Simultaneous Ascending Bid Auctions with Budget Constraints*

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Abstract

We analyze the impact of budget constraints on collusion in simultaneous ascending bid auctions with multiple objects. The known presence of binding budget constraints shrinks the equilibrium set. In many cases, a family of equilibria with a high degree of competition, as well as a family of equilibria with a high degree of collusion, disappear. If the bidders' budget levels are privately known, then even a very small probability of having a budget constrained bidder may have a dramatic impact on the equilibrium set, eliminating the most competitive equilibria.

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1 Introduction

Budget constraints have only recently received attention in auction theory. Pioneering work in this area is due to Che and Gale [4] and [5]. They have analyzed single-object standard auctions, and have characterized revenue maximizing mechanisms for selling one object to one buyer, in situations where buyers have private information about both their willingness and their (possibly lower) ability to pay. One important insight that emerges from Che and Gale's work is that having a buyer with maximum budget w and value v for an object is not generally the same as having a buyer with a value $\min\{v, w\}$.

Multiple objects auctions with budget constrained bidders have been studied in Benoît and Krishna [3], but only under the assumption of complete information. [Fang and Parreiras [6] and [7], and Zheng [11].]

In the present paper we analyze multi-unit simultaneous ascending-bid auctions under Che and Gale's information structure, i.e. under the assumption that each bidder has private information about both her willingness to pay and her budget. Since 1994, the governments of the US and of several European countries have used multi-unit simultaneous ascending-bid auctions numerous times to sell licenses for the use of parts of the electromagnetic spectrum. In a previous paper [2] we have shown that, for a large class of information and preference structures, multi-unit simultaneous ascending bid auctions provide the bidders with ample opportunities for collusion. The basic idea is that competing for two objects may yield less expected surplus than buying a single object at a relatively low price. Formally, we have identified Perfect Bayesian equilibria in which the bidders are able to divide the object among themselves while keeping prices at relatively low levels. The focus of the present paper will be the effect that the possibility of binding budget constraints has on the opportunities for tacit collusion. In particular, we will concentrate on 'least collusive' and 'most collusive' equilibria.

Intuitively, the presence of budget constraints can affect the equilibrium set of the auction in several ways. First, it is clear that bidders with low budget levels cannot place high bids. (We are assuming that each bid must be backed by "money on the table," hence no bidder can make bids whose total is above her budget.) Thus 'noncollusive' equilibria, which would generate socially efficient outcomes without budget constraints, now yield both a lower seller's expected revenue and a lower social surplus.

However tacit collusion in multi-unit simultaneous ascending bid auctions can only be sustained with a credible threat of reverting to non-collusive behavior, which increases the total payment of a deviating bidder. But, with sufficiently tight budget constraints, the punishing bidders are not able to increase prices to sufficiently high levels. Hence the presence of budget constraints may also hinder collusion.

Finally, in situations where the bidders' budgets are commonly known but asymmetric, a third effect, similar to the 'demand reduction' seen in uniform price auctions (see Ausubel and Cramton [1]), arises. Once prices reach levels at which a low-budget bidder is unable to buy more than one object, then a high-budget bidder can end the auction at any time by simply letting the low-budget bidder win one object. Again, this can be more profitable than trying to buy both objects.

This effect can play an important role if the bidders have private information about their budget levels: we will show that, as long as there is any positive probability that all bidders may be budget-constrained, then, in any noncollusive equilibrium, a nonempty set of types with no budget constraint but with relatively low values for the objects mimics the behavior of budget-constrained types, thus they bid less aggressively (i.e. reducing demand) and implicitly provide opportunities for splitting the objects. Thus an arbitrarily small probability of having a budget-constrained bidder may have a first order effect on efficiency and revenue.

The rest of the paper is organized as follows. In Section 2 we describe the model. For simplicity, our analysis is focuses on the case with two bidders and two objects, and each bidder assigns the same value to the each object. In Section 3 we study the case with commonly known budget levels, considering first the symmetric case, and then the case with different budget levels. In Section 4 we introduce private information about the budget levels. Section 5 concludes, and an appendix collects all proofs.

2 The Model

There are two objects, and a set $N := \{1, ..., i, ..., n\}$ of bidders. Each bidder $i \in N$ is characterized by a type $\theta_i := (v_{i1},\ v_{i2},\ k_i,\ w_i)$, where v_{ij} denotes the amount of money that she is willing to pay to own object j alone, k_i is her "complementarity premium," i.e. the additional value derived from owning both objects, and w_i is the maximum amount of money that the she can spend in the auction. Bidder i's type θ_i is drawn from a distribution with support $[0,1]^2 \times K \times W_i$, where K and W_i are compact subsets of the real line.

The objects are sold using a "simultaneous ascending bid auction", which is a natural extension of the standard one-object English auction to environments with multiple objects. The rules of the auction, described in detail in Brusco and Lopomo (2001), are essentially as follows. The auction proceeds in rounds. In each round $t = 1, 2, \ldots$, for each object $j = 1, 2$, each bidder i can either stay silent or raise the highest bid of the previous round by at least a minimum amount $\varepsilon > 0$. Formally, i's bid on object j in round t, denoted by $b_j^{(i)}(t)$, can either be $-\infty$, which is to be interpreted as "stay silent", or must be a number in the interval $[b_j(t-1) + \varepsilon, +\infty)$, where $b_j(t-1)$ denotes the "current outstanding bid", defined recursively by:

$$b_j(0) = 0 \quad \text{and} \quad b_j(t) := \max\left\{b_j(t-1),\ b_j^{(i)}(t);\ i \in N\right\}.$$

If at least one bidder increases the outstanding bid on at least one object, i.e. if $b_j(t) > b_j(t-1)$ for some j, then for each of these objects the new highest bid is identified, a potential winner is selected among the bidders who have made the new highest bid, and the auction moves to the next round, with the potential winner of all other objects unchanged. If instead all bidders stay silent on all objects, the auction ends, and each object is sold to the winner selected at the end of the previous round, for her last bid.

In our analysis we will consider the minimum bid increment ε negligibly small. Formally, we will focus on the limit of the equilibrium set as ε tends to 0. By "equilibrium" we will always mean "perfect Bayesian equilibrium," and to keep the formal statements of our results as simple as possible, we will often write that a given strategy profile σ "forms an equilibrium" to mean that there exists a consistent belief system μ such that the pair (σ, μ) constitutes a perfect Bayesian equilibrium. In most cases, given a strategy profile σ it will be easy to find a consistent belief system which supports σ as an equilibrium. We will be explicit about the belief system that goes together a given strategy profile only in some of our proofs.

In order to make the analysis manageable, we limit our attention to the case with two bidders. Furthermore, we assume that for each bidder the two objects identical and there are no complementarities, i.e. $v_{i1} = v_{i2} := v_i$ for each $i = 1, 2$, and $K = \{0\}$. Under these restrictions, bidder i's type is identified by the pair $(v_i, w_i) \in [0, 1] \times W_i$. We will assume that the four variables (v_1, v_2, w_1, w_2) are independently distributed. Each variable v_i is drawn by a distribution with differentiable density f, and support $[0, 1]$. We will denote the corresponding c.d.f. by $F(v_i)$.

If $2 \leq \min W_i$ for each $i = 1, 2$, the model is a special case of the model studied in Brusco and Lopomo [2], because it is common knowledge that each bidder's budget constraint is not binding. In that paper we have established the existence of collusive equilibria which are sustained by the threat of reverting to non-collusive continuation strategies. Here our focus is on the effect that the possible presence of budget constraints has on the auction's equilibrium set. Thus we assume, without loss of generality, that $\min W_i < 2$, for some $i = 1, 2$.

3 Commonly Known Budgets

In this section we assume that the two bidders have an identical and commonly known budget level, i.e. $W_1 = W_2 = \{w\}$. The assumption $\min W_i < 2$ in this case implies $w < 2$.

3.1 Identical Budgets: The Noncollusive Equilibrium

Before stating a formal definition of the noncollusive equilibrium strategies, we describe the bidders' behavior on the equilibrium path. For simplicity, we will ignore the (zero probability) event in which the bidders' values are equal. The bidders start by increasing both outstanding bids at the same pace, until they reach the minimum among the two bidders' values and the threshold $h := \frac{w}{2}$. More precisely, for each $i = 1, 2$, bidder i raises by ε the outstanding bid on object i in any odd round, and on object $3 - i$ in any even round, up to $\min\{v_i, h\}$. Thus the auction progresses with each bidder being the potential winner of one object in each round, until the outstanding bids reach either $\min\{v_1, v_2\}$ or h. In the first case, i.e. if $\min\{v_1, v_2\} < h$, the bidder with the highest value buys both objects and pays twice her opponent's value. Otherwise, each bidder buys one object and pays h.

The next proposition establishes that this behavior can be supported as an equilibrium.

Proposition 1 *(Noncollusive equilibrium with known and equal budgets) If $w_1 = w_2 = w$, the following strategy forms a symmetric equilibrium: at any stage $t + 1$, type v_i of bidder i raises by ε the outstanding bid*

- *of object i, if the current outstanding bids are equal, she is not the winner on any object, and*

$$b_i(t) < \min\{v_i, w\}; \qquad (1)$$

- *of the object with the lowest outstanding bid, if the current outstanding bids are different, she is not the winner on any object, and*

$$\min\{b_1(t), b_2(t)\} < \min\{v_i, w\}; \tag{2}$$

- *of object j only, if she is the winner on object $3-j$ only, and*

$$b_j(t) < \min\{v_i, \ w - b_{3-j}(t), \ h\}; \tag{3}$$

- *of no object, otherwise.*

Without budget constraints, i.e. if $2 < w$, the equilibrium of Proposition 1 collapses to the "separated English auctions" equilibrium characterized in Brusco and Lopomo [2] — i.e. each bidder bids on each object up to its valuation. The presence of commonly known budget constraints affects the noncollusive equilibrium strategy in two ways. First, it is clear that the sum of the outstanding bids of the objects on which a bidder is the potential winner cannot exceed her budget. This constraint is captured by the presence of w in inequalities (1) and (2), and $w - b_{3-j}(t)$ in inequality (3). The budget constraints however are binding only off the equilibrium path.

The effect of the budget constraints on the bidders' equilibrium behavior is captured by the presence of h in inequality (3): as Proposition 1 implies, it is optimal for each bidder to stop trying to buy both objects once the outstanding bids have reached only half of her budget. To see this, first notice that no bidder will let the auction end if she is losing both objects and the lowest outstanding bid is below the minimum between her value and her budget; therefore, if a bidder buys both objects, her total payment cannot be lower than twice her opponent's value. Now suppose that in round t, each bidder is winning one object, say bidder i is winning object i, and the outstanding bid $b_2(t)$ is above h. If bidder 2 is following the strategy of Proposition 1, bidder 1 infers that $h < v_2$, i.e. $w < 2v_2$, hence she concludes that she cannot afford to buy both objects. Since each bidder is already winning one object, there is no point in raising any current outstanding bid.

3.2 Identical Budgets: Collusive Equilibria

The noncollusive equilibrium of Proposition 1 can be used to construct a family of 'collusive' equilibria, which we label γ-equilibria. For each $\gamma \in [0, h]$, a γ-strategy is defined as follows:

each bidder uses the noncollusive strategy of Proposition 1 until either the opponent stays silent or the outstanding bids reach the level γ, and stay silent in the next round. After any deviation each bidder reverts to the noncollusive strategy.

