PPP relationship does not hold. However, the estimated unrestricted relationship is found to be remarkably close to the theoretical one(1,-1.5,0.9 instead of 1,-1,1). Relevant asymptotic results are stated, proved, and evaluated using Monte Carlo simulations. The asymptotic results are general and may hence be used in similar empirical contexts using the same structure. Parametric bootstrap inference is used in order to deal with test size distortions.

Key words: Panel data, long-run purchasing power parity, multivariate cointegration analysis, bootstrap inference.

JEL Classification: F30, C15, C32.

1 Introduction

Does purchasing power parity hold in the long run? Are real exchange rates mean-reverting? A reading of the voluminous literature on this matter appears to give the following conclusions. If one applies unit root tests to real exchange rate data spanning long periods of time (say, close to a century or more) then evidence of long-run PPP is most often found (see e.g. Frankel 1986, Abuaf and Jorion 1990 and Lothian and Taylor 1996). However, when examining the recent post-Bretton Woods period of floating exchange rates the answer is less

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clear-cut. Conventional unit-root tests do not find evidence of PPP, while other approaches, e.g. using panel data, have provided evidence in favor of PPP.¹

In this paper we re-examine the issue of PPP for several reasons. First, many earlier studies of PPP have typically analyzed the real exchange rate using various univariate techniques.² In contrast, we cast the analysis in terms of multivariate panel cointegration. The advantage of such a framework, as we see it, is that the information set can be considerably enlarged. Moreover, our approach captures the interdependence of foreign exchange markets. A second reason is more practical, yet very important. It concerns size distortion, i.e., the erroneous rejection of a true null hypothesis due to an inappropriate asymptotic approximation. There are two sources for this. Firstly, Engel (1999) argues that the unit-root tests referred to above may have serious size biases due to the fact that any stationary process can be made arbitrarily close to a nonstationary process. Secondly, as shown in Lyhagen (2000), using panel unit root test in the context of PPP gives invalid inference, i.e., the size of the test tends to one when the number of countries increases. This is due to that a commonly shared common trend is not taken account of when calculating the critical values. Both these effects lead to the false conclusion of a stationary real exchange rate. In this paper, asymptotic tests are augmented with parametric bootstrap analogues, whereby we reduce the effect, if not eliminate, the size distortion typically present in small-sample applications of asymptotic tests. As we bootstrap the multivariate model the problem of commonly shared common trend is also solved.

We examine monthly data for the post-Bretton Woods years 1974-1999 for France, Germany, Italy and the United Kingdom, and the results of our analysis are the following. We do find evidence of cointegration between nominal exchange rates and prices; in fact the number of cointegrating vectors is exactly what PPP predicts. But the coefficients in the cointegrating vectors are not compatible with PPP, although we find that the cointegrating vector in all panels is the same. Hence, we reject PPP.³ We discuss this result in the concluding section, Section 6. Prior to that, Section 2 explains the implications of PPP in terms of cointegration and the asymptotic results for the proposed tests are presented in Section 3. Section 4 contains the cointegration analysis and Section 5 investigates the small sample properties of the test statistics derived in Section 3 by use of Monte Carlo simulation. Proofs of the theorems are spelled out in the Appendix.

¹This interpretation of the post-Bretton Woods period is not self-evident. Cheung and Lai (1998), using more efficient unit-root tests, report evidence in favor of PPP. On the other hand, O'Connell (1998) provides a critical assessment of the evidence from panel studies.

²In addition to the references cited in the text, influential papers include Diebold, Husted and Rush (1991), Glen (1992) and Edison (1987).

³Some would actually interpret our results as evidence of 'weak form' PPP; see e.g. MacDonald (1993). We prefer to associate PPP with the stricter requirement that the cointegrating relations satisfy certain linear restrictions. This is explained more in Section 2.

Multivariate framework

In contrast with most earlier studies of purchasing power parity, we cast the analysis in terms of multivariate panel cointegration. The multivariate nature of the framework offers two advantages. *First*, we are able to test for (bilateral) PPP between all countries in one system, meaning that the interdependent nature of the foreign exchange markets is taken into explicit account. Ideally, such an analysis should include prices and exchange rates of all large economies in order to fully account for the simultaneity. But doing so one would of course run into problems with degrees of freedom. Hence we have restricted the number of countries in the analysis to the four mentioned above, concentrating on what we believe to be major economies/currencies in Europe of the twentieth century. Furthermore, in this multivariate setup we will test not only individual bilateral PPP relations, but also whether all bilateral PPP relations hold simultaneously. *Second*, nominal exchange rates and prices enter separately into the analysis. Hence no a priori restrictions are imposed on the joint behavior of prices and exchange rates (i.e. the so-called symmetry and proportionality conditions are not imposed, but instead subsequently tested for).⁴

Size and power issues in tests of long-run PPP

In the empirical PPP literature there has been much concern with issues of statistical power of the tests used when examining whether real exchange rates are mean-reverting (see e.g. Cheung and Lai 1998). On the other tack, Engel (1999) has shown that these tests may in fact have serious size biases when applied to random variables that contain a stationary but persistent component and a non-stationary component. On the panel unit root front Lyhagen (2000) has shown that the usually used critical values are wrong as they do not properly take care of the common common trend implied by PPP.

There is reason to believe that the usefulness of multivariate maximum likelihood cointegration analysis can be severely hampered by the curse of dimensionality, i.e., a large number of parameters in relation to a small number of observations. One undesirable effect is that the use of asymptotic critical values may jeopardize the validity of inference. This has been empirically verified in Jacobson, Vredin and Warne (1998). Gredenhoff and Jacobson (2000) have confirmed the presence and examined the nature of size distortion for likelihood ratio tests of linear restrictions on cointegrating vectors. However, they also found that parametric bootstrap testing is a robust alternative to asymptotic approximations, eliminating size distortions even for quite large systems and as few observations as 60. In this paper, all asymptotic tests (not only those

⁴Earlier studies using the multivariate cointegration setup to analyze long-run PPP — Cheung and Lai (1993), Kugler and Lenz (1993), Johansen and Juselius (1992), MacDonald (1993) and Edison, Gagnon and Melick (1997) — have used data from the post-Bretton Woods period. Furthermore, these studies have examined PPP in series of trivariate systems (an exception is Nessén 1996). The typical result in these studies (and Nessén 1996) is that evidence of cointegration is found, but that the cointegrating relations fail comply with the restrictions implied by PPP.

of linear restrictions on cointegrating vectors) are augmented by parametric bootstrap analogues.⁵

2 PPP and linear restrictions on prices and exchange rates

We examine long-run PPP between four large European economies in a multivariate panel setting. The purpose of this section is to show how such a system is set up and to identify the restrictions implied by long-run PPP.

Denote the natural logarithm of the nominal British pound exchange rate of country i (that is, the number of currency i per unit British pound) by e_t^i . Further, let p_t^i be the natural logarithm of the price level in country i . Further, let p_t^* denote the price level in our numeraire country, the United Kingdom. Define

$$X_{it} = \begin{bmatrix} e_t^i \\ p_t^i \end{bmatrix}$$

and then

$$X_t = \begin{bmatrix} e_t^1 \\ p_t^1 \\ \vdots \\ e_t^N \\ p_t^N \\ p_t^* \end{bmatrix}$$

where N is the number of countries except the base country, in our case three.

