Lecture 4:

Algebra, Geometry, and Complexity of the Simplex Method

Reading: Sections 2.6.4, 3.5, 10.2 10.5

Summary of the Phase I/Phase II Simplex Method

We write a typical simplex tableau as

z	x_1	x_2	$\ldots$	x_n	rhs
0	$\bar{a}_{11}$	$\bar{a}_{12}$	$\ldots$	$\bar{a}_{1n}$	$\bar{b}_1$
0	$\bar{a}_{21}$	$\bar{a}_{22}$	$\ldots$	$\bar{a}_{2n}$	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	$\bar{a}_{m1}$	$\bar{a}_{m2}$	$\ldots$	$\bar{a}_{mn}$	$\bar{b}_m$
1	$\bar{a}_{m+1,1}$	$\bar{a}_{m+1,2}$	$\ldots$	$\bar{a}_{m+1,n}$	$\bar{b}_{m+1}$

where $\bar{a}_{m+1,j} = -\bar{c}_j$, $\bar{b}_{m+1} = \bar{z}_0$ if this is a max problem, and $\bar{a}_{m+1,j} = \bar{c}_j$, $\bar{b}_{m+1} = -\bar{z}_0$ if it is a min problem.

Generic Simplex Pivot for Row r:

P1: Choose as **entering variable** any x_j with $\bar{a}_{rj} < 0$

P2: Choose as **leaving variable** any x_{B_i} with $i < r$, $\bar{a}_{ij} > 0$ and

$$\bar{b}_i/\bar{a}_{ij} = \min\{\bar{b}_k/\bar{a}_{kj} \mid \bar{a}_{kj} > 0,\ k = 1, \ldots, r-1\}$$

P3: Perform a pivot on entry $\bar{a}_{ij}$.

Steps of the Phase I/Phase II Method

Initialize: Perform a standard Gauss-Jordan reduction of the original equality system. If a redundant row is found delete it, and if an inconsistent row is found **STOP**, the LP is **infeasible**.

Phase I: For row $r = 1, \ldots, m$

while $\bar{b}_r < 0$ do the following:

1. Attempt a generic simplex pivot.
2. If Step P1 fails, then **STOP**, the LP is **infeasible**
3. If Step P2 fails **pivot on element** $\bar{a}_{rj}$. Go to the next row.
4. If neither P1 or P2 fails, perform the pivot and repeat.

When rows $1, \ldots, m$ have been successfully processed, the LP is **feasible**.

Phase II: For $r = m + 1$ do the following:

1. Attempt a generic simplex pivot.
2. If Step P1 fails, then **STOP**, the current solution is **optimal**.
3. If Step P2 fails **STOP**, the LP is **unbounded**.
4. If neither P1 or P2 fails, perform the pivot and repeat.

Matrix form of a basic tableau

Consider a general equality form LP tableau

z	x_1	x_2	$\ldots$	x_n	rhs
0	a_{11}	a_{12}	$\ldots$	a_{1n}	b_1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	a_{m1}	a_{m2}	$\ldots$	a_{mn}	b_m
1	$-c_1$	$-c_2$	$\ldots$	$-c_n$	0

or

z	x	rhs
0	A	b
1	$-c$	0

We wish to give a matrix description of an arbitrary basic tableau associated with basic variables x_B. To do this, we first partition of the tableau into basic and nonbasic columns:

z	x_B	x_N	rhs
0	B	N	b
1	$-c_B$	$-c_N$	0

Then the associated basic tableau will look like this:

basis	z	x_{B_1}	$\ldots$	x_{B_m}	x_{N_1}	$\ldots$	$x_{N_{n-m}}$	rhs
x_{B_1}	0	1	$\ldots$	0	$\bar{a}_{1,N_1}$	$\ldots$	$\bar{a}_{1,N_{n-m}}$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{B_m}	0	0	$\ldots$	1	$\bar{a}_{m,N_1}$	$\ldots$	$\bar{a}_{m,N_{n-m}}$	$\bar{b}_m$
z	1	0	$\ldots$	0	$-\bar{c}_{N_1}$	$\ldots$	$-\bar{c}_{N_{n-m}}$	$\bar{z}_0$

or

basis	z	x_B	x_N	rhs
x_B	0	I	$\bar{N}$	$\bar{b}$
z	1	0	$-\bar{c}_N$	$\bar{z}_0$

We wish to describe $\bar{N}$, $\bar{b}$, $\bar{c}_N$, and $\bar{z}_0$ in terms of B, N, b, c_B, and c_N.