Clearly, the set of all γ-strategies, $\gamma \in [0, h]$, can be ordered according to their implied degree of collusion. The 0-strategy is the most collusive: it induces each bidder to buy one object at the lowest possible price. The least collusive γ-strategy is the h-strategy, i.e. the noncollusive strategy of Proposition 1.

To identify the conditions under which a given γ-strategy forms a symmetric equilibrium, we start by writing the bidders' interim expected surplus functions corresponding to an arbitrary symmetric equilibrium. Let $Q_j(v)$ denote the probability that type v of bidder 1 is awarded object j, and let $M(v)$ denote her expected payment (by symmetry, these functions are the same for both bidders), and define $Q(v) \equiv Q_1(v) + Q_2(v)$, and $S(v) \equiv v\,Q(v) - M(v)$. By standard mechanism design techniques, we have that incentive compatibility implies $S(v) = \int_0^v Q(t)\,dt$, all $v \in [0, 1]$.

Now in the γ-equilibrium we have (with the obvious notation):

$$Q(v\,|\gamma) = \begin{cases} 2F(v), & v \in [0, \gamma]; \\ \\ 1 + F(\gamma), & v \in (\gamma, 1]; \end{cases}$$

This is because, if bidder 1 has value $v \leq \gamma$, she buys both objects if her opponent has a lower value, and no object otherwise. Since the probability of the first event is $F(v)$, we have $Q(v\,|\gamma) = 2F(v)$. If instead $v > \gamma$, then the bidder always wins at least one object. Furthermore, if the opponent's value is below γ, she wins the other object too. Thus in this case $Q(v\,|\gamma) = 1 + F(\gamma)$. The associated surplus function is:

$$S(v\,|\,\gamma) = \begin{cases} \int_0^v 2F(t)\,dt, & v \in [0, \gamma]; \\ \\ \int_0^\gamma 2F(t)\,dt + [1 + F(\gamma)](v - \gamma), & v \in (\gamma, 1]. \end{cases}$$

Now let $\Delta(v\,|\,\gamma, h)$ denote the expected surplus that the γ-strategy profile generates for type v in excess of the surplus obtained in the equilibrium of Proposition 1, i.e. $\Delta(v\,|\,\gamma, h) \equiv S(v\,|\,\gamma) - S(v\,|\,h)$. Clearly, the γ-strategy forms an equilibrium if and only if

$$\Delta(v\,|\,\gamma, h) \geq 0 \quad \text{for all } v \in [0, 1], \tag{4}$$

i.e. each type $v \geq \gamma$ is willing to split the objects when the prices reach γ rather than reverting to the noncollusive equilibrium of Proposition 1. Substituting the expressions for $S(v \mid \gamma)$ and $S(v \mid h)$ we have:

$$\Delta(v \mid \gamma, h) = \begin{cases} 0, & v \in [0, \gamma]; \\ [1 + F(\gamma)](v - \gamma) - \int_\gamma^v 2F(t)\, dt, & v \in (\gamma, h); \\ h - \gamma + F(\gamma)(v - \gamma) - F(h)(v - h) - \int_\gamma^v 2F(t)\, dt, & v \in [h, 1]; \end{cases}$$

It is easy to see that $\Delta(\cdot \mid \gamma, h)$ is concave in the interval $[\gamma, 1]$; hence the equilibrium condition in (4) is equivalent to

$$\Delta(1 \mid \gamma, h) \geq 0.$$

It is then sufficient to check that for the highest type it is not profitable to trigger the reversion to the noncollusive strategies, once the prices reach the level γ. We record this conclusion in the next proposition.

Proposition 2 *(Collusive equilibria with known and equal budget levels) A γ-strategy is part of a γ-equilibrium if and only if*

$$V(\gamma) \geq V(h), \tag{5}$$

where

$$V(\gamma) \equiv S(1 \mid \gamma) = \int_0^\gamma 2F(t)\, dt + [1 + F(\gamma)](1 - \gamma).$$

Inequality (5) shows that the set of all γ-equilibria is determined by the budget level w via the function $V(\cdot)$, which in turn depends on the distribution F of the bidders' types. It is easy to see that, depending on the nature of F, decreasing the budget level w may facilitate or hinder collusion in the sense that a given γ-strategy may form an equilibrium for a budget level $w' \leq 2$, but not for a smaller level w''; or viceversa, it may form an equilibrium for w'' and not for w'. This is because a reduction of the budget level may increase or decrease the expected surplus of the highest type in the noncollusive equilibrium, i.e. the derivative

$$V'(h) = (1 - h)\, f(h) - [1 - F(h)] \tag{6}$$

may be positive or negative. Thus the set

$$\Gamma(h) := \{\gamma \in [0, h] \mid V(\gamma) \geq V(h)\}, \tag{7}$$

of all price levels at which the bidders can split the objects in equilibrium may get larger or shrink as h decreases.

To interpret the two terms which make up the derivative $V'(h)$ in (6), consider an increment of the threshold h, say from h to $h+\delta$. This has no effect on the highest type's expected surplus if her opponent's value is below h. Otherwise:

i) if the opponent's value is between h and $h+\delta$, she can now buy two objects instead of one, hence her expected surplus increases by

$$2[1-\phi(\delta)] - (1-h) = 1 - 2\phi(\delta) + h,$$

where $\phi(\delta) := E[v \mid h < v < h+\delta]$;

ii) if the opponent's value is above $h+\delta$, she still buys only one object, but her payment increases by δ.

Multiplying each term by its probability and adding yields the overall change in her expected surplus

$$\Delta V = [F(h+\delta) - F(h)][1 - 2\phi(\delta) + h] - [1 - F(h+\delta)]\,\delta,$$

from which one obtains the derivative in (6) after dividing by δ and taking the limit for $\delta \to 0$, since $\lim_{\delta \to 0} \phi(\delta) = h$. Thus the first term in (6) is due to the additional surplus deriving from the purchase of a second object, while the second term captures the negative effect due to the higher degree of competition. Depending on which of the two effects prevails, a reduction in the bidders' budget level may increase or decrease their surplus in the non-collusive equilibrium, and this in turn restricts or enlarges the set of prices at which the bidders can split the objects in equilibrium.

We end this section with a complete characterization of the equilibrium price correspondence Γ defined in (7) for each of two broad classes of distributions. The first class consists of all distributions with *single-peaked* densities, i.e. such that f is increasing on $[0,x)$ and decreasing on $(x,1]$, for some $x \in [0,1]$.

Proposition 3 *(f single-peaked.) Suppose that there exists a point $x \in [0,1]$ such that $0 < f'(v)$ for all $v \in [0,x)$ and $f'(v) < 0$ for all $v \in (x,1]$. In this case,*

a) *if $f(0) \geq 1$, then $\Gamma(h) = \{h\}$, for each $h \in [0,1]$;*

b) *if instead $f(0) < 1$, there are two sub-cases:*

1. *if $\frac{1}{2} \leq E(v)$, then*

$$\Gamma(h) = \begin{cases} [0,h], & h \in [0,h_0], \\ [0,\widehat{\gamma}(h)] \cup \{h\}, & h \in (h_0,1], \end{cases}$$

where $h_0 := \arg\min_{\gamma \in [0,1]} V(\gamma)$, and the function $\widehat{\gamma} : (h_0,1] \to [0,h_0]$ is defined by the equality $V(\gamma) = V(h)$;

2. *if $E(v) < \frac{1}{2}$, then*

$$\Gamma(h) = \begin{cases} [0,h], & h \in [0,h_0], \\ [0,\widetilde{\gamma}(h)] \cup \{h\}, & h \in (h_0,h_1], \\ \{h\}, & h \in (h_1,1]; \end{cases}$$

where the function $\widetilde{\gamma} : (h_0,h_1] \to [0,h_0]$ is defined by the equality $V(\gamma) = V(h)$, and h_1 is the unique solution of the equation $1 = V(h)$ in $(0,1]$.

To understand the result of Proposition 3, observe that, for any density f, we have

$$V''(\gamma) = (1-\gamma)\, f'(\gamma), \text{ and } V'(1) = 0. \tag{8}$$

Therefore, if f is single-peaked, V is convex on $[0, x)$, and is both concave and increasing on $[x, 1]$. Moreover, we have

$$V'(0) = f(0) - 1. \tag{9}$$

Thus, if $f(0) \geq 1$, V is also increasing on $(0, x)$, hence increasing on $(0, 1)$. In this case, for any $h \in [0, 1]$, the noncollusive strategy generates more expected surplus for highest type than any γ-strategy (with $\gamma \leq h$). Thus $\Gamma(h) = \{h\}$, and is it clear that a lower budget level can only decrease the seller's expected revenue, as the level of competition in the non-collusive equilibrium decreases.

If instead $f(0) < 1$, then V is first decreasing and then increasing, i.e. the minimizer $h_0 := \arg\min_{\gamma \in [0,1]} V(\gamma)$ is positive. Since V is decreasing on $[0, h_0)$, for each $h \in [0, h_0]$ any γ-strategy with $\gamma \leq h$ forms an equilibrium, i.e. $\Gamma(h) = [0, h]$.

For h above h_0, the nature of the equilibrium correspondence Γ depends on whether, without budget constraints, it is an equilibrium for the bidders to split the objects immediately, i.e. on how $V(0)$ compares with $V(1)$. If $E(v) < \frac{1}{2}$, so that

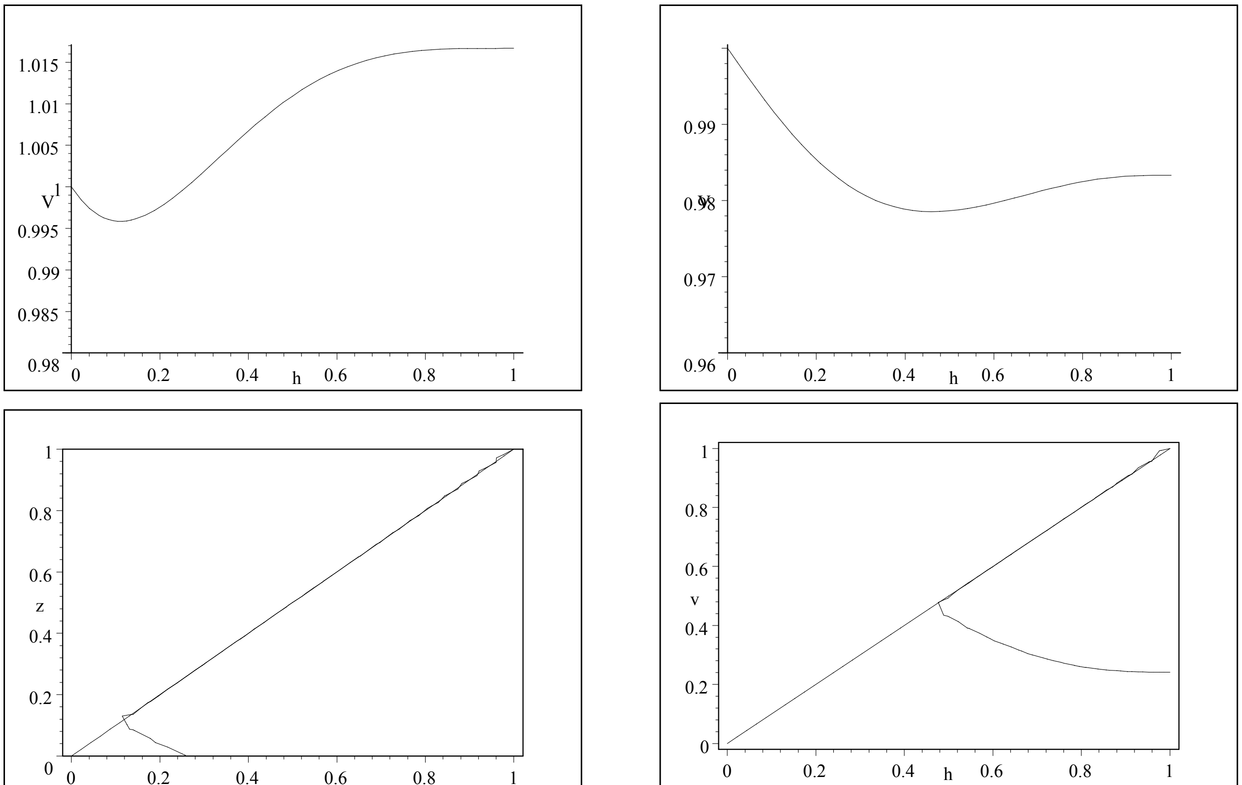
$$1 < 2[1 - E(v)] \Leftrightarrow V(0) < V(1),$$