Now, if long-run bilateral PPP holds then the real exchange rates between all pairs of countries are stationary, or integrated of order 0, $I(0)$. This may be expressed as

$$q_t^i \equiv e_t^i - p_t^i + p_t^* \sim I(0) \quad i = 1, \dots, N$$

where q_t^i is the real exchange rate between country i and Great Britain. These N equations can be summarized as:

$$\begin{bmatrix} q_t^1 \\ q_t^2 \\ \vdots \\ q_t^N \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e_t^1 \\ p_t^1 \\ \vdots \\ e_t^N \\ p_t^N \\ p_t^* \end{bmatrix} \sim I(0) \quad (1)$$

⁵Edison et al. (1997) are also concerned about inappropriate use of asymptotic approximations in the context of multivariate maximum likelihood cointegration analysis of PPP. Analyzing post-Bretton Woods data they find only weak support for PPP, despite the use of small-sample critical values in the hypothesis testing.

It is easily recognized that the choice of base country is arbitrary. Pre-multiply the relationship with the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 1 & & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 1 \end{bmatrix}$$

where the column of -1 is in the position of the new base country, gives the desired result. Note that the eigenvalues are $N - 1$ ones and the last is minus one so the new relationships span the same space as the original one.

The equations in (1) can be evaluated in a vector error correction model on the form

$$\Delta X_t = \alpha \beta' X_{t-1} + \sum_{j=1}^{m-1} \Gamma_j \Delta X_{t-j} + \mu + \varepsilon_t, \quad (2)$$

where $\alpha'_{\perp} \mu \neq 0$, with α_{\perp} such that $\alpha'_{\perp} \alpha = 0$ and (α, α_{\perp}) has full rank. (This means that μ is not restricted to the cointegration space.) Moreover, α and β are $N_p \times Nr$, where $N_p \equiv Np + 1$ and β is given by

$$\beta = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{bmatrix}, \quad (3)$$

where for $i = 1, \dots, N$, the β_i are $p \times r$ and the $\beta_{N+1,i}$ are $1 \times r$. No restrictions are imposed on the α , Γ_j ($N_p \times N_p$) and Σ ($N_p \times N_p$) matrices, the latter being the covariance matrix of ε_t ($N_p \times 1$). Assume that observations are taken at $t = 1, \dots, T$. Note that this model is similar to the one in Larsson and Lyhagen (1999) with the addition of the last row in the β matrix, and the estimation procedures follows those outlined there.

If PPP holds, we have $p = 2$, $r = 1$ and $[\beta'_i, \beta'_{N+1,i}] = [1, -1, 1]$. This is a restriction in model (2) that we will test in three steps. First, we will estimate r , using the sequential testing Johansen (1995) procedure, i.e., first test if $r = 0$ against $r = p$, then if the null hypothesis is rejected test if $r = 1$, and so on, until the null hypothesis cannot be rejected. This null hypothesis then gives us the estimated r . If the estimated r turns out to be 1, we will go on and test if all $[\beta'_i, \beta'_{N+1,i}]'$ span the same (cointegration) space. Finally, if the hypothesis of a common cointegration space is not rejected, we will test that all $[\beta'_i, \beta'_{N+1,i}] = c_i [1, -1, 1]$, where the c_i are constants.

The asymptotics of these tests is considered in the next section. The asymptotic results are general, and can be used in other contexts with the same model structure.

3 Asymptotic results

The hypotheses to be discussed in this section are

\mathcal{H}_4 : $\text{rank}(\Pi) = Np + 1$,

\mathcal{H}_3 : $\Pi = \alpha\beta'$ where α and β are $(Np + 1) \times Nr$ as above, but with no restrictions on β ,

\mathcal{H}_2 : as \mathcal{H}_3 but where β is as in (3),

\mathcal{H}_1 : as \mathcal{H}_2 but where all $(\beta'_i, \beta'_{N+1,i})'$ span the same space,

\mathcal{H}_0 : as \mathcal{H}_1 but where all $(\beta'_i, \beta'_{N+1,i})' = H_i\psi_i$, where the H_i are known $(p + 1) \times s$ matrices and the ψ_i are $s \times r$, unknown and span the same space. Obviously, $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \mathcal{H}_4$. For $i < j$, we will denote the maximum likelihood ratio between \mathcal{H}_i and \mathcal{H}_j by Q_{ij} . The theorems will give the asymptotic distributions of, in turn, $-2\log Q_{24}$, $-2\log Q_{12}$ and $-2\log Q_{01}$.

We will need the following assumption, which is typical for this kind of theory. (The matrix β_\perp is defined in a similar fashion as α_\perp above.) This assumption guarantees that X_t is an $I(1)$ process (cf Johansen, 1995, p.49).

Assumption A The roots of the characteristic equation corresponding to (2) have modulus > 1 or are equal to 1, and $\alpha'_\perp \Gamma \beta_\perp$ has full rank, where $\Gamma = I_{Np} - \sum_{i=1}^{m-1} \Gamma_i$.

We now turn to the asymptotics of the test for cointegrating rank, which is used for the sequential rank estimation procedure. This is the test of \mathcal{H}_2 against \mathcal{H}_4 . The main idea is to write $Q_{24} = Q_{23}Q_{34}$, implying

$$-2\log Q_{24} = -2\log Q_{23} - 2\log Q_{34}.$$

The result is that, as $T \rightarrow \infty$, $-2\log Q_{23}$ converges weakly to the χ^2 variate V , while $-2\log Q_{34}$ tends to U which has a Dickey-Fuller type distribution as given in the formulation of the theorem. Furthermore, $-2\log Q_{23}$ and $-2\log Q_{34}$ are asymptotically independent. Observe that if $r = 0$, the χ^2 variate disappears, and we have the usual Johansen trace test. Moreover, observe the short-hand notation of integrals, i.e. $\int FF' = \int_0^1 F(t)F'(t)'dt$, etcetera.

Theorem 1 Under \mathcal{H}_2 , assumption A and if $\alpha'_\perp \mu \neq 0$ and $r > 0$, we have that as $T \rightarrow \infty$, the maximum likelihood ratio test of cointegrating rank r , Q_{24} , satisfies

$$-2\log Q_{24} \xrightarrow{w} U + V,$$

where, defining $B(t)$ to be an $\{N(p - r) + 1\}$ -dimensional standard Wiener process (with mean zero and identity covariance matrix),

$$U = \text{tr} \left\{ \int dB F' \left(\int FF' \right)^{-1} \int F dB' \right\},$$

and where V is χ^2 with $N(N - 1)(p - r)r$ degrees of freedom, independent of U . The process F is $\{N(p - r) + 1\}$ -dimensional with components

$$F_i(u) \equiv \begin{cases} B_i(u) - \int_0^1 B_i(t) dt, & i = 1, \dots, N(p - r), \\ u - \frac{1}{2}, & i = N(p - r) + 1, \end{cases}$$

where the $B_i(t)$ are components of $B(t)$.