We have already described the $\bar{N}$ matrix and $\bar{b}$ vector, using the equation

$$Ax = Bx_B + Nx_N = b$$

or

$$B^{-1}Ax = x_B + B^{-1}Nx_N = B^{-1}b$$

so that $\bar{N} = B^{-1}N$ and $\bar{b} = B^{-1}b$.

How about $\bar{c}_N$ and $\bar{z}_0$? Writing $z = c_Bx_B + c_Nx_N$ and substituting $x_B = B^{-1}b - B^{-1}Nx_N$, we get

$$\begin{aligned} z &= c_B(B^{-1}b - B^{-1}Nx_N) + c_Nx_N \\ &= c_BB^{-1}b + (c_N - c_BB^{-1}N)x_N \end{aligned}$$

or extending to all variables,

$$z = c_BB^{-1}b + (c - c_BB^{-1}A)x$$

so that the complete tableau is

basis	z	x_B	x_N	rhs
x_B	0	I	$B^{-1}N$	$B^{-1}b$
z	1	0	$-c_N + c_BB^{-1}N$	$c_BB^{-1}b$

or simply

basis	z	x	rhs
x_B	0	$B^{-1}A$	$B^{-1}b$
z	1	$-c + c_BB^{-1}A$	$c_BB^{-1}b$

We define an important set of values $\bar{y}_1, \ldots, \bar{y}_m$ for this tableau, called the **shadow prices**, by setting

$$\bar{y} = c_B B^{-1}$$

Using this, and writing $B^{-1} = \bar{S} = \begin{pmatrix} \bar{s}_{11} & \ldots & \bar{s}_{1m} \\ \vdots & & \vdots \\ \bar{s}_{m1} & \ldots & \bar{s}_{mm} \end{pmatrix}$, the tableau can be written

basis	z	x_B	x_N	rhs
x_B	0	I	$\bar{S}N$	$\bar{S}b$
z	1	0	$-c_N + \bar{y}N$	$\bar{y}b$

or

basis	z	x	rhs
x_B	0	$\bar{S}A$	$\bar{S}b$
z	1	$-c + \bar{y}A$	$\bar{y}b$

More on the role of shadow prices later.

Formulae for a Basic Tableau

We can now give the formulae for the current tableau in terms of the original tableau and the $\bar{S}$ and $\bar{y}$ values:

$$
\begin{array}{llcll}
(1) & \bar{a}_{ij} & = & \bar{s}_{i1}a_{1j} + \bar{s}_{i2}a_{2j} + \ldots + \bar{s}_{im}a_{mj} & \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array} \\
(2) & \bar{b}_i & = & \bar{s}_{i1}b_1 + \bar{s}_{i2}b_2 + \ldots + \bar{s}_{im}b_m & i = 1, \ldots, m \\
(3) & \bar{c}_j & = & c_j - \bar{y}_1 a_{1j} - \bar{y}_2 a_{2j} - \ldots - \bar{y}_m a_{mj} & j = 1, \ldots, n \\
(4) & \bar{z}_0 & = & \bar{y}_1 b_1 + \bar{y}_2 b_2 + \ldots + \bar{y}_m b_m &
\end{array}
$$

Example: Consider the following LP:

$$
\begin{aligned}
\max cx & \\
Ax &= b \\
x &\geq 0
\end{aligned}
$$

Where A, b, and c are given below:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	0	3	3	1	−1	1	−5	3
A	1	3	−1	0	−1	1	3	4
	1	2	0	1	2	0	−2	5
c	2	3	−4	3	1	−4	6	

Suppose we want to reconstruct the tableau with starting basis $x_B = (x_4, x_6, x_1)$.