there exists a unique point $h_1 \in (h_0, 1]$ such that $1 = V(h_1)$. For each $h \in (h_0, h_1]$ there is a unique $\widetilde{\gamma}(h) \in [0, h_0)$ such that $V(\gamma) = V(h)$, thus any γ-strategy with $\gamma \leq \widetilde{\gamma}(h)$ forms an equilibrium, hence $\Gamma(h) = [0, \widetilde{\gamma}(h)] \cup \{h\}$. For $h \in (h_1, 1]$, we have $\Gamma(h) = \{h\}$. In this case, a reduction of the budget level lowers the seller's expected revenue in two possible ways: competition in the non-collusive equilibrium decreases, and collusive equilibria become possible.

Finally, if $\frac{1}{2} \leq E(v)$, so that $V(0) \geq V(1)$, then, for each $h \in (h_0, 1]$ there is a unique $\widehat{\gamma}(h) \in [0, h_0)$, such that $V(\gamma) = V(h)$; thus any γ-strategy with $\gamma \leq \widehat{\gamma}(h)$ forms an equilibrium, hence $\Gamma(h) = [0, \widehat{\gamma}(h)] \cup \{h\}$. In this case immediate collusion is viable — i.e. the 0-strategy forms an equilibrium — for any $h \in [0, 1]$.

In sum, when f is single-peaked lower budget levels can only generate a lower seller's expected revenue.

Figure 1 below provides examples for cases b1 and b2 in Proposition 3. The two graphs above show the function V, for $f(v) = 12v(1 - v)^2$ (left), and $f(v) = v^2(1 - v)$ (right), and the graphs directly below show the associated equilibrium correspondence Γ.



$E(v) < \frac{1}{2}$ $\qquad$ $\frac{1}{2} < E(v)$

f single peaked

The next proposition shows that, if $-f$ is single peaked, i.e. if f is first decreasing and then increasing, tightening the bidders' budget constraints may hinder collusion, either by eliminating all collusive equilibria or by increasing the minimum collusive price,

Proposition 4 *(-f single-peaked.) Suppose that there exists a point* $x \in [0,1]$ *such that* $f'(v) < 0$ *for all* $v \in [0,x)$ *and* $0 < f'(v)$ *for all* $v \in (x,1]$. *In this case,*

a) *if* $f(0) \leq 1$, *then* $\Gamma(h) = [0,h]$, *for each* $h \in [0,1]$;

b) *if instead* $f(0) > 1$, *there are two sub-cases:*

1. *if* $E(v) < \frac{1}{2}$*, then*

$$\Gamma(h) = \begin{cases} [0,h], & h \in [0,h_*], \\ & \\ [0,\widehat{\gamma}(h)] \cup \{h\}, & h \in (h_*,1], \end{cases}$$

   where $h := \arg\max V(\gamma)$, *and the function* $\widehat{\gamma} : (h_*,1] \to [0,h_*]$ *is defined by the equality* $V(\gamma) = V(h)$;

2. *if* $\frac{1}{2} \leq E(v)$, *then*

$$\Gamma(h) = \begin{cases} [0,h], & h \in [0,h_*], \\ & \\ [0,\widetilde{\gamma}(h)] \cup \{h\}, & h \in (h_*,h_1], \\ & \\ \{h\}, & h \in (h_1,1]; \end{cases}$$

   where the function $\widetilde{\gamma} : (h_*,h_1] \to [0,h_*]$ *is defined by the equality* $V(\gamma) = V(h)$, *and* h_1 *is the unique solution of the equation* $1 = V(h)$ *in* $(0,1]$.

The analysis is very similar to the one of the previous proposition, but the shape of the V curve is the opposite: it is first concave and then both convex and decreasing. If $f(0) \leq 1$, then V is always decreasing, and any the objects can be divided at any $\gamma \in [0,h]$, for each $h \in [0,1]$. If instead $f(0) > 1$, collusive equilibria exist only for sufficiently high values of h. In particular, if $\frac{1}{2} \leq E(v)$, the 0-strategy forms an equilibrium if and only if $h \in [h_1,1]$. Thus, if $-f$ is single peaked, tightening the bidders' budget constraints can actually reduce the scope for collusion.

3.3 The Asymmetric Case

Before analyzing the symmetric model with privately known budget levels, it is useful to study the case with commonly known, but different, budget levels. Without loss of generality let $w_2 < w_1$, and define $h_i := \frac{w_i}{2}$, $i = 1,2$. We assume $h_2 < 1$, that is the bidder with the lower budget level may face a binding budget constraint. However, we make no assumptions on h_1. Thus, bidder 1 may or may not be budget constrained, i. e. h_1 may be greater or smaller than 1.

We begin by observing that the noncollusive strategy defined for the symmetric case (Proposition 1) cannot form an equilibrium in this case. In particular, this strategy is not a best reply to itself for some types of bidder 1 above h_2. To see this, suppose that the values of both bidders are above h_2, so that, following the strategy of Proposition 1, the bidders raise both outstanding bids up to h_2. Beyond this point bidder 2 cannot afford to bid on more than one object; but bidder 1 can, and according to the strategy of Proposition 1, she should still try to win both objects, until the bids reach h_1. Under what conditions is it optimal for bidder 1 to do this? To generalize a little bit the analysis, assume that the outstanding bid is $b \geq h_2$ for both objects, and that the bidders are winning one object each. Also assume that we are on the equilibrium path, and that the probability distribution on the opponent's type is $G(v_2|\, b) = \frac{F(v_2)-F(b)}{1-F(b)}$ with support $[b, 1]$. By continuing to bid on both objects up to any level $s \in [b, h_1]$, her expected surplus (conditional on the fact that the outstanding bids have reached b) is:

$$U(\,v_1, s|\, b) \;= 2\int_b^s (v_1 - v_2)\; dG(v_2|\, b) \;+ (v_1 - s)\left[1 - G(s|\, b)\right] \tag{10}$$

In fact, when the current bids for the two objects have reached $b \geq h_2$, bidder 1 knows that the other bidder is content with ending the auction if she is allowed to win one object. Therefore, bidder 1 can buy a single object at the current price. If bidder 1 pushes up the bids to $b+\varepsilon$ then bidder 2 will give up, allowing bidder 1 to win both objects, if $v_2 \in [b, b+\varepsilon]$. But if $v_2 > b+\varepsilon$ then bidder 2 will make a bid on one of the objects, and in this case bidder 1 has increased the price and is still unable to get both objects. In general, by setting a maximum price (or 'stopping time') s bidder 1 wins both objects at price v_2 whenever $v_2 < s$, and one object at price s otherwise.

Taking the derivative with respect to s we have:

$$\frac{\partial U(v_1, s\,|b)}{\partial s} = (v_1 - s)\; G'(s|\, b) \;- \left[1 - G(s|\, b)\right]$$

where the first term denotes the gain coming from the probability of winning an extra object when s is marginally increased, while the second term is the probability of ending up paying more and still win a single object. At an optimal value of s the two effects should balance.[1]

[1] Ignoring corner solutions.

The value of s prescribed by the strategy of Proposition 1 is $\min\{v_1, h_1\}$, so that a type $v_1 \in (h_2, h_1)$ should stop at v_1. But setting $s = v_1$ we have $\frac{\partial U(v_1, v_1 | b)}{\partial s} = -[1 - G(v_1 | b)] < 0$ whenever $v_1 < 1$. Therefore, setting $s = v_1$ cannot possibly be optimal, and in general bidder 1 will want to stop before. The point is pretty simple. Suppose the price for each object has reached the level $v_1 - \varepsilon$, and bidder 1 is winning only one object. Bidding also on the other object does not make sense, since winning two objects at price v_1 gives zero utility, while buying one object at $v_1 - \varepsilon$ gives ε. More in general, it is clear that bidder 1 will have an incentive to stop before the bids get too close to v_1. We conclude that the noncollusive strategy defined in Proposition 1 for the symmetric case does not form an equilibrium if the bidders' budget levels are different.

The problem confronting bidder 1, once the outstanding bids have reached h_2, is similar to a monopsonist's profit maximization problem: the essential feature in both problems is the trade-off between buying a single object for a given price and buying two objects for a higher unit price. In light of this analogy, one should not be surprised to find that the outcome is in general inefficient; i.e. the stopping time s is in general below bidder 1's value v_1, so that whenever bidder 2's value is between s and v_1, the objects are split instead of going both to bidder 1, despite the fact that bidder 1 has money enough to buy both objects even if the prices reach the opponent's valuation. This 'demand reduction' effect has been noted before for other auction formats (see Ausubel and Cramton, 1998) in the absence of budget constraints. However, for the simultaneous ascending bid auction the 'demand reduction' effect only appears when budget constraints are imposed. Absent budget constraints, the 'competitive' equilibrium has no demand reduction.

We now define a strategy profile which generalizes the noncollusive equilibrium to the case of asymmetric budgets. For bidder 2 (the low-budget bidder) the strategy is the one described in Proposition 1. For bidder 1 (the high-budget bidder) we amend the strategy defining the optimal "stopping time" $s_1 : [0,1] \times [h_2, 1] \to [h_2, \min\{1, h_1\}]$ to be used on the equilibrium path.