Proof. See the appendix. ■

The next theorem gives the asymptotic distribution of Q_{12} .

Theorem 2 *Under \mathcal{H}_1 , assumption A and if $\alpha'_{\perp}\mu \neq 0$ and $r > 0$, we have that as $T \rightarrow \infty$, the maximum likelihood ratio test of common cointegrating space, Q_{12} , fulfills that $-2\log Q_{12}$ is asymptotically χ^2 with $(N - 1)(p + 1 - r)$ degrees of freedom.*

Proof. See the appendix. ■

In particular, in the PPP case, $-2\log Q_{12}$ is asymptotically χ^2 with $2(N - 1)$ degrees of freedom.

Our final object is to test if, given cointegrating rank $r = 1$ and $p = 2$, the cointegrating relation is $(\beta'_i, \beta'_{N+1,i}) = c_i(1, -1, 1)$ for all i and constants c_i . This is a special case of \mathcal{H}_0 with $r = s = 1$, all $H_i = (1, -1, 1)'$ and all $\psi_i = c_i$.

Theorem 3 *Under \mathcal{H}_0 , assumption A and if $\alpha'_{\perp}\mu \neq 0$ and $r > 0$, we have that as $T \rightarrow \infty$, the maximum likelihood ratio test of the restriction $(\beta'_i, \beta'_{N+1,i})' = H_i\psi_i$ for all i , where the H_i are known $(p + 1) \times s$ matrices and the ψ_i are $s \times r$ and unknown, against the hypothesis of common cointegrating space, Q_{01} , fulfills that $-2\log Q_{01}$ is asymptotically χ^2 with $(p + 1 - s)r$ degrees of freedom.*

Proof. See the appendix. ■

Note that in the PPP case, $-2\log Q_{01}$ is asymptotically $\chi^2(2)$.

4 The empirical analysis

Our database contains monthly observations of wholesale prices and nominal exchange rates (vs the British pound) for Germany, France, Italy and Great Britain for the years 1974 - 2000, i.e. $N = 3$ and $T = 314$. The Appendix contains a fuller description of the data and sources, and also graphs, figures (1)-(3), of the exchange rate, real exchange rate and wholesale price series.

In the subsequent sections we use the framework outlined in previous sections in the following way: First we estimate the number of cointegrating relations in a VECM of our seven-variable data set that satisfies standard specification tests. Second, we test hypotheses about the cointegration vectors. First we test if the cointegrating vectors span the same space and then if the theoretical relationship is within this space.

4.1 Specification and mis-specification analysis

The number of lags is specified using the information criterion proposed by Schwarz (1978) where a upper limit of five lags are pre-specified. The results suggest that $k = 2$ would be appropriate. (The LM test suggests that there are no autocorrelations in the residuals.) Given a lag of 2 the likelihood ratio test of

the three null $r = 0, 1, 2$ is calculated with the alternative of full rank. Instead of using the asymptotic distribution we use the method discussed above, i.e. a parametric bootstrap as it was used in Gredenhoff and Jacobson (1998). Note that data is generated under the null and with lags, so the parameter uncertainty is dealt with as well. A nominal size of 5% is used and the number of bootstrap replicates is 1000. The test statistics with the corresponding critical values are given in Table (1). The null of $r = 0$ is rejected while the null of $r = 1$ is not, hence, we conclude that one cointegrating relationship per country is sufficient. The normalized cointegrating vectors are displayed in Table (2).

Table (1) in here

Table (2) in here

4.2 Testing linear restrictions

Having found support for the necessary condition for PPP, we now turn to the sufficient conditions. The multivariate setup used in this paper actually enables us to test for PPP in different ways. First, we test whether *all three* bilateral PPP relations span the same space, i.e. the four countries share the same economic laws but not necessarily the one outlined above. The test statistic is 17.4 with a bootstrapped critical value of 21.6 at a 5% nominal size, hence, we do not reject the null of a common cointegrating space. The normalized (with respect to β_{ix}) common cointegrating vector is $[1.00, -1.52, 0.885]'$ and has the correct signs and does not seem to be far from the relationship implied by PPP. To test if PPP holds the likelihood ratio test with $[1, -1, 1]'$ as null is tested against common cointegrating space. The test statistic is 60.8 with a bootstrapped critical value of 12.0, hence we reject the null.

In summary, we have found support for our hypothesis that the variables in x_t can be characterized by an error correction model like equation (2). This implies that they are driven by a limited number of common stochastic trends and therefore are tied together in the long run. There are three long-run, cointegrating, relations. However, none of these long-run relations can be interpreted in terms of PPP although they span the same space.

5 Small sample properties

Although we have used bootstrap based inference in the empirical sections above it is of interest to show how well the asymptotic distributions work in small samples. To analyze this a Monte Carlo simulation is performed. The data generating process (DGP) is the empirical model estimated in the previous section. We are interested in five different null hypotheses. The first three consider the rank: $r = 0, 1$ or 2 , and the remaining two are tests on the cointegrating space: test of common space and test that the cointegrating vector is the theoretical PPP relationship, $[1, -1, 1]'$. The alternative for the first

three models are the usual full rank model and for the last two an unrestricted cointegrating model with rank one. For the very last model the alternative of a common cointegrating space is also considered. The largest eigenvalues of the DGP's are displayed in Table (3). (See further the discussion below.)

Table (3) in here

The Monte Carlo setup is as follows. First we generate data according to the model under the null, then we estimate the models under the null and the alternative and calculate the likelihood ratio statistic. We then compare with the asymptotic critical value and note if the test rejects or does not reject the null. This is repeated 1,000 times and the proportion of rejections is the estimated size, which should be compared with a nominal size of 5%. The size adjusted power, i.e., the power when the simulated small sample critical values are used, is also of interest. For the null models with $r = 0, 1$ or 2 , the DGP's have $r = 1, 2$ and full rank respectively. Regarding the cointegrating space tests, the DGP is the $r = 1$ model. We also investigate the power when the null is the theoretical PPP but the data is generated from a model with common cointegrating space. The Monte Carlo simulation is done for sample sizes $T = 100, 200, 400, 800$ and 1600 , and the number of replicates is 1000 . The results are displayed in Table (4) and Table (5).

[Table (4) about here.]

[Table (5) about here.]

The results show the well known problem in cointegration analysis that for larger systems with many parameters the small sample critical values tend very slowly to the asymptotic (see e.g. Gredenhof and Jacobson (2001)). In a panel setting similar to the one in the present paper, Larsson and Lyhagen (1999) obtain size problems for the test of cointegration rank, but not for tests on restrictions. Here, however, in the rank one model some eigenvalues of the companion matrix are very close to the unit circle (see table (3)), indicating "closeness" to $I(2)$. This will make size problems even more severe, even for the restriction tests. This result shows the very need for the bootstrap or other size adjusting measures. The power properties are very satisfying for the larger sample sizes. For the sample size closest ($T = 400$) to the one used in the empirical part the power properties are good.