We have

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad c_B = (3, -4, 2)$$

and so

$$\begin{aligned} \bar{S} &= B^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \\ \bar{y} &= c_B\bar{S} = (-3/2, -5/2, 9/2). \end{aligned}$$

The final tableau has

$$\begin{aligned} \bar{A} = \bar{S}A &= \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 3 & 3 & 1 & -1 & 1 & -5 \\ 1 & 3 & -1 & 0 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 0 & -5 \\ 0 & 2 & 1 & 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 & 3 \end{pmatrix} \\ \bar{b} = \bar{S}b &= \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
\bar{c} = c - \bar{y}A &= (2, 3, -4, 3, 1, -4, 6) \\
&\quad -(-3/2, -5/2, 9/2)\begin{pmatrix} 0 & 3 & 3 & 1 & -1 & 1 & -5 \\ 1 & 3 & -1 & 0 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 & 0 & -2 \end{pmatrix} \\
&= (0, 6, -2, 0, -12, 0, 15) \\
\bar{z}_0 &= (-3/2, -5/2, 9/2)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 8
\end{aligned}$$

So the current tableau is

basis	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
x_4	0	0	1	2	1	1	0	−5	2
x_6	0	0	2	1	0	−2	1	0	1
x_1	0	1	1	−2	0	1	0	3	3
z	1	0	−6	2	0	12	0	−15	8

We apply the formulae to a few entries:

$$\begin{aligned}
\bar{a}_{23} &= \bar{s}_{21}a_{13} + \bar{s}_{22}a_{23} + \bar{s}_{23}a_{33} \\
&= (1/2)(3) + (1/2)(-1) + (-1/2)(0) = 1
\end{aligned}$$

$$\begin{aligned}
\bar{b}_3 &= \bar{s}_{31}b_1 + \bar{s}_{32}b_2 + \bar{s}_{33}b_3 \\
&= (-1/2)(3) + (1/2)(4) + (1/2)(5) = 3
\end{aligned}$$

$$\begin{aligned}
\bar{c}_5 &= c_5 - \bar{y}_1a_{15} - \bar{y}_2a_{25} - \bar{y}_3a_{35} \\
&= 1 - (-3/2)(-1) - (-5/2)(-1) - (9/2)(2) = -12
\end{aligned}$$

$$\begin{aligned}
\bar{z}_0 &= \bar{y}_1b_1 + \bar{y}_2b_2 + \bar{y}_3b_3 \\
&= (-3/2)(3) + (-5/2)(4) + (9/2)(5) = 8
\end{aligned}$$

Matrix Form for Canonical Max Tableau

The tableaus associated with a canonical max problem has some interesting special features. Consider the canonical max LP

$$\begin{array}{rcrcccrcl}
\max z & = & c_1x_1 & + & \ldots & + & c_nx_n & & \\
 & & a_{11}x_1 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & & \vdots & & & & & & \vdots \\
 & & a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & x_1 \geq 0 & & \ldots & & x_n \geq 0 & &
\end{array}$$

This has starting tableau

z	x_1	x_2	$\ldots$	x_n	s_1	$\ldots$	s_m	rhs
0	a_{11}	a_{12}	$\ldots$	a_{1n}	1	$\ldots$	0	b_1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	a_{m1}	a_{m2}	$\ldots$	a_{mn}	0	$\ldots$	1	b_m
1	$-c_1$	$-c_2$	$\ldots$	$-c_n$	0	$\ldots$	0	0

or

z	x	s	rhs
0	A	I	b
1	$-c$	0	0

where $s = (s_1, \ldots, s_m)$ is the vector of slack variables for the inequalities.