Suppose that the current pair of bids is (b, b), with $b \geq h_2$. Let $U(v_1, s | b)$ be the function defined in (10), that is the utility obtained by a high-budget bidder who has beliefs given by $F(v_2 | v_2 \geq b)$ on the opponent and sets a 'stopping time' of s. Then we define $r(v_1, b)$ as:

$$r(v_1, b) = \begin{cases} \arg\max_{s \in [b, \min\{1, h_1\}]} U(v_1, s | b) & \text{if } b \leq v_1; \\ \\ \{v_1\}, & \text{if } v_1 < b. \end{cases}$$

and we notice that $r(v_1, b)$ is a compact set for each pair (v_1, b). In the non-trivial case in which the current bids are inferior to v_1, the set $r(v_1, b)$ is the set of optimal stopping times. If the current bids are above v_1 then the only sensible action is to stop immediately, so we set $r(v_1, b) = \{v_1\}$. There may be multiple stopping times, so we have to specify a tie-breaking rule. Since we are looking for the most competitive equilibrium, we select the highest one. Thus, we define the function:

$$s_1(v_1, b) = \max r(v_1, b). \tag{11}$$

The function $s_1(v_1, b)$ has the following shape. First, for each $b \in [h_2, 1]$ the function $s_1(v_1, b)$, is nondecreasing in v_1. Second, consider the function $s_1(v_1, h_2)$, which establishes the optimal stopping time for bidder 1 when h_2 is reached. The interval $[h_2, 1]$ can be partitioned in three subintervals. First, all types in a lower interval $[h_2, v_1']$ let the auction end at h_2. These are types who have a value close to h_2 and prefer to stop as soon as possible, securing one object at the lowest possible price. This lower interval $[h_2, v_1']$ is always nonempty, that is it is always the case that $v_1' > h_2$. The reason is that, since f is bounded, we can find $\delta > 0$ such that, for all $v_1 \in [h_2, h_2 + \delta]$, we have

$$\frac{\partial U(v_1, s\,|h_2)}{\partial s} = \frac{(v_1 - s)\ f(s) - (1 - F(s))}{1 - F(h_2)} < 0, \quad s \in [h_2, \min\{1, h_1\}].$$

At the opposite side of the spectrum, all types in an upper interval $[v_1'', 1]$ push the bids on both objects up to h_1. These are types with a high value for the object, who are willing to pay up to h_1 in order to have the chance of getting them both. Notice that it must be $v_1'' > h_1$, since the optimal stopping time cannot be too close to the value v_1. Thus, a necessary condition for this interval not to be empty is that $h_1 < 1$, so that the budget constraints can be binding also for the high-budget bidder.

All types in the remaining middle interval (v_1', v_1'') have a stopping time which satisfies the first order condition:

$$(v_1 - s)\ f(s)\ - [1 - F(s)] = 0.$$

Notice that only the first interval $[h_2, v_1']$ is surely not empty. In fact, it may well be the case that $v_1' = 1$, so that all types stop at h_2. Also notice that if the middle interval (v_1', v_1'') is non-empty then the stopping function must be strictly increasing in v_1 at all points on this interval such that $f'(s(v_1)) \neq 0$. This follows from differentiating the first order condition:

$$\frac{ds}{dv_1} = -\frac{f(s)}{(v_1 - s) f'(s)}$$

noting that the second order condition requires $(v_1 - s) f'(s) < 0$ at an optimal point.

This completes the analysis of strategic behavior on the equilibrium path. Basically, the low-budget bidder tries to get both objects up to the value $\min\{h_2, v_2\}$. When the bids reach (h_2, h_2) then she adopts a defensive strategy, trying to buy the lowest priced object as long as the price remains below $\min\{w_2, v_2\}$. The high budget bidder follows a similar strategy, except that she tries to get both objects up to an optimally chosen 'stopping bid' $s_1(v_1, h_2)$ (which in general differs from $\min\{h_1, v_1\}$), and tries to get a single object when bids reach that level.

To complete the analysis of the equilibrium, we have now to consider what happens out of the equilibrium path. We will consider the following belief-formation rule: At any pair of bids (b_1, b_2), let $\widehat{v}$ be the highest bid ever made by the opponent at any round. Then beliefs on the opponent i are given by $F(v_i | v_i \geq \widehat{v})$. In general, there is no problem in describing the optimal strategy when a bidder is either winning both objects or losing both. Remember that we are assuming that the opponent always bids on one object at a time, increasing the bid by the minimum amount. In this case, there is no point in bidding on both objects or by more than the minimum amount at any round, so we can limit attention to strategies involving these features. If the bidder is winning both objects than staying silent is always a best reply. If the bidder is losing both objects then staying silent would terminate the auction. This is optimal if both bids are above the value or the budget. If not, that is if $\min\{b_1, b_2\} < \min\{v_i, w_i\}$, then it is a best reply to increase the bid by the minimum amount on the lowest-priced object.

We are therefore left with the task of specifying the strategy when the bidder is winning exactly one object.

Consider bidder 1 (the high-budget bidder) and suppose that beliefs on the opponent are given by $F(v_2 | v_2 \geq \widehat{v})$. Suppose that bidder 1 is currently winning object $3-j$, and she has to decide whether to bid on object j. We first observe that if $b_j \geq \min\{h_1, w_1 - b_{3-j}\}$ then the bidder should stop bidding immediately, and try to get a single object. If $b_j \geq w_1 - b_{3-j}$ this is the only feasible action. If $b_j \geq h_1$ then $\widehat{v} > h_1$; this in turn means that it is impossible to win both objects paying less than h_1, which makes it impossible for bidder 1 to win both objects. Therefore, bidder 1 should stop immediately and get one object.

Assume therefore $b_j < \min\{h_1, w_1 - b_{3-j}\}$, so that bidding on j is feasible and possibly optimal. We want to see what is the optimal stopping price in this situation, for each possible pair of prices (b_j, b_{3-j}). In the analysis we can assume that $b_j \leq \widehat{v}$, since the opponent is winning object j at b_j and $\widehat{v}$ is the highest offer made by bidder j.

Case 1. $b_{3-j} \leq \widehat{v}$. Suppose first $b_{3-j} < b_j$. In this case, as soon as bidder 1 tries to get j, bidder 2 will raise the bid on $3-j$. Therefore, any attempt to get both objects will first increase the price of object $3-j$ to b_j, and then it will increase the bids simultaneously. This creates a discontinuity. By bidding on j, bidder 1 becomes the winner on the highest priced object. The alternative is between winning one object at $s = b_{3-j}$ or both, and in this case it must be $s > \widehat{v}$. The objective function can be written as follows:

$$2\int_{\widehat{v}}^{\max\left\{s,\widehat{v}\right\}} (v_1 - v_2)\, dF\left(v_2 \left| v_2 \geq \widehat{v}\right.\right) + (v_1 - s)\left(1 - F\left(s \left| v_2 \geq \widehat{v}\right.\right)\right)$$

Notice that it is therefore never optimal to choose a stopping time $s \in (b_{3-j}, \widehat{v})$, since such choice is dominated by $s = b_{3-j}$.

Suppose next that $b_j < b_{3-j}$. In this case, any attempt to get object j will only increase the price of object j up to b_{3-j} (remember that we are considering the case $b_{3-j} \leq \widehat{v}$, and bidder 2 will not concede both objects unless both prices reach $\widehat{v}$). After that, the two prices will increase simultaneously. This means that we can assume without loss of generality $b_j = b_{3-j}$ and write the utility function exactly as before. Therefore, the analysis is identical to the sub-case $b_{3-j} < b_j$.

Case 2. $\widehat{v} < b_{3-j}$. Again, there is no hope of getting both objects for a price inferior to $\widehat{v}$. When $s \in [\widehat{v}, b_{3-j})$ then giving up means that 1 pays b_{3-j} for the object she is currently winning, while the other bidder pays s. If $v_2 \in (\widehat{v}, s)$ then 1 wins both objects, paying them v_2 and b_{3-j} respectively, while if $v_2 > s$ then bidder 1 wins a single object and pays it b_{3-j} . If $s \geq b_{3-j}$ then bidder 1 wins both objects paying a total of $v_2 + \max\left\{v_2, b_{3-j}\right\}$ if $v_2 \leq s$, while if $v_2 > s$ then bidder 1 wins a single object which is paid s. We can summarize the objective function as:

$$\int_{\widehat{v}}^{s} \left(2v_1 - v_2 - \max\left\{v_2, b_{3-j}\right\}\right) dF\left(v_2 \left| v_2 \geq \widehat{v}\right.\right) + \left(v_1 - \max\left\{s, b_{3-j}\right\}\right)\left[1 - F\left(s \left| v_2 \geq \widehat{v}\right.\right)\right]$$

The analysis of the two cases can be unified considering the following objective function for bidder 1:

$$U\left(v_1, s \right| b_{3-j}, \widehat{v}) = 2\int_{\widehat{v}}^{\max\left\{s,\widehat{v}\right\}} \left(2v_1 - v_2 - \max\left\{v_2, b_{3-j}\right\}\right) dF\left(v_2 \left| v_2 \geq \widehat{v}\right.\right) +$$

$$+ \left(v_2 - \max\left\{s, b_{3-j}\right\}\right)\left[1 - F\left(s \left| v_2 \geq \widehat{v}\right.\right)\right]$$

At last, we observe that the same reasoning applies when any of the two prices is below h_2. It is clear that in the case $\max\{b_1, b_2\} < h_2$ then setting an optimal stopping time of v_1 is the only sensible thing to do, since the other bidder tries to go for both objects. If $\max\{b_1, b_2\} \geq h_2$, this is enough for the other bidder to stop immediately when winning one object. The best reply to this strategy is to adopt the optimal stopping time as in the case in which both bids are above h_2.

We can now characterize the equilibrium. We first define the optimal stopping time for bidder 1 when she is winning one object as the highest stopping time that maximizes $U_1(v_1, s|\, b_{3-j}, \widehat{v})$. Therefore we define:

$$s_1(v_1; b_{3-j}, \widehat{v}) = \max\left\{\arg \max_{s \in \left[\min\left\{b_{3-j}, \widehat{v}\right\}, \min\{h_1, 1\}\right]} U(v_1, s|\, b_{3-j}, \widehat{v})\right\}. \qquad (12)$$

We notice that the stopping function defined along the equilibrium path is obtained as a special case of $s_1(v_1, b_{3-j}, \widehat{v})$ when $b_{3-j} = \widehat{v}$. Also, to save notation in the statement of the equilibrium strategies, let:

$$s_2(v_2; b_{3-j}, \widehat{v}) \equiv v_2. \qquad (13)$$

We are now prepared to describe the strategy profile.

Proposition 5 *Let $w_2 < w_1$, and let s_1 and s_2 be defined as in (12) and (13). The following strategy pair forms an equilibrium. Let (b_1, b_2) be the pair of outstanding bids at round t. Then at round $t+1$, each type v_i of bidder i raises by ε the outstanding bid:*

- *on object i, if she is not the winner on any object, the current outstanding bids are equal, and*
$$b_i < \min\{v_i,\, w_i\};$$
- *on the object with the lowest outstanding bid, if she is not the winner on any object, the current outstanding bids are different, and*
$$\min\{b_1, b_2\} < \min\{v_i,\, w_i\};$$
- *on object j only, if she is the winner on object $3-j$ only, and if*
$$b_j < \min\{s_i(v_i; b_{3-j}, \widehat{v}),\, h_i,\, w_i - b_{3-j}\},$$
where $\widehat{v}$ is the highest bid ever made on any object by bidder $3-i$.

- *on no object, otherwise.*

The strategy profile described in proposition 5 can be seen as the 'most competitive' equilibrium when there are asymmetric and known budget constraints. The main message of the proposition is that there are additional forces limiting competition besides the pure effect on the ability to pay. In fact, the proposition shows that a non-constrained bidder ($h_1 > 1$) will always have some types behaving in a non-competitive way when facing a budget-constrained opponent. By this we mean that the bidder will stop trying to get both objects before the bids reach the value v_1.

In the next section we will show that this effect plays an important role also in the case of privately known budget. In fact, we will show that the simple *possibility* that a bidder be budget constrained generates a competition-restraining effect similar to the one described in Proposition 5.