6 Conclusions

Previous studies of long-run purchasing power parity have predominantly used univariate techniques (e.g. unit-root tests) and have often found support for long-run PPP. We, on the other hand, use a multivariate panel approach, and arrive at a different conclusion. We do find cointegrating vectors between nominal exchange rates and prices - and just the number that PPP would predict - but none of these can be interpreted in terms of PPP. An interesting result is that all the cointegrating vectors share the same space which indicates that the

same economic law is valid for all four countries investigated, France, Germany, Italy and Great Britain.

It is difficult to reconcile the evidence given by traditional unit-root tests with the results provided in this study. What can explain this striking difference in conclusion? Possible explanations are offered by Engel (1999) (and Lyhagen (2000)), who argues that the traditional (panel) unit-root tests are greatly oversized. The reliability of our results is enhanced by what we believe to be a well-specified statistical model and by the fact that all the asymptotic tests have been replaced by robust bootstrap inference.

Now, whereas the bootstrap test can be expected to be approximately correct in size, it should be noted that its power will not be higher, nor lower, than the power of a size-adjusted asymptotic test. This has been theoretically predicted for the general case by Davidson and McKinnon (1996) and verified for the likelihood ratio test of linear restrictions on cointegrating vectors by Gredenhoff and Jacobson (1998) using Monte Carlo simulation. Moreover, the results in Gredenhoff and Jacobson suggest that the power of the likelihood ratio test in a complex model based on relatively few observations, such as the one at hand, cannot be expected to be high. Despite this we do reject the null of PPP. Had we not, low test power could very well have driven that result. In other words, the bootstrap procedure ensures a proper size for the test and the insufficient power only strengthens the rejection result.

The conclusion arising from our analysis is that real exchange rates are non-stationary, even when examining data stretching over long periods of time. Hence shocks to real exchange rates do not subside with time, but instead have infinitely long-lived effects. This suggests that permanent real shocks are the predominant sources of real exchange rate movements. A natural suggestion for future research is thus to develop models of real exchange rate behavior that focuses mainly on real factors.

Figure 1: Monthly WPI for Great Britain, Germany, France and Italy

Appendix

6.1 Description of data

The database is comprised of three nominal exchange rates and four wholesale price indices. The frequency is monthly and the series run from 1974 to 1999. See Figures 1 -3. The exchange rates are the price of British pounds in German mark, French franc and Italian lire respectively. The WPI's are from row 63 in the IFS-tapes.

6.2 Omitted proofs

This section contains omitted proofs of theorems 1-3. However, we start out by proving a theorem about the distribution of the estimated cointegrating space. This theorem and its proof is useful when proving theorems 1-3. The proofs follow closely the proof of theorem C.1 of Johansen (1991), which gives the corresponding result for any smooth hypothesis on β .

Figure 2: Monthly exchange rates against the British pound for Germany, France and Italy

Figure 3: Monthly real exchange rate for Germany, France and Italy using Great Britain as a base country.

6.2.1 The distribution of the estimated cointegrating space

At first, define the $\{N(p+1)\} \times (Np+1)$ matrix

$$R \equiv \begin{pmatrix} (I_p, 0) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & (I_p, 0) \\ (0, 1) & \cdots & & (0, 1) \end{pmatrix}'.$$

Then,

$$\beta = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix} = R'\varphi,$$

where

$$\varphi \equiv \text{diag}(\varphi_1, \dots, \varphi_N),$$

with

$$\varphi_i \equiv \begin{pmatrix} \beta_i \\ \beta_{N+1,i} \end{pmatrix},$$

which are $(p+1) \times r$, for $i = 1, \dots, N$. Thus, we may re-write (2) as

$$\Delta X_t = \alpha\varphi'RX_{t-1} + \sum_{i=1}^{m-1} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t.$$

In the limit results needed in the sequel, this formulation enables us to use RX_{t-1} in place of X_{t-1} . Now, define $C \equiv \beta_{\perp}(\alpha'_{\perp}\Gamma\beta_{\perp})^{-1}\alpha'_{\perp}$, where $\Gamma \equiv I_p - \sum_{i=1}^{m-1} \Gamma_i$. Granger's representation theorem (theorem 4.2 of Johansen (1995)) reads

Lemma 4 *If assumption A holds, we have the representation*

$$X_t = C \left(\mu t + \sum_{j=1}^t \varepsilon_j \right) + Y_t,$$

where Y_t is an $I(0)$ process.

Because of the lemma, the dominating deterministic trend of RX_t is $\tau \equiv RC\mu$, where $\Gamma \equiv I_{Np} - \sum_{i=1}^{m-1} \Gamma_i$. Hence, τ is $N(p+1) \times 1$, and we may write $\tau = (\tau'_1, \dots, \tau'_N)'$, where for $i = 1, \dots, N$, τ_i is $(p+1) \times 1$. Then, for each i , choose γ_i orthogonal to φ_i and τ_i . Then, γ_i is $p \times (p-r)$ and $\gamma \equiv \text{diag}(\gamma_1, \dots, \gamma_N)$ is orthogonal to φ and τ . Moreover, let $\{W(t)\}$ be a p -dimensional Wiener process with expectation 0 and covariance matrix . The modified version of lemma A.2 of Johansen (1991) reads

Lemma 5 As $T \rightarrow \infty$, we have for $u \in [0, 1]$ that

$$\begin{aligned} T^{-1/2} \bar{\gamma}' R X_{[Tu]} &\xrightarrow{w} \bar{\gamma}' R C W(u), \\ T^{-1} \bar{\tau}' R X_{[Tu]} &\xrightarrow{w} u. \end{aligned}$$

Now, put

$$\begin{aligned} \tilde{G}_1(t) &\equiv \bar{\gamma}' R C \left\{ W(t) - \int_0^1 W(u) du \right\}, \\ \tilde{G}_2(t) &\equiv t - \frac{1}{2}, \end{aligned}$$

$\tilde{G}(t) \equiv \left\{ \tilde{G}_1(t)', \tilde{G}_2(t) \right\}'$, and $V(t) \equiv \alpha' {}^{-1} W(t)$. Observe that the \tilde{G} process is defined in a slightly different way than the G process in Johansen (1991), where there is no R matrix, and a bit different $\bar{\gamma}$ matrix. With this modification, the same limit results hold here, and we do not re-iterate them. Furthermore, define the matrix

$$S \equiv \left(H_1^{(r)} \quad \tilde{H}_1, \dots, H_N^{(r)} \quad \tilde{H}_N \right)$$

with, for $i = 1, \dots, N$ and n arbitrary,

$$H_i^{(n)} \equiv (0, \dots, 0, I_n, 0, \dots, 0)',$$

which is a $Nn \times n$ matrix with I_n as the i th block, and the $\tilde{N} \times (p+1-r)$ matrix, where $\tilde{N} \equiv N(p-r)+1$,

$$\tilde{H}_i \equiv \begin{pmatrix} H_i^{(p-r)} & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that S is $Nr\tilde{N} \times \kappa$, where $\kappa \equiv N(p+1-r)r$. We may now formulate our theorem. (From now on, we use short-hand notation for our integrals, i.e. $\int \tilde{G}\tilde{G}'$ instead of $\int_0^1 \tilde{G}(t)\tilde{G}(t)' dt$.)

Theorem 6 As $T \rightarrow \infty$, the ML estimate of φ , $\hat{\varphi}$, satisfies

$$\left(T\gamma, T^{3/2}\tau \right)' \text{vec}(\hat{\varphi} - \varphi) \xrightarrow{w} S \left\{ S' \left(\alpha' {}^{-1} \alpha \quad \int \tilde{G}\tilde{G}' \right) S \right\}^{-1} S' \text{vec} \left\{ \int \tilde{G}(dV)' \right\}.$$

Proof: As a preparation, write $\varphi = \text{diag}(\varphi_1, \dots, \varphi_N)$, where $\varphi_i = \left(\varphi_i^{(1)'}, \varphi_i^{(2)' \right)'$, where $\varphi_i^{(1)}$ is $r \times r$, for $i = 1, \dots, N$. Then, it follows that $\alpha\varphi' = \alpha^*\varphi^{*'}$, where $\alpha^* \equiv \alpha \text{diag}(\varphi_1^{(1)'}, \dots, \varphi_N^{(1)'})$ and $\varphi^* \equiv \text{diag}(\varphi_1^*, \dots, \varphi_N^*)$ with $\varphi_i^* \equiv (I_r, \vartheta_i)'$, $\vartheta_i \equiv \varphi_i^{(2)} \left\{ \varphi_i^{(1)} \right\}^{-1}$ for $i = 1, \dots, N$. (The ϑ_i are $(p+1-r) \times r$.) Hence, regarding $\varphi^* = \varphi^*(\vartheta)$ as a matrix-valued function of the elements of the vector

$\vartheta \equiv \text{vec}(\vartheta_1, \vartheta_2, \dots, \vartheta_N)$, the derivatives are, denoting the elements of ϑ_i by ϑ_i^{jk} , where $j = 1, \dots, p$, $k = 1, \dots, r$, and similarly for φ ,

$$\frac{\partial \varphi_{ii}^{*jk}}{\partial \vartheta_i^{lm}} = 1$$

if $\{(j, k) = (l, m)\}$, and 0 otherwise. (The derivatives of non diagonal block elements of φ^* w.r.t. any elements of ϑ are all zero.) Hence, the derivative in the direction \tilde{u} , where $\tilde{u} \equiv (u_1, \dots, u_\kappa)'$ is a vector of the same structure as ϑ , i.e. $\kappa \times 1$ where $\kappa = N(p+1-r)r$, is the block diagonal $N(p+1-r) \times Nr$ matrix $D\varphi(\tilde{u})$ with elements

$$\sum_{s=1}^{\kappa} u_s \frac{\partial \varphi_{ii}^{*jk}}{\partial \vartheta_s} = u_{s^*},$$

where s^* is the s corresponding to element (j, k) of the matrix ϑ_{ii} . In other words, $D\varphi(\tilde{u})$ has the same structure as φ^* , except that the I_r matrices are replaced by 0.

Next, choose $\tilde{u}_1, \dots, \tilde{u}_\kappa$ orthogonal in \mathcal{R}^κ , such that $\tau' D\varphi(\tilde{u}_i) = 0$ for $i = 1, \dots, \kappa_1$ and $\tau' D\varphi(\tilde{u}_i) \neq 0$ for $i = \kappa_1 + 1, \dots, \kappa$. Furthermore, define the matrix $D\tilde{\varphi}$ with i th column given by

$$\begin{aligned} D\tilde{\varphi}_i &= \text{vec} \{(\gamma, 0)' D\varphi(\tilde{u}_i)\}, \quad i = 1, \dots, \kappa_1, \\ D\tilde{\varphi}_i &= \text{vec} \{(0, \tau)' D\varphi(\tilde{u}_i)\}, \quad i = \kappa_1 + 1, \dots, \kappa. \end{aligned}$$

Here, $(\gamma, 0)$ and $(0, \tau)$ are $Np \times \tilde{N}$, and so, $(\gamma, 0)' D\varphi(\tilde{u}_i)$ and $(0, \tau)' D\varphi(\tilde{u}_i)$ are $\tilde{N} \times Nr$ and $D\tilde{\varphi}$ is $Nr\tilde{N} \times \kappa$. To understand the structure of $D\tilde{\varphi}$, write

$$\begin{aligned} \text{vec} \{(\gamma, 0)' D\varphi(\tilde{u}_i) I_{Nr}\} &= \begin{Bmatrix} I_{Nr} & (\gamma, 0)' \end{Bmatrix} \text{vec} \{D\varphi(\tilde{u}_i)\}, \\ \text{vec} \{(0, \tau)' D\varphi(\tilde{u}_i) I_{Nr}\} &= \begin{Bmatrix} I_{Nr} & (0, \tau)' \end{Bmatrix} \text{vec} \{D\varphi(\tilde{u}_i)\}, \end{aligned}$$

so that

$$D\tilde{\varphi} = [\{I_{Nr} \quad (\gamma, 0)'\} L_1, \{I_{Nr} \quad (0, \tau)'\} L_2], \quad (4)$$

where

$$\begin{aligned} L_1 &\equiv [\text{vec} \{D\varphi(\tilde{u}_1)\}, \dots, \text{vec} \{D\varphi(\tilde{u}_{\kappa_1})\}], \\ L_2 &\equiv [\text{vec} \{D\varphi(\tilde{u}_{\kappa_1+1})\}, \dots, \text{vec} \{D\varphi(\tilde{u}_\kappa)\}]. \end{aligned}$$

Next, for $i = 1, \dots, N$ define $H_i^{(r)}$ as above, which is orthogonal to the $Nr \times (N-1)r$ matrix

$$H_{i\perp}^{(r)} = \begin{pmatrix} I_{r(i-1)} & \mathbf{0}_{r(i-1) \times r(N-1-i)} \\ \mathbf{0}_{r \times r(i-1)} & \mathbf{0}_{r \times r(N-1-i)} \\ \mathbf{0}_{r(N-1-i) \times r(i-1)} & I_{r(N-1-i)} \end{pmatrix},$$

for $i = 1, \dots, N$. Furthermore, let

$$\tilde{H}_i \equiv \begin{pmatrix} H_i^{(p-r)} \\ 0 \end{pmatrix},$$

which is $\tilde{N} \times (p-r)$, and

$$S_{\perp} \equiv \left(H_{1\perp}^{(r)} \quad \tilde{H}_1, \dots, H_{N\perp}^{(r)} \quad \tilde{H}_N \right),$$

which is $Nr\tilde{N} \times N(N-1)(p-r)r$ and orthogonal to S defined above. Now, we find via (4) that