4 Privately Known Budgets

We now introduce private information on budget levels. A type of each bidder $i = 1, 2$ is now identified by the pair $(v_i, w_i) \in [0, 1] \times W$. We analyze the simplest model in which each bidder may or may not be budget constrained. Specifically, we assume that for each $i = 1, 2$, the distribution of w_i is independent of all other random variables, has support $W = \{w_L, w_H\}$ with $1 < w_L < 2 < w_H$, and $\Pr[w_i = w_L] := \lambda \in (0, 1)$. We will call any type (v_i, w_i) 'low-budget' or 'budget constrained' if $w_i = w_L$, and 'high-budget' or 'budget unconstrained' if $w_i = w_H$. Notice that we are assuming that even the budget constrained bidder can always bid up to her value for a single object ($w_L > 1$); the assumption is not essential, but it simplifies the analysis.

As in the previous sections, we begin looking for the 'most competitive' equilibrium, i.e. an equilibrium in which bidders come as close as possible to pushing the bids up to their values for the objects. One thing that is immediately clear is that there is no equilibrium in which the budget unconstrained bidder always bids up to the value of each object. That is, even in the most competitive equilibrium some 'demand reduction' occurs.

To see this, assume that all unconstrained bidders behave competitively and push up bids up to the value. Let $h_L := \frac{w_L}{2}$, and suppose that bids have reached the level (h_L, h_L) with each bidder winning one object. At this point, low-budget bidders are happy to terminate the auction, and do not bid on the objects they are not winning. Consider now a high budget bidder of type $h_L + \varepsilon$. If this type does not bid, the auction stops with probability λ; she wins one object, and the expected utility is $\lambda\varepsilon$. This is an expected utility surely higher than what would be obtained trying to buy both objects. Therefore, at least some high-budget types must stop bidding before their value is reached. In fact, what we observe is a *multiplier effect*. If some high-budget types stop bidding and accept sharing the object at h_L, then the expected utility of stopping at h_L is even greater. This in turn convinces more high-budget types to accept splitting the objects at h_L. Eventually, a threshold value $v_* > h_L$ is determined, and in equilibrium all high-budget types $v_1 < v_*$ accept to split the object at h_L. The value of the threshold v_* depends on the distribution F and on the probability of facing a low budget opponent λ. However, it turns out that even as λ goes to zero, the value of v_* remains bounded away from h_L. Therefore, even if the probability of facing a budget constrained opponent is vanishing, there will be first-order effects on expected welfare and the expected revenue for the seller. In some cases the effects are quite dramatic, as we will shortly see analyzing the uniform distribution case.

Before providing a formal definition of the equilibrium strategy, we describe the bidders' behavior on the equilibrium path. The auction starts with both outstanding bids increasing at the same pace, up to the minimum among the bidders' values and the threshold. More precisely, for each $i = 1, 2$, bidder i increases the outstanding bid, by the minimum increment ε, on object i in any odd round, and on object $3 - i$ in any even round, up to $\min\{v_i, h_L\}$. The auction progresses with each bidder being the potential winner of one object in each round, until the outstanding bids reach either the lowest of the bidders' values, or h_L. In the first case, the bidder with the lowest value stays silent, and the auction ends with her opponent winning both objects. Otherwise, i.e. if $h_L < \min\{v_1, v_2\}$, the behavior of each bidder depends on whether her type is "tough," i.e. high-budget and with value above a threshold v_* (which is strictly above h_L), or "soft," i.e. either low-budget, or high-budget and with value between h_L and v_*. All soft types stay silent. All tough types instead continue to raise the outstanding bid on any object which is assigned to the opponent, up to a threshold which depends on the opponent's behavior.

Thus, if both bidders are soft, they stay silent for another round and the auction ends with each bidder buying one object and paying h_L. If both bidders are tough, the bidding continues as in the initial phase and the bidder with the highest value wins both objects. Finally, if one bidder is tough and the other is soft, the soft bidder starts to bid "defensively," i.e. she bids on one of the objects with the lowest current outstanding bid if she is losing both objects, and stays silent otherwise; while the tough bidder tries to win both objects until the bids reach an optimally chosen threshold. The auction then ends with the tough bidder buying both objects if her threshold is above her opponent's value, and with the bidders splitting the objects otherwise.

We now provide a formal definition of the strategy just described, i.e. a specification of each bidder's behavior at any information set. As in the previous case, we have to define the "stopping" function $s(v_i, w_i, b_1, b_2)$ which determines the highest price that a bidder of type (v_i, w_i) who has observed a pair of bids (b_1, b_2) is willing to pay in order to get both objects. We will see that, except for the definition of the function s, the strategies involved in our equilibrium are identical to the strategies defined in Proposition 5. However, the way in which the function s is determined in the case of privately known budgets is conceptually different from what we have seen in the previous section.

When there is no private information on budgets, a high-budget bidder faces a straightforward single person decision problem, as discussed in the previous section. When there is incomplete information on budgets, the optimal stopping time depends on the type and behavior of the opponent. It is clear that a low-budget opponent will bid defensively, but the behavior of a high-budget opponent will depend on her beliefs. In fact, some types of a high-budget opponent always decide to mimic the behavior of low-budget bidders. The bottom line of this reasoning is that bidder i needs to formulate a conjecture on the stopping function of the other bidder in order to compute her own stopping function. Therefore, the stopping functions have to be computed as the solution to a fixed point problem, rather than as the solution to a single-person decision problem.

We start computing the optimal stopping function on the equilibrium path, and in particular when the bids reach the level (h_L, h_L). For simplicity of exposition, we will analyze the behavior of bidder 1.

Suppose that the current bids are (b, b), with $b > h_L$. We assume that bidder 1 played 'tough' when the bids reached the level (h_L, h_L), meaning that he made an offer for $h_L + \varepsilon$ on the object she was not winning, while the opponent played 'soft', that is she was silent at the round after the bids reached (h_L, h_L). In this case the belief of bidder 2 on bidder 1 is $\Pr(w_H = 1)$, while the c.d.f. on the types is determined by the optimal stopping strategy.

The beliefs of player 1 about the opponent's budget are as follows:

$$\Pr(w_L|\,\text{soft}) = \frac{\lambda[1-F(b)]}{\lambda[1-F(b)]+(1-\lambda)\max\{F(v_*)-F(b),0\}}$$

$$\Pr(w_H|\,\text{soft}) = \frac{(1-\lambda)\max\{F(v_*)-F(b),0\}}{\lambda[1-F(b)]+(1-\lambda)\max\{F(v_*)-F(b),0\}}$$

and the conditional densities are:

$$f(v_2|\,v_2\geq b, w_L) = \begin{cases} \frac{f(v)}{1-F(b)} & v_2\in[b,1] \\ 0 & \text{otherwise.} \end{cases}$$

$$f(v_2|\,v_2\geq b, w_H) = \begin{cases} \frac{f(v)}{F(v_*)-F(b)} & v_2\in[\min\{b,v_*\},v_*] \\ 0 & \text{otherwise.} \end{cases}$$

The expected utility of a stopping time s is therefore:

$$U(v_1,s;v_*,b) =$$

$$\Pr(w_L|\,\text{soft})\left[2\int_b^s (v_1-v_2)\,f(v_2|\,b,w_L)\,dv_2+(v_1-s)(1-F(s|\,b,w_L))\right]+$$

$$\Pr(w_H|\,\text{soft})\left[2\int_b^{\min\{s,v_*\}} (v_1-v_2)\,f(v_2|\,b,w_H)\,dv_2+(v_1-s)(1-F(s|\,b,w_H))\right]$$

After substitutions and ignoring multiplicative constants we can write the objective function as:

- When $b<v_*$:

$$U(v_1,s;v_*,b)=\lambda\left[2\int_b^s (v_1-v_2)\,f(v_2)\,dv_2+(v_1-s)(1-F(s))\right]$$
$$+(1-\lambda)\left[2\int_b^{\min\{s,v_*\}}(v_1-v_2)\,f(v_2)\,dv_2+(v_1-s)(F(v_*)-F(\min\{s,v_*\}))\right]$$

- When $b\geq v_*$:

$$U(v_1,s;v_*,b)=2\int_b^s (v_1-v_2)\,f(v_2)\,dv_2+(v_1-s)(1-F(s))$$

Define the 'best stopping time' correspondence as:

$$r(v_1;b,v_*)=\arg\max_s U(v_1,s;v_*,b).$$

We will be interested in the 'most competitive' equilibrium, so that we will focus on the 'highest' stopping time function:

$$s(v_1;b,v_*)=\max r(v_1;b,v_*).$$

Before proceeding to a more general analysis, we will examine the uniform case.

4.1 The Uniform Case

Suppose that types are uniformly distributed, that is $F(v) = v$ for $v \in [0,1]$. In this case, when the bids reach the level (h_L, h_L), and the threshold value is $v_* \in (h_L, 1]$, the problem of the unconstrained bidder facing a soft opponent is:

$$\max_s \lambda \left[2\int_{h_L}^{s} (v_1 - v_2)\, dv_2 + (v_1 - s)(1 - s) \right]$$

$$+ (1-\lambda) \left[2\int_{h_L}^{\min\{s, v_*\}} (v_1 - v_2)\, dv_2 + (v_1 - s)(v_* - \min\{s, v_*\}) \right]$$

We want to prove that, for the uniform case, the only threshold value v_* which can possibly be part of an equilibrium is $v_* = 1$. Showing that a threshold value v_* is part of an equilibrium involves showing that all high-budget types $v_1 < v_*$ prefer to play 'soft' when the bids reach (h_L, h_L), while all types $v_1 \geq v_*$ prefer to play 'tough'. We will show that when $v_* < 1$ then the type v_* strictly prefers to play 'soft', a contradiction.

Thus, suppose $v_* < 1$ and consider type $v_1 = v_*$. The derivative of the objective function for $s < v_*$ is:

$$(v_* - s) - (\lambda + (1-\lambda) v_* - s) = -\lambda (1 - v_*) < 0$$

so that the optimal stopping time is $s = h_L$. This clearly remains true for types $v_* + \delta'$, with δ' small enough. Therefore, a set of types $(v_*, v_* + \delta)$, with $\delta > 0$, will choose a stopping time of h_L. Consider now the utility of playing tough and of playing soft for a type $v_1 \in (v_*, v_* + \delta)$. Playing soft gives at least:

$$U(v_1, \text{soft}) \geq (\lambda + (1-\lambda) F(v_* + \delta)) (v_1 - h_L)$$

where $(\lambda + (1-\lambda) F(v_* + \delta))$ is the probability that the opponent will also play soft or that will play tough but give up immediately after. Opening tough gives:

$$U(v_1, \text{tough}) = (\lambda + (1-\lambda) F(v_*)) (v_1 - h_L) + 2(1-\lambda) \int_{v_*}^{v_1} (v_1 - v_2)\, dv_2$$

It is clear that when F is strictly increasing and δ strictly positive then

$$U(v_1, \text{soft}) > U(v_1, \text{tough})$$

when v_1 is sufficiently close to v_*. This is a contradiction, since we have found a type $v_1 > v_*$ who prefers to play soft. We conclude that the only possible candidate for an equilibrium is $v_* = 1$.

In fact, we can readily check that this is an equilibrium. If $v_* = 1$ then the expected utility of playing soft is $(v_1 - h_L)$. Playing tough is now an out of equilibrium action. Let us assume that, faced with a tough bidder, an opponent plays defensively, trying to get one object (this is optimal for beliefs assigning high probability to high values of the opponent). Then the highest utility which can be obtained by opening tough is obtained solving:

$$\max_s \left[2 \int_{h_L}^{s} (v_1 - v_2)\, dv_2 + (v_1 - s)(1 - s) \right]$$

The optimal value of s is h_L whenever $v_1 < 1$. Therefore, deviating to 'tough' is not profitable.