$$\begin{aligned} & S'_{\perp} \{I_{Nr} \quad (\gamma, 0)'\} L_1 \\ &= \begin{pmatrix} H_{1\perp}^{(r)'} & \tilde{H}'_1(\gamma, 0)' \\ \vdots & \\ H_{N\perp}^{(r)'} & \tilde{H}'_N(\gamma, 0)' \end{pmatrix} [\text{vec}\{D\varphi(\tilde{u}_1)\}, \dots, \text{vec}\{D\varphi(\tilde{u}_{\kappa_1})\}], \\ &= \begin{pmatrix} \text{vec}\left(\tilde{H}'_1(\gamma, 0)' D\varphi(\tilde{u}_1) H_{1\perp}^{(r)}\right) & \cdots & \text{vec}\left(\tilde{H}'_1(\gamma, 0)' D\varphi(\tilde{u}_{\kappa_1}) H_{1\perp}^{(r)}\right) \\ \vdots & & \vdots \\ \text{vec}\left(\tilde{H}'_N(\gamma, 0)' D\varphi(\tilde{u}_1) H_{N\perp}^{(r)}\right) & \cdots & \text{vec}\left(\tilde{H}'_N(\gamma, 0)' D\varphi(\tilde{u}_{\kappa_1}) H_{N\perp}^{(r)}\right) \end{pmatrix} = 0, \end{aligned}$$

where the third equality follows since for all i and j , $\tilde{H}'_i(\gamma, 0)' D\varphi(\tilde{u}_j) H_{i\perp}^{(r)}$ picks out only non-diagonal blocks of $(\gamma, 0)' D\varphi(\tilde{u}_j)$, which are 0. Similarly, $S'_{\perp} \{I_{Nr} \quad (0, \tau)'\} L_2 = 0$. Now, let S be as above. Then, the identity

$$I_{Nr\tilde{N}} = \overline{S} \overline{S}' + \overline{S}_{\perp} S'_{\perp}$$

and (4) imply

$$D\tilde{\varphi} = \left(\overline{S} \overline{S}' + \overline{S}_{\perp} S'_{\perp} \right) [\{I_{Nr} \quad (\gamma, 0)'\} L_1, \{I_{Nr} \quad (0, \tau)'\} L_2] = SM,$$

where

$$M \equiv \overline{S}' [\{I_{Nr} \quad (\gamma, 0)'\} L_1, \{I_{Nr} \quad (0, \tau)'\} L_2],$$

which is a $\kappa \times \kappa$ matrix. Now, if M is non-singular, eq. (C.7) of Johansen (1991) yields (with the Kronecker product twisted around, due to different notational conventions),

$$\begin{aligned} & \left(T\gamma, T^{3/2}\tau \right)' \text{vec}(\hat{\varphi} - \varphi) \\ & \xrightarrow{w} D\tilde{\varphi} \left\{ D\tilde{\varphi}' \left(\alpha' \quad -1\alpha \quad \int \tilde{G}\tilde{G}' \right) D\tilde{\varphi} \right\}^{-1} D\tilde{\varphi}' \text{vec} \left\{ \int \tilde{G}(dV)' \right\} \\ &= SM \left\{ M'S' \left(\alpha' \quad -1\alpha \quad \int \tilde{G}\tilde{G}' \right) SM \right\}^{-1} M'S' \text{vec} \left\{ \int \tilde{G}(dV)' \right\} \\ &= S \left\{ S' \left(\alpha' \quad -1\alpha \quad \int \tilde{G}\tilde{G}' \right) S \right\}^{-1} S' \text{vec} \left\{ \int \tilde{G}(dV)' \right\}, \end{aligned}$$

as was to be shown. The non-singularity of M follows since

$$\begin{aligned}
& S' \{ I_{Nr} \quad (\gamma, 0)' \} L_1 \\
&= \begin{pmatrix} H_1^{(r)'} & \tilde{H}'_1(\gamma, 0)' \\ \vdots & \\ H_N^{(r)'} & \tilde{H}'_N(\gamma, 0)' \end{pmatrix} [\text{vec} \{ D\varphi(\tilde{u}_1) \}, \dots, \text{vec} \{ D\varphi(\tilde{u}_{\kappa_1}) \}], \\
&= \begin{pmatrix} \text{vec} \left(\tilde{H}'_1(\gamma, 0)' D\varphi(\tilde{u}_1) H_1^{(r)} \right) & \cdots & \text{vec} \left(\tilde{H}'_1(\gamma, 0)' D\varphi(\tilde{u}_{\kappa_1}) H_1^{(r)} \right) \\ \vdots & & \vdots \\ \text{vec} \left(\tilde{H}'_N(\gamma, 0)' D\varphi(\tilde{u}_1) H_N^{(r)} \right) & \cdots & \text{vec} \left(\tilde{H}'_N(\gamma, 0)' D\varphi(\tilde{u}_{\kappa_1}) H_N^{(r)} \right) \end{pmatrix}
\end{aligned}$$

which is a block diagonal matrix with Nr diagonal blocks of dimension $(p+1-r) \times (p+1-r)$. For example, the first block is given by

$$\begin{aligned}
& \left(\text{vec} \left(\tilde{H}'_1(\gamma, 0)' D\varphi(\tilde{u}_1) H_1^{(r)} \right), \dots, \text{vec} \left(\tilde{H}'_1(\gamma, 0)' D\varphi(\tilde{u}_{p+1-r}) H_1^{(r)} \right) \right) \\
&= (\gamma'_1, 0) (0, I_{p+1-r})'.
\end{aligned}$$

Indeed, this is the form of the r uppermost blocks, then comes the block $(\gamma'_2, 0) (0, I_{p+1-r})'$, etcetera. Similarly, $S' \{ I_{Nr} \quad (0, \tau)' \} L_2$ is block diagonal with Nr diagonal blocks of dimension $(p+1-r) \times (p+1-r)$, given by $(0, \tau_1) (0, I_{p+1-r})'$, etcetera. Hence,

$$M(I_{\kappa_1}, I_{\kappa-\kappa_1})' = [S' \{ I_{Nr} \quad (\gamma, 0)' \} L_1, S' \{ I_{Nr} \quad (0, \tau)' \} L_2] (I_{\kappa_1}, I_{\kappa-\kappa_1})'$$

is block diagonal with Nr diagonal blocks of dimension $(p+1-r) \times (p+1-r)$, given by $(\gamma_1, \tau_1) (0, I_{p+1-r})'$, etcetera. Since these blocks are non-singular, it follows that $M(I_{\kappa_1}, I_{\kappa-\kappa_1})'$, and hence M , is non-singular.