The equilibrium has the remarkable property that it does not depend on λ, the fraction of budget-constrained players. That is, for *any* $\lambda > 0$ the most competitive equilibrium has all the high-budget bidders mimicking the low-budget bidders when the bids reach (h_L, h_L). This implies a discontinuity in the equilibrium set. When $\lambda = 0$ then a 'competitive' equilibrium exists in which each bidder pushes up the bid on each object up to their value. However, for any $\lambda > 0$ this equilibrium disappears, and it becomes impossible to induce competition among bidders at prices higher than h_L.

4.2 The General Case

In order to complete the analysis we have to accomplish two tasks. First, we have to show that a threshold value v_* exists, i.e. we have to show that a fixed point exists. That is, it must be true that when bidder 1 conjectures a threshold value v_* for the opponent, then all types $v_1 < v_*$ are willing to play soft and all types $v_1 > v_*$ are willing to play tough. As we will see, this requires some further assumptions on the environment. Second, we have to describe the out of equilibrium behavior.

For the moment, let it be taken for granted that a threshold value v_* exists, so that an optimal stopping function $s(v_i, w_i; v_*)$ can be computed. We now proceed to generalize the bidding behavior for any arbitrary pair (b_1, b_2). There are two simple cases.

1. $\max\{b_1, b_2\} < h_L$. In this case we set $s = v_i$, since whenever prices are below h_L the strategies are as in the standard 'competitive' equilibrium.

2. $\max\{b_1, b_2\} \geq h_L$ <u>and</u> (b_1, b_2) can be reached on the equilibrium path. In this case the beliefs are updated using the Bayes' rule. The stopping rule for bidders with

$v_i \geq v_*$ remains the same. Those with $v_i < v_*$ play defensively and have no interest in triggering the 'competitive' equilibrium, since the opponent has a type higher than their own.

We now consider the beliefs and strategies when $\max\{b_1, b_2\} \geq h_L$ and (b_1, b_2) is out of the equilibrium path. We assume that whenever a bidder observes the other deviating then she puts probability 1 on $w_i = w_H$, and this belief is maintained in case further deviations are observed.

There are 3 cases to be distinguished, depending on how many bidders have deviated.

a) Both bidders deviated from the prescribed strategy. In this case both bidders assign probability 1 to the fact that the other bidder has a high budget, and this fact is common knowledge. In this case the 'competitive' equilibrium is triggered. Therefore we set $s(v_i, w_H, b_1, b_2) = v_i$.

b) If the other bidder deviated then bidder i assigns probability 1 to $v_{3-i} = \widehat{b}$, where $\widehat{b}$ is the highest bid ever made by bidder $3-i$, and assumes that bidder $3-i$ will never make a bid on any object if she becomes convinced that the type of the other bidder is higher. Since by making a bid on the other object bidder i signals that her type is greater than $\widehat{b}$ (remember that $\max\{b_1, b_2\} \geq \widehat{b}$ and i bids on both objects) then it is rational for i to bid myopically on both objects, i.e. assuming that the other bidder will not make any further bid. This in turn justifies a myopic behavior on the part of bidder $3-i$. Notice that this cannot make a deviation profitable, since by deviating bidder $3-i$ only obtains a more aggressive behavior on the part of bidder i. We can therefore set $s(v_i, w_H, b_1, b_2) = v_i$ in this case as well.

c) The last case we have to deal with is the one in which a deviation occurred only on part of agent i. Since $\max\{b_1, b_2\} > h_L$ any counterbid by $3-i$ signals that she is of type w_H and $v_{3-i} \geq \widehat{b}(i)$, where $\widehat{b}(i)$ is the highest bid ever made by i up to that round. Also, in that case bidder $3-i$ starts bidding myopically. Then bidding myopically is a best reply on part of agent i. The conclusion is that in this case as well we can set $s(v_i, w_H, b_1, b_2) = v_i$.

The bottom line is that out of the equilibrium path the bidders raise the bids whenever the value of the object is superior to their current bids. Along the equilibrium path, the bidders adopt optimal stopping times.

We now come to the issue of the existence of a threshold value v_*. Let $\mu(v_1) = \frac{1-F(v_1)}{f(v_1)}$ be the inverse hazard rate. We make the following assumption.

Assumption 1 *For each $v_1 > h_L$, we have:*

$$2f(v_1) \geq f(v_1 + \mu(v_1))(1 + \mu'(v_1))$$

whenever $v_1 + \mu(v_1) < 1$ and $1 + \mu'(v_1) > 0$.

The assumption is immediately satisfied when either $v_1 + \mu(v_1) \geq 1$ or $1 + \mu'(v_1) \leq 0$ for each $v_1 \geq h_L$. This happens, for example, in the uniform case. Another example of a distribution satisfying assumption 1 is the truncated negative exponential:

$$f(v_1) = \begin{cases} \frac{\lambda e^{-\lambda v_1}}{1-e^{-\lambda}} & \text{if } v_1 \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

In this case we have:

$$\mu(v_1) = \frac{1 - e^{\lambda(v_1 - 1)}}{\lambda}$$

so that:

$$v_1 + \mu(v_1) = v_1 + \frac{1 - e^{\lambda(v_1 - 1)}}{\lambda} \geq 1$$

for each $\lambda > 0$ and $v_1 \in [0,1]$

More in general, consider distributions such that $\mu'(v_1) \leq 0$ (monotone increasing hazard rate). Then a sufficient condition for assumption 1 to hold becomes that for all $v_1 > h_L$ we have:

$$2f(v_1) \geq f(x) \text{ for each } x > v_1.$$

The condition is satisfied for distribution without large peaks. In particular, if $v_{\min}$ and $v_{\max}$ are respectively the points at which the density achieves the maximum and the minimum over the interval $[h_L, 1]$ then a sufficient condition is:

$$2f(v_{\min}) \geq f(v_{\max}).$$

The next proposition characterizes the noncollusive equilibrium.

Proposition 6 *Assume that assumption 1 is satisfied. Then there exists a value v_* and a corresponding stopping function $s^*(v_i, w_i, b_1, b_2)$ such that the following strategy profile is an equilibrium. At any stage in which the current outstanding bids are b_1 and b_2, each type (v_i, w_i) of bidder i increases the bid by the minimum increment:*

- *on object i, if she is not the winner on any object, the current outstanding bids are equal, and*
$$b_i < \min\{v_i, w_i\};$$
- *on the object with the lowest outstanding bid, if she is not the winner on any object, the current outstanding bids are different, and*
$$\min\{b_1, b_2\} < \min\{v_i, w_i\};$$
- *on object j, if she is winning object $3-j$ only, and*
$$b_j < \min\left\{s(v_i, w_i, b_1, b_2), \frac{w_i}{2}, w_i - b_{3-j}\right\};$$
- *on no object, otherwise.*

As it is apparent from the definition, the equilibrium with symmetric bidders and private information on budget constraints is similar to the equilibrium with known and asymmetric budgets. In both cases a 'demand reduction' effect is present.

5 Conclusions

[TO BE WRITTEN]

Appendix

Proof of Proposition 1. We establish the proposition by showing that, if bidder 2 is following the prescribed strategy, then bidder 1 has no profitable deviation. We start by observing that it is never optimal for bidder 1 to raise the bid on any object by more than the minimum increment. This is because bidder 2's behavior at any stage depends only on the current outstanding bids and the assignment of the objects; it does not depend on how the current outstanding bids and assignment have been reached. Therefore jump-bidding can never be optimal for bidder 1. In fact, we can restrict attention to strategies such that at any round bidder 1 raises the bid by the minimum increment on at most one object.

We first show that, on the equilibrium path, the strategies are optimal even ex-post, i.e. for any realization of the bidders' values v_1, v_2. We partition the type space in three subsets. Case 1: $h \leq \min\{v_1, v_2\}$. If both bidders follow the prescribed strategy, each buys one object and pays h. Because of the budget constraint, bidder 1 can win both objects only if at least one price is below h; but in this case bidder 2 would not let the auction end before pushing each price above h. Also, if each bidder gets one object, say bidder i pays p_i for object i, then we must have that $h \leq p_1$. To see this, observe that since bidder 2 is following the prescribed strategy and the auction ends at (p_1, p_2) assigning object 2 to bidder 2 and object 1 to bidder 1, we must have $\min\{h, w - p_2\} < p_1$; if not, bidder 2 would bid on object 1 as well and the auction would not be over. If $p_1 < h$ then this implies $w - p_2 < h$, i.e. $h < p_2$. This is impossible because, by following the prescribed strategy, bidder 2 cannot push the bid on object 2 above h, unless the bid on the first object goes above h as well.
Case 2: $v_2 < \min\{h, v_1\}$. By following the equilibrium strategy, bidder 1 obtains both objects at the price v_2. Given the strategy of bidder 2, it is impossible for bidder 1 to win any object for a unit price lower than v_2.
Case 3: $v_1 < \min\{h, v_2\}$, or $v_1 = v_2 < h$. By following the prescribed strategy bidder 1 obtains a utility of zero. Whatever the deviation, it is impossible to buy an object for less than v_2, thus implying that a utility zero is the maximum which can be attained.

We have proved that no deviation is profitable along the equilibrium path. We next show that the strategy is optimal also out of the equilibrium path.

Thus, suppose now that bidder 1 is facing an out-of-equilibrium pair of bids (b_1, b_2) and allocation of objects. If bidder 1 is not winning any object, then it is optimal to stop if $\min\{b_1, b_2\} \geq \min\{v_1, w\}$. If the reverse inequality is true, and given bidder 2's strategy, then it is optimal to make a bid on the object with the lowest value. This is true for any belief on v_2. Similarly, if bidder 1 is winning both objects it is optimal to avoid increasing the bids.

We are left with the case in which bidder 1 is winning object $3 - j$ but not object j. If $b_j \geq w - b_{3-j}$ then bidder 1 cannot bid on object j, and given the strategy of agent 2 it doesn't make sense to increase the bid on object $3 - j$. Similarly, if $b_j \geq v_1$ then bidding on object j is suboptimal. We conclude that it is optimal to stop bidding whenever $b_j \geq \min\{v_1, w - b_{3-j}\}$. Consider now the case $b_j < \min\{v_1, w - b_{3-j}\}$. If $b_j \geq h$ then bidder 1 believes that the type of agent 2 is in the interval $[b_j, 1]$. Any attempt to buy object j will cause bidder 2 to bid on object b_{3-j} until the price reaches at least h. Therefore bidder 1 cannot get both objects, and it is optimal to stop bidding. Summing up, it is optimal to stop bidding whenever $b_j \geq \min\{v_1, w - b_{3-j}, h\}$. The last case we have to deal with is $b_j < h \leq \min\{v_1, w - b_{3-j}\}$. Notice that this implies $w - b_{3-j} \geq h$, or $h \geq b_{3-j}$.

If $v_2 > h$ then bidder 2 will bid on object $3 - j$ as well and the final outcome is that each bidder wins one object at h. Increasing the bid on j is therefore weakly optimal. If $v_2 \in (b_j, h)$ then we distinguish two cases.

If $b_{3-j} \geq v_2$ then bidder 2 will not bid on $3 - j$ any longer, and it will bid up to v_2 for object j. Therefore it is optimal for bidder 1 to try to win object j increasing the bid.

If $b_{3-j} < v_2$ then bidder 2 will try to win both objects, always bidding on any object she is not currently winning. Therefore, no object can be bought for less than v_2. The prescribed strategy allows bidder 1 to buy both objects at the lowest possible price, given the strategy of bidder 2. ■

Proof of Proposition 2. A γ-equilibrium can be sustained if $\Delta(v \mid \gamma, h) \geq 0$ for each $v \in [\gamma, 1]$. Substituting the expressions for $S(v; \gamma)$ and $S(v; h)$ we have:

$$\Delta(v \mid \gamma, h) = \begin{cases} [1 + F(\gamma)](v - \gamma) - \int_\gamma^v 2F(t)\, dt, & v \in [\gamma, h); \\ h - \gamma + F(h)(h - v) + F(\gamma)(v - \gamma) - \int_\gamma^v 2F(t)\, dt & v \in [h, 1]; \end{cases}$$

The function $\Delta(\cdot \mid \gamma, h)$ is continuous, increasing in the interval $(\gamma, \gamma + \delta)$ for any sufficiently small $\delta > 0$, and concave in the interval $[\gamma, 1]$. This implies that either Δ is always increasing on the interval or it is first increasing and then decreasing. Since $\Delta(\gamma \mid \gamma, h) = 0$, we have that $\Delta(1 \mid \gamma, h) \geq 0$ implies $\Delta(v \mid \gamma, h) \geq 0$ for all $v \in [\gamma, 1]$. Thus a given γ-strategy forms an equilibrium if and only if it is not profitable for the highest type of each bidder to trigger the noncollusive strategies, that is $V(\gamma) \geq V(h)$. ■

Proof of Proposition 5. As in the proof of Proposition 1, in order to prove that a bidder's strategy is a best response, we can restrict attention to strategies in which at each round a bid is made only on (one of) the lowest priced object. Optimality for agent 2 basically follows from the arguments used in Proposition 1. It suffices to observe that the strategy is a best reply whenever bidder 1 adopts a 'stopping time' function $s(v_1)$ such that $s(v_1) = v_1$ when $v_1 \leq h_2$.

Assume now that bidder 2 is following the prescribed strategy. The proof that bidder 1's strategy is optimal at any information set is broken in two steps. First, without loss of generality we restrict attention to pure strategies[2], and show that for any type v_1, and for any pair of outstanding bids $(b_1,\ b_2)$, given that bidder 2 is following the noncollusive strategy, any strategy of bidder 1 induces a partition of bidder 2's type space $[0, 1]$ in at most three subintervals: all types with value v_2 below a given threshold τ_L lose both objects, all types with value v_2 above a second threshold τ_H win both objects, and all types with value v_2 between τ_L and τ_H win one object.

The second step consists in showing that the prescribed strategy maximizes bidder 1's expected surplus over all outcomes such that the objects are assigned according to a three-interval partition as specified above and bidder 1's total payments are the lowest among the ones which can be obtained given bidder 2's strategy.

The strategy prescribed for bidder 1 is clearly optimal when she is winning both objects or no object. In the second step we therefore deal only with the case in which bidder 1 is winning only one object.

Step 1. Consider any round t and any pair of outstanding bids $b_1$and b_2. Suppose that, given both bidder's (pure) strategies, the outcome entails bidder 2 losing both objects. Since bidder 2 is following the noncollusive strategy, this can happen only if both final outstanding bids are at least as large as v_2. But then any type of bidder 2 with a lower value $v_2' < v_2$ also loses both objects. We conclude that the set of types of bidder 2 which lose both objects is an interval with infimum zero. Let τ_L denote its supremum. Notice also that bidder 1's total payment in this case cannot be lower than $\max\{v_2, b_1\} + \max\{v_2, b_2\}$, since the final price of each object j cannot be lower than $\max\{v_2, b_j\}$.

Now suppose that, given both bidder's (pure) strategies, bidder 1 loses both objects. This can happen only if bidder 2's value v_2 is above both final bids. But then all types with value $v_2' > v_2$ behave identically, and win both objects. The set of types of bidder 2 which win both objects is thus also an interval, this time with supremum 1. Let τ_H denote its infimum. Bidder 1's total payment in this case is zero.

[2]Recall that we are only looking for a best reply for bidder 1 to bidder 2's strategy. Therefore if the best reply correspondence includes a mixed strategy, any pure strategy in its support is also optimal.

All types in the remaining middle interval, with value between τ_L and τ_H, must win only one object. Bidder 1's total payment in this case is at least $\min\{b_1, b_2\}$.

Step 2. Suppose bidder 1 is winning object $3-j$ at b_{3-j}, and the other object has price b_j. We first observe that the belief at round $t+1$ that bidder 2's type is distributed according to $F(v_2 | v_2 \geq \widehat{v})$, where $\widehat{v}$ is the highest bid ever made on any object in the t preceding rounds, is consistent both on and off the equilibrium path. Given this belief and the fact that the types of bidder 2 can be divided in intervals as described in Step 1, the optimal strategy of bidder 1 must involve an optimal stopping time. The function $s_1(v_1, b_{3-j}, \widehat{v})$ is clearly optimal. ■

Proof of Proposition 6. In order to prove that an equilibrium exists, we have to prove that it is possible to find a type $v_* \in [h_L, 1]$ such that all types $v_1 \in [h_L, v_*)$ prefer to play soft when prices reach h_L, while all types $[v_*, 1]$ prefer to play tough. We start establishing a few results.

Lemma 1 *Fix a pair $\left(\overline{b}, \overline{v}_*\right)$. The correspondence $r\left(v_1; \overline{b}, \overline{v}_*\right)$ is upper hemicontinuous and compact valued. Similarly, fix a pair $\left(\overline{v}_1, \overline{b}\right)$. Then the correspondence $r\left(\overline{v}_1, \overline{b}; v_*\right)$ is upper hemicontinuous and compact valued. It is always the case that if $s \in s(v_1; b, v_*)$ then $s < v_1$ whenever $v_1 > b$.*

Proof. The first part comes from the Maximum Theorem and continuity of $U(v_1, s; v_*, b)$. It is obvious that $s(v_1; b, v_*) \leq v_1$. To see that $s(v_1; b, v_*) < v_1$ whenever $v_1 > b$ observe that $\left.\frac{\partial U(v_1,s;v_*,b)}{\partial s}\right|_{s=v_1} < 0$ for each $v_1 < 1$. ■

For each given pair (b, v_*) define now the single valued function:

$$s(v_1; b, v_*) = \max \qquad r(v_1; b, v_*) .$$

We will make use of the following result.

Lemma 2 *Fix a pair (b, v_*). Then the function $s(v_1; b, v_*)$ is non-decreasing. Let $K \subset [b, 1]$ be the set of points of discontinuity of $s(v_1; b, v_*)$ and let ψ be a measure defined on $[b, 1]$ which is absolutely continuous with respect to the Lebesgue measure. Then $\psi(K) = 0$.*

Proof. The function $U(v_1, s, v_*, b)$ satisfies increasing differences in $(s; v_1)$, and this implies that $s(v_1; b, v_*)$ is non-decreasing in v_1. A non-decreasing function defined on a compact set has at most countably many points of discontinuity (Kolmogorov and Fomin, page 316, Theorem 3), and a countable set has Lebesgue measure zero. ■

Take $b = h_L$ and consider a bidder of type $v_1 \in [h_L, 1]$. Take a value v_* and assume that bidder 1 conjectures that a high-budget opponent will play soft when the type is below v_* and tough when the type is above v_*. Let $G(v_2) = F(v_2 | v_2 \geq h_L)$.

We first compute the payoff of playing tough against a budget constrained opponent. If a stopping time s is chosen then the payoff is:

$$T_L(v_1; s, v_*) = 2\int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)(1 - G(s))$$

If the opponent is not budget constrained, then with probability $(1 - G(v_*))$ she also opens tough. In this case the two bidders play a standard English auction on both objects, with payoff $2\max\{v_1 - v_2, 0\}$. If the opponent opens soft then the utility is $2(v_1 - v_2)$ if $v_2 < s$ and $v_1 - s$ otherwise. We can summarize the expected payoff for this case as:

$$T_H(v_1; s, v_*) = 2\int_{h_L}^{\min\{v_*, s\}} (v_1 - v_2)\, dG(v_2) + \int_{\min\{v_*, s\}}^{v_*} (v_1 - s)\, dG(v_2)$$
$$+2\int_{v_*}^{1} \max\{v_1 - v_2, 0\}\, dG(v_2)$$

Then the expected payoff of opening tough and then selecting a stopping time s is:

$$T(v_1; s, v_*) = [\lambda T_L(v_1; s, v_*) + (1 - \lambda) T_H(v_1; s, v_*)].$$

Define:

$$T(v_1; v_*) = \max_{s \in [h_L, 1]} [\lambda T_L(v_1; s, v_*) + (1 - \lambda) T_H(v_1; s, v_*)]$$

and:

$$T(v_*) = T(v_*; v_*)$$

The function $T(v_*)$ gives the expected utility of opening tough for a type v_* when it is conjectured that the opponent opens tough when $v_2 > v_*$ and the budget is high.

Lemma 3 *The function $T(v_*)$ is continuous.*

Proof. This follows from the Maximum Theorem and the fact that the function $\lambda T_L(v_1; s, v_*) + (1 - \lambda) T_H(v_1; s, v_*)$ is continuous in s and v_*. Furthermore, $T(v_1; v_*)$ is continuous in v_1. Therefore $T(v_*)$ is continuous. ■

Consider now the expected outcome of playing soft. If the opponent is budget constrained she will also play soft, and one object is won at h_L, so that utility is $(v_1 - h_L)$. The same happens if the opponent is high budget but her type is lower than v_*. The probability of playing against a soft type is $\lambda + (1-\lambda) G(v_*)$.

Now suppose that the opponent opens tough, which happens when her opponent has high budget and her type is above v_*. In this case the opponent increases the bids up to $s(v_2; h_L, v_*)$. Since bidder 1 has opened soft, she does not bid when winning one object. Then, when the opponent stops, the game ends and each bidder is allocated one object. Our equilibrium specifies that if, after having opened soft, a bidder tries to get both objects then the opponent reacts behaving as in a standard English auction equilibrium and trying to buy both objects.