6.2.2 Proof of theorem 1

Consider the three hypotheses $\mathcal{H}_4 : \text{rank}(\Pi) \leq Np$, $\mathcal{H}_3 : \Pi = \alpha\beta'$ where α, β are $Np \times Nr$, of full rank and \mathcal{H}_2 : as \mathcal{H}_3 but where β is block-diagonal with $p \times r$ -dimensional blocks. Denoting the maximum likelihood ratio between \mathcal{H}_i and \mathcal{H}_j ($\mathcal{H}_i \subset \mathcal{H}_j$) by Q_{ij} , we then have $Q_{24} = Q_{23}Q_{34}$, i.e.

$$-2 \log Q_{24} = -2 \log Q_{23} - 2 \log Q_{34}.$$

Johansen (1995b) has showed that the asymptotic distribution of $-2 \log Q_{34}$ equals the distribution of U as defined in the theorem. (The fact that β has the specific form under our hypothesis under test, \mathcal{H}_2 , does not affect this result.) Now, to prove our theorem, our plan is

- 1) To show the convergence of $-2 \log Q_{23}$ to the χ^2 distribution.
 - 2) To show the asymptotic independence between $-2 \log Q_{23}$ and $-2 \log Q_{34}$.
- 1) It follows from theorem C.1 of Johansen (1991) that $-2 \log Q_{23}$ is asymptotically $\chi^2(k-s)$, where k is the number of free parameters under the

alternative hypothesis \mathcal{H}_3 , and where s is the number of free parameters under the null hypothesis \mathcal{H}_2 . In other words, $k - s$ is the difference between the numbers of free parameters of $\alpha\beta'$ under \mathcal{H}_3 and \mathcal{H}_2 , respectively. Now, under \mathcal{H}_2 ,

$$\begin{aligned} \beta &= \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \beta_1^{(1)} \\ \beta_1^{(2)} \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} \beta_N^{(1)} \\ \beta_N^{(2)} \end{pmatrix} \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix} \\ &= \tilde{\beta} \text{diag}(\beta_1^{(1)}, \dots, \beta_N^{(1)}), \end{aligned}$$

where

$$\tilde{\beta} \equiv \begin{pmatrix} \tilde{\beta}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \tilde{\beta}_N \\ \tilde{\beta}_{N+1,1} & \cdots & & \tilde{\beta}_{N+1,N} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} I_r \\ \vartheta_1 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} I_r \\ \vartheta_N \end{pmatrix} \\ \vartheta_{N+1,1} & \cdots & & \vartheta_{N+1,N} \end{pmatrix}, \quad (5)$$

with $\vartheta_i \equiv \beta_i^{(2)} \{\beta_i^{(1)}\}^{-1}$ and $\vartheta_{N+1,i} \equiv \beta_{N+1,i}^{(2)} \{\beta_{N+1,i}^{(1)}\}^{-1}$ for $i = 1, \dots, N$. (The $\beta_i^{(1)}$ are $r \times r$ and the $\beta_i^{(2)}$ are $(p-r) \times r$.) Then, $\alpha\beta' = \tilde{\alpha}\tilde{\beta}'$, where $\tilde{\alpha} \equiv \alpha \text{diag}(\beta_1^{(1)'}, \dots, \beta_N^{(1)'})$. Here, for each i , the numbers of free parameters of $\tilde{\beta}_i$ and $\tilde{\beta}_{N+1,i}$ are $(p-r)r$ and r , respectively. Consequently, under \mathcal{H}_2 , the number of free parameters of $\alpha\beta'$ is $N^2pr + N(p-r)r + Nr$. Under \mathcal{H}_3 , the corresponding trick yields $N^2pr + N^2(p-r)r + Nr$ free parameters. Hence,

$$\begin{aligned} k - s &= \{N^2pr + N^2(p-r)r + Nr\} - \{N^2pr + N(p-r)r + Nr\} \\ &= N(N-1)(p-r)r, \end{aligned}$$

as was to be shown.

2) Introduce the extra hypothesis $\mathcal{H}_0 : \Pi = \alpha\beta'$ where β is fixed. It follows as above that

$$-2 \log Q_{23} = -2 \log Q_{03} - (-2 \log Q_{02}).$$

Here, following Johansen (1991), p. 1576, we find as in the preceding proof

that

$$\begin{aligned}
& -2 \log Q_{02} \\
& \xrightarrow{w} \text{vec} \left\{ \int \tilde{G} (dV)' \right\}' D\tilde{\beta} \left\{ D\tilde{\beta}' \left(\alpha' \quad -^1\alpha \quad \int \tilde{G}\tilde{G}' \right) D\tilde{\beta} \right\}^{-1} D\tilde{\beta}' \text{vec} \left\{ \int \tilde{G} (dV)' \right\} \\
& = \text{vec} \left\{ \int \tilde{G} (dV)' \right\}' SM \left\{ M'S' \left(\alpha' \quad -^1\alpha \quad \int \tilde{G}\tilde{G}' \right) SM \right\}^{-1} M'S' \text{vec} \left\{ \int \tilde{G} (dV)' \right\} \\
& = \text{vec} \left\{ \int \tilde{G} (dV)' \right\}' S \left\{ S' \left(\alpha' \quad -^1\alpha \quad \int \tilde{G}\tilde{G}' \right) S \right\}^{-1} S' \text{vec} \left\{ \int \tilde{G} (dV)' \right\}.
\end{aligned}$$

As for $-2 \log Q_{03}$, we simply replace $D\tilde{\beta}$ by an identity matrix, to obtain

$$\begin{aligned}
& -2 \log Q_{03} \\
& \xrightarrow{w} \text{vec} \left\{ \int \tilde{G} (dV)' \right\}' \left(\alpha' \quad -^1\alpha \quad \int \tilde{G}\tilde{G}' \right)^{-1} \text{vec} \left\{ \int \tilde{G} (dV)' \right\} \\
& = \text{vec} \left\{ \int \tilde{G} (dV)' \right\}' \left(\alpha' \quad -^1\alpha \quad \int \tilde{G}\tilde{G}' \right)^{-1} \text{vec} \left\{ \int \tilde{G} (dV)' \right\}.
\end{aligned}$$

Consequently,

$$-2 \log Q_{23} \xrightarrow{w} \text{vec} \left\{ \int \tilde{G} (dV)' \right\}' P \text{vec} \left\{ \int \tilde{G} (dV)' \right\}, \quad (6)$$

where

$$P \equiv J^{-1} - S (S'JS)^{-1} S'$$

with

$$J \equiv \alpha' \quad -^1\alpha \quad \int \tilde{G}\tilde{G}'.$$

Then, it follows that

$$P = J^{-1} S_{\perp} (S'_{\perp} J^{-1} S_{\perp})^{-1} S'_{\perp} J^{-1}. \quad (7)$$