Therefore, when playing against a tough opponent, the expected utility for a player of type $v_1 \leq v_*$ is:

$$S_H(v_1; v_*) = \int_{v_*}^{1} \max\{v_1 - s(v_2; v_*), 0\} \, dG(v_2)$$

The expected payoff of opening soft is therefore:

$$S(v_1; v_*) = (\lambda + (1-\lambda) G(v_*)) (v_1 - h_L) + (1-\lambda) S_H(v_1; v_*)$$

Also define:

$$S(v_*) = S(v_*; v_*)$$

Lemma 4 *The function $S(v_*)$ is continuous.*

Proof. The only problem is to show that $S_H(v_*) := S_H(v_*; v_*)$ is continuous. Let $S_H(v_*) = S_H(v_*; v_*)$ and $\widehat{s}(v_2; v_*) = \min\{s(v_2; v_*), v_*\}$. Then we can write::

$$S_H(v_*) = v_*(1 - G(v_*)) - \int_{v_*}^{1} \widehat{s}(v_2; v_*) \, dG(v_2)$$

In order to prove the continuity of S_H it is enough to prove the continuity of the function:

$$H(v_*) = \int_{v_*}^{1} \widehat{s}(v_2; v_*) \, dG(v_2)$$

We want to prove that $\lim_{n\to\infty} H(v^n) = H(v_*)$ whenever $v^n \to v_*$. For simplicity, take an increasing sequence $v^n \uparrow v_*$ (the proof for a generic sequence is identical with minor notational adjustments). Observe:

$$H(v^n) - H(v_*) = \int_{v^n}^{1} \widehat{s}(v_2; v_n) \, dG(v_2) - \int_{v_*}^{1} \widehat{s}(v_2; v_*) \, dG(v_2)$$

$$= \int_{v^n}^{v_*} (\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*))\, dG(v_2) + \int_{v_*}^{1} (\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*))\, dG(v_2)$$

Since the quantity $(\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*))$ is bounded and $G(v_2)$ is atomless, we have:

$$\lim_{n\to\infty} \int_{v^n}^{v_*} (\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*))\, dG(v_2) = 0$$

Therefore, we are done if we can prove that

$$\lim_{n\to\infty} \int_{v_*}^{1} (\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*))\, dG(v_2) = 0$$

Observe

$$\left| \int_{v_*}^{1} (\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*))\, dG(v_2) \right| \leq \int_{v_*}^{1} |\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*)|\, dG(v_2)$$

Therefore we are done if we can prove that:

$$\lim_{n\to\infty} \quad |\widehat{s}(v_2; v_n) - \widehat{s}(v_2; v_*)| = 0$$

except at most on a set of measure zero. Suppose that at a point v_2 we have

$$\lim_{n\to\infty} \widehat{s}(v_2; v_n) = s^* \neq \widehat{s}(v_2; v_*)$$

We first observe that it must be $s^* \in r(v_2; v_*)$, since the correspondence $r(v_2; v_n)$ is upper hemicontinuous in v_n (lemma 1). This in turn implies $s^* < s(v_2; v_*)$, since $s(v_2; v_*)$ is the maximum of $r(v_2; v_*)$. Therefore, v_2 must be a point of discontinuity of $s(v_2; v_*)$. But we have already established in lemma 2 that the set of v_2 at which the function is discontinuous has measure zero. ■

Now suppose that the opponent opens tough, which happens when her opponent has high budget and her type is above v_*. In this case the opponent increases the bids up to $s(v_2; h_L, v_*)$. Since bidder 1 has opened soft, she is supposed to stop when the other bidder stops. We will assume that when this does not happen, so that an out of equilibrium situation is created, the opponent reacts behaving as in a standard English auction equilibrium and trying to buy both objects. This is a maximizing behavior if the opponent believes that bidder 1 is high budget and going for both objects.

Upon observing the opponent stopping at s, bidder 1 will update her beliefs and decide whether to accept a single object at s or try to buy both objects. In the first case, utility is $v_1 - s$. In the second case, utility is $2E\left(\max\left\{v_1 - z, 0\right\}\middle| s\left(z; v_*\right) = s\right)$. Therefore, when playing against a tough opponent utility is:

$$S_H\left(v_1; v_*\right) = \int_{v_*}^{1} \max\left\{v_1 - \overline{s}\left(v_2; v_*\right), 2E\left(\max\left\{v_1 - z, 0\right\}\middle| s\left(z; v_*\right) = s\left(v_2; v_*\right)\right), 0\right\} dG\left(v_2\right)$$

The expected payoff of opening soft is therefore:

$$S\left(v_1; v_*\right) = \left(\lambda + \left(1 - \lambda\right) G\left(v_*\right)\right)\left(v_1 - h_L\right) + \left(1 - \lambda\right) S_H\left(v_1; v_*\right)$$

Also define:

$$S\left(v_*\right) = S\left(v_*; v_*\right)$$

and:

$$D\left(v_*\right) = S\left(v_*\right) - T\left(v_*\right)$$

We now prove some preliminary results.

Lemma 5 *For each $v_* > h_L$ there exists a $\delta > 0$ such that all types $v_1 \in \left(h_L, h_L + \delta\right)$ prefer playing soft to playing tough.*

Proof. For any v_* we have $T\left(h_L; v_*\right) = S\left(h_L; v_*\right) = 0$. Assume $v_* > h_L$ and consider δ such that $h_L + \delta < v_*$. Then the utility of playing soft is:

$$S\left(v_1; v_*\right) = \left(\lambda + \left(1 - \lambda\right) G\left(v_*\right)\right)\left(v_1 - h_L\right) + \left(1 - \lambda\right) \int_{v_*}^{1} \max\left\{v_1 - \overline{s}\left(v_2; v_*\right), 0\right\} dG\left(v_2\right)$$

The utility of playing tough for a type $h_L + \delta$ is:

$$\begin{aligned} H_L\left(v_1; s, v_*\right) = \lambda &\left[2\int_{h_L}^{s} \left(v_1 - v_2\right) dG\left(v_2\right) + \left(v_1 - s\right)\left(1 - G\left(s\right)\right)\right] \\ + \left(1 - \lambda\right) &\left[2\int_{h_L}^{s} \left(v_1 - v_2\right) dG\left(v_2\right) + \left(v_1 - s\right)\left(G\left(v_*\right) - G\left(s\right)\right)\right] \end{aligned}$$

Now observe that, for a small enough δ and for all types $v_1 < h_L + \delta$:

$$\begin{aligned} \frac{\partial H_L}{\partial s} &= \lambda\left[\left(v_1 - s\right) g\left(s\right) - \left(1 - G\left(s\right)\right)\right] \\ &+ \left(1 - \lambda\right)\left[\left(v_1 - s\right) g\left(s\right) - \left(G\left(v_*\right) - G\left(s\right)\right)\right] < 0 \end{aligned}$$

for each $s \in \left[h_L, h_L + \delta\right]$. Then the utility of playing tough is exactly:

$$H_L\left(v_1; s, v_*\right) = \left(\lambda + \left(1 - \lambda\right) G\left(v_*\right)\right)\left[\left(v_1 - h_L\right)\right]$$

This is less than or equal to $S\left(v_1; v_*\right)$. ■

Lemma 6 *If v_* is an equilibrium threshold and $s(v_1; v_*)$ the corresponding stopping function defined for $v_1 \in [v_*, 1]$. Then*

$$\lim_{v_1 \downarrow v_*} s(v_1; v_*) > h_L$$

Proof. Suppose not, so that $\lim_{v_1 \downarrow v_*} s(v_1; v_*) = h_L$. It must be the case that all types $v_1 > v_*$ prefer playing tough to playing soft. When we consider the utility of playing soft we have:

$$\begin{aligned} \lim_{v_1 \downarrow v_*} S(v_1; v_*) &= (\lambda + (1-\lambda) G(v_*)) (v_* - h_L) \\ &\quad + (1-\lambda) \int_{v_*}^{1} \max \{v_* - s(v_2; v_*), 0\} \, dG(v_2) \end{aligned}$$

while when we look at the utility of playing tough we have:

$$\lim_{v_1 \downarrow v_*} T(v_1; v_*) = (\lambda + (1-\lambda) G(v_*)) (v_* - h_L)$$

Since there is a set with positive measure such that $\overline{s}(v_2; v_*) < v_*$, we conclude that:

$$\lim_{v_1 \downarrow v_*} S(v_1; v_*) > \lim_{v_1 \downarrow v_*} T(v_1; v_*)$$

a contradiction. ■

Lemma 7 *For every possible equilibrium threshold value v_* there is a set $[h_L, h_L + \delta]$ such that for each $v_1 \in [h_L, h_L + \delta]$ we have*

$$S(v_1; v_*) = T(v_1; v_*)$$

Proof. By lemma 6 for every possible equilibrium threshold v_* there is $\delta' > 0$ such that $\lim_{v_1 \downarrow v_*} s(v_1; v_*) = h_L + \delta'$. This implies that all types $v_1 \in [h_L, h_L + \delta']$, when playing soft or tough can possibly win something only if they meet a soft type. Therefore

$$S(v_1; v_*) = (\lambda + (1-\lambda) G(v_*)) (v_1 - h_L).$$

When we look at the utility of playing tough, we know by lemma 5 that for a set of types $[h_L, h_L + \delta]$ the optimal stopping time is h_L. Therefore, for this set:

$$T(v_1; v_*) = S(v_1; v_*)$$

■

Define now:

$$A = \{\delta \in [h_L, 1] \,|\, S(v_1; \delta) \geq T(v_1; \delta) \text{ for each } v_1 \in [h_L, \delta]\}$$

We know that the set A is non-empty. By continuity of S and T it must also be a closed interval. Then we define:

$$v_* = \sup A$$

If $v_* = 1$ then we are done.

Suppose now $v_* < 1$. Our problem is to show that for all $v_1 > v_*$ we have $T(v_1; v_*) \geq S(v_1; v_*)$. It is clear by definition of v_* that there is a non-empty interval $(v_*, v_* + \varepsilon)$ such that $T(v_1; v_*) > S(v_1; v_*)$ for every v_1 in the interval. Our final lemma shows that this is indeed the case when assumption 1 is satisfied.

Lemma 8 *An equilibrium exists if assumption 1 is satisfied.*

Proof. By the previous analysis we have:

$$\frac{\partial T}{\partial v_1} - \frac{\partial S}{\partial v_1} = Z(v_1) =$$

$$\begin{cases} G(s(v_1; v_*)) + (1-\lambda)[2G(v_1) - G(v_*) - G(v^a(v_1))] & \text{if } \quad s(v_1; v_*) < v_*^\lambda \\ \lambda G(s(v_1; v_*)) + (1-\lambda)[2G(v_1) - G(v_*) - G(v^a(v_1))] & \text{if } \quad s\left(v_1; v_*^\lambda\right) \geq v_*^\lambda \end{cases}$$

By definition of v_* it must be the case that $Z(v_*) > 0$. Therefore, we are done if we can prove that the function $Z(v_1)$ is non-decreasing. Since $s(v_1; v_*)$ is always a non-decreasing function, it is sufficient to prove that the function:

$$\phi(v_1) = 2G(v_1) - G(v_*) - G(v^a)$$

is non-decreasing. We remind the reader that:

$$v^a = \sup\{v_2 \,|\, s(v_2; v_*) \leq v_1\}$$

The issue is therefore the behavior of the function $v^a(v_1)$. If $v^a(v_1)$ is constant in a right neighborhood of v_1, then $\phi'(v_1) = g(v_1) > 0$. If $v^a(v_1)$ is not constant, then it must be strictly increasing and therefore the following first order condition has to be satisfied:

$$(v^a - v_1)\, g(v_1) = 1 - G(v_1)$$

This is the condition ensuring that v_1 is the optimal stopping time for type v^a. Notice that we are looking at $v_1 > v_*$, so that the FOC we apply is the one relative to the case $s > v_*$ This yields:

$$v^a = v_1 + \mu(v_1).$$

Since μ is assumed differentiable and at v_1 the function is strictly increasing, it must be the case that $v_1 + \mu(v_1) < 1$ and the function is differentiable, with

$$\frac{dv^a}{dv_1} = 1 + \mu'(v_1) > 0.$$

Therefore, $v_1 + \mu(v_1) < 1$ and $1 + \mu'(v_1) > 0$ are necessary condition ensuring that $v^a(v_1)$ is strictly increasing at v_1. At last observe:

$$\begin{aligned} \phi(v_1) &= 2g(v_1) - g(v^a(v_1))\frac{dv^a}{dv_1} \\ &= 2g(v_1) - g(v_1 + \mu(v_1))(1 + \mu'(v_1)) \geq 0 \end{aligned}$$

where the inequality follows from assumption 1. ■

■

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