Now, conditional on \tilde{G} , $\text{vec} \left(\int \tilde{G} dV' \right)$ is normal with expectation 0 and covariance matrix J , and so, $S'_{\perp} J^{-1} \text{vec} \left(\int \tilde{G} dV' \right)$ is normal with expectation 0 and covariance matrix $S'_{\perp} J^{-1} S_{\perp}$. Hence, by (7), the r.h.s. of (6) is $\chi^2(1)$, conditional on \tilde{G} . Moreover, since this distribution is independent of \tilde{G} , this property holds also unconditionally. Consequently, the quantity on right-hand side of (6) is independent of \tilde{G} , a fact that will be useful below. Furthermore, from Johansen (1995b), p. 158-160, we deduce the representation

$$\begin{aligned}
& -2 \log Q_{34} \\
& \xrightarrow{w} \text{tr} \left\{ \left(\int GG' \right)^{-1} \int G dW' \alpha_{\perp} (\alpha'_{\perp} \quad \alpha_{\perp})^{-1} \alpha'_{\perp} \left(\int G dW' \right)' \right\} \\
& = \text{vec} \left(\int G dW' \alpha_{\perp} \right)' \left(\alpha'_{\perp} \quad \alpha_{\perp} \quad \int GG' \right)^{-1} \text{vec} \left(\int G dW' \alpha_{\perp} \right) \quad (8)
\end{aligned}$$

where W is as above. The processes G and \tilde{G} are defined in slightly different ways, but the stochastic parts of them are the same. In fact, defining

$$Z(t) \equiv \left(W(t)' - \int_0^1 W(u)' du, \quad t - \frac{1}{2} \right)',$$

we have $\tilde{G}(t) = \tilde{A}Z(t)$, where $\tilde{A} \equiv \text{diag}(\bar{\gamma}'RC, 1)$. Similarly, $G(t) = AZ(t)$, where A is as \tilde{A} , but with no R and a slightly different $\bar{\gamma}$. Now, we need to show that the right hand side terms of (6) and (8), M_1 and M_2 say, are independent. To this end, let us condition on Z . Then, $\int \tilde{G}(dV)' = \tilde{A} \int Z dW' \alpha$ and $\int G dW' \alpha_{\perp} = A \int Z dW' \alpha_{\perp}$ are both normals, each with expectation zero, and the covariance between them is

$$\tilde{A}E \left\{ \int Z dW' \alpha \left(\int Z dW' \alpha_{\perp} \right)' \right\} A' = 0,$$

showing that $\int \tilde{G}(dV)'$ and $\int G dW' \alpha_{\perp}$ are conditionally independent given Z . Hence, M_1 and M_2 must also be conditionally independent given Z . Furthermore, as we saw above, M_1 is independent of G , hence also of Z . Thus we get, denoting the densities for M_1 and M_2 by f_1 and f_2 , their simultaneous density by $f_{1,2}$, the density of Z by f_Z and the corresponding conditional densities by $f_{1|Z}$ etcetera,

$$f_{1,2} = \int f_{1,2|Z} f_Z = \int f_{1|Z} f_{2|Z} f_Z = f_1 \int f_{2|Z} f_Z = f_1 f_2,$$

where the integrals are over the support of the Z density. This shows the independence between M_1 and M_2 , and we are done.

6.2.3 Proof of theorem 2

Again, the theorem is a special case of theorem C.1 of Johansen (1991). Hence, the asymptotic distribution is χ^2 , and the number of degrees of freedom is the difference between the number of free parameters of $\alpha\beta'$ under \mathcal{H}_2 and \mathcal{H}_1 , respectively. As we saw in the previous proof, the former number is $N^2pr + N(p+1-r)r$. Similar arguments lead to the corresponding number $N^2pr + (p+1-r)r$ under \mathcal{H}_1 . Consequently, the number of degrees of freedom for the test is the difference, $(N-1)(p+1-r)r$, as was to be shown.

6.2.4 Proof of theorem 3

As above, the asymptotic distribution is χ^2 . We need to find the number of degrees of freedom. Now, under \mathcal{H}_0 , we may without loss of generality assume that all ψ_i are equal. Consequently, with $\psi \equiv (\psi'_0, \psi'_1)'$ where ψ_0 is $r \times r$, we

may write

$$\begin{aligned}
\beta &= \left(\left(\begin{array}{cc} I_p & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{array} \right) \left(\begin{array}{c} \beta_1 \\ \beta_{N+1,1} \end{array} \right), \dots, \left(\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ I_p & 0 \\ 0 & 1 \end{array} \right) \left(\begin{array}{c} \beta_N \\ \beta_{N+1,N} \end{array} \right) \right) \\
&= \left(\left(\begin{array}{cc} I_p & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{array} \right) H_1, \dots, \left(\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ I_p & 0 \\ 0 & 1 \end{array} \right) H_N \right) \left\{ I_N \quad \left(\begin{array}{c} I_r \\ \psi_1 \psi_0^{-1} \end{array} \right) \right\} (I_N \quad \psi_0) \\
&\equiv \beta^* (I_N \quad \psi_0),
\end{aligned}$$

implying $\alpha\beta' = \alpha^*\beta^{*'}$, where $\alpha^* \equiv \alpha (I_N \quad \psi_0')$. Here, β^* has $(s-r)r$ free parameters, and so, the number of free parameters of $\alpha\beta'$ is $N^2pr + (s-r)r$. Hence, in the manner as in the previous proofs, the number of degrees of freedom for the test of \mathcal{H}_0 against \mathcal{H}_1 is

$$N^2pr + (p+1-r)r + r - \{N^2pr + (s-r)r\} = (p+1-s)r.$$

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$r =$	0	1	2
Test stat.	217	69.5	0.467
Crit. value	135	70.8	7.31

Table 1: Test statistics and bootstrapped critical values (1000 replicates), a 5 % nominal size.

	Germany	France	Italy
β_{ix}	1.00	1.00	1.00
β_{ip}	-5.12	9.87	-2.25
β_{ib}	0.671	-8.78	0.528

Table 2: Normalized unrestricted estimates of the cointegrating vectors

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$r = 0$	1	2
1.00	1.00	1.00
1.00	1.00	0.991
1.00	1.00	0.974
1.00	1.00	0.974
1.00	0.991	0.896
1.00	0.966	0.890
1.00	0.966	0.890
0.827	0.403	0.455
0.398	0.403	0.407
0.301	0.242	0.302
0.288	0.242	0.302
0.288	0.281	0.262
0.203	0.281	0.218
0.156	0.141	0.218

Table 3: Absolute values of the eigenvalues of the companion matrix for $r = 0, 1, 2$

Null\T	100	200	314	400	800	1600
$r = 0$	0.625	0.271	0.159	0.134	0.078	0.082
$r = 1$	0.578	0.534	0.454	0.423	0.311	0.213
$r = 2$	0.494	0.487	0.256	0.142	0.074	0.086
Common $r = 1$	0.692	0.460	0.278	0.177	0.067	0.031
PPP $r = 1$ vs unrestricted	0.648	0.363	0.237	0.179	0.101	0.072
PPP $r = 1$ vs common	0.926	0.705	0.432	0.198	0.127	0.074

Table 4: Size of PPP related panel tests

Null\T	100	200	314	400	800	1600
$r = 0$	0.688	0.999	1.000	1.000	1.000	1.000
$r = 1$	0.111	0.547	0.953	0.999	1.000	1.000
$r = 2$	0.018	0.022	0.168	0.404	0.921	0.988
Common $r = 1$	0.050	0.241	0.700	0.936	1.000	1.000
PPP $r = 1$ vs unrestricted	0.101	0.485	0.951	0.994	1.000	1.000
PPP $r = 1$ vs common	0.083	0.662	0.981	1.000	1.000	1.000

Table 5: Size adjusted power of PPP related panel tests