A general simplex tableau will look like this:

z	x_1	x_2	$\ldots$	x_n	s_1	$\ldots$	s_m	rhs
0	$\bar{a}_{11}$	$\bar{a}_{12}$	$\ldots$	$\bar{a}_{1n}$	$\bar{a}_{1,n+1}$	$\ldots$	$\bar{a}_{1,n+m}$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	$\bar{a}_{m1}$	$\bar{a}_{m2}$	$\ldots$	$\bar{a}_{mn}$	$\bar{a}_{m,n+1}$	$\ldots$	$\bar{a}_{m,n+m}$	$\bar{b}_m$
1	$-\bar{c}_1$	$-\bar{c}_2$	$\ldots$	$-\bar{c}_n$	$-\bar{c}_{n+1}$	$\ldots$	$-\bar{c}_{n+m}$	$\bar{z}_0$

and if we apply the matrix formula to the extended system with $\hat{A} = (A\ I)$ and $\hat{c} = (c, 0)$, we get tableau

basis	z	(x, s)	rhs
x_B	0	$\bar{S}(A\ I)$	$\bar{S}b$
z	1	$-[(c, 0) - \bar{y}(A\ I)]$	$\bar{y}b$

=

basis	z	x	s	rhs
x_B	0	$\bar{S}A$	$\bar{S}$	$\bar{S}b$
z	1	$-c + \bar{y}A$	$\bar{y}$	$\bar{y}b$

so that $\bar{S}$ is precisely the matrix under the slack variables, and $\bar{y}$ is precisely the set of objective row values under the slack variables.

Note: If the objective were a **min** (constraints still $\leq$) the tableau will look like

basis	$-z$	x	s	rhs
x_B	0	$\bar{S}A$	$\bar{S}$	$\bar{S}b$
$-z$	1	$c - \bar{y}A$	$-\bar{y}$	$-\bar{y}b$

and so the values under the slack variables will be the **negatives** of the shadow prices.

Example

Woody's 3-variable problem has starting tableau

basis	z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
x_4	0	8	12	16	1	0	0	120
x_5	0	0	15	20	0	1	0	60
x_6	0	3	6	9	0	0	1	48
z	1	-35	-60	-75	0	0	0	0

The optimal tableau is

basis	z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
x_5	0	0	0	-10	$15/4$	1	-10	30
x_2	0	0	1	2	$-1/4$	0	$2/3$	2
x_1	0	1	0	-1	$1/2$	0	-1	12
z	1	0	0	10	$5/2$	0	5	540

so that the shadow prices are $(5/2, 0, 5)$ — which incidentally are also the *negative* of the reduced costs on the slack variables. The reduced costs on the x-variables are

$$(\bar{c}_1, \bar{c}_2, \bar{c}_3) = (35, 60, 75) - (5/2, 0, 5) \begin{pmatrix} 8 & 12 & 16 \\ 0 & 15 & 20 \\ 3 & 6 & 9 \end{pmatrix} = (0, 0, -10)$$

which appears negated in the optimal tableau.

Change Vectors and Ascent Directions

Consider an LP in equality form:

$$(P) \qquad \begin{array}{rrcl} \max & z = cx & & \\ & Ax & = & b \\ & x & \geq & 0 \end{array}$$

and suppose $\hat{x}$ and $\hat{x}'$ are two feasible (not necessarily basic) solutions for (P) for which $\hat{x}'$ has a larger objective function than $\hat{x}$. We will concentrate on the **change vector** v such that $\hat{x}' = \hat{x} + v$.

Two key properties of the change vector v are:

(a) $Av = 0$

(b) $cv > 0$

Proof:

(a) $A\hat{x} = b$ and $A\hat{x}' = b \Rightarrow A\hat{v} = A\hat{x}' - A\hat{x} = b - b = 0$

(b) $c\hat{x}' > c\hat{x} \Rightarrow cv = c\hat{x}' - c\hat{x} > 0$

Pivot Directions as Change Vectors

The simplex method involves moving from some BFS $\hat{x}$ by choosing an entering variable x_j in the associated tableau, and then increasing x_j to some value $\Delta \geq 0$, leaving all other nonbasic variables at 0. This resulted in new point x^Δ defined by

$$\begin{aligned} \hat{x}_j^\Delta &= \Delta, \\ \hat{x}_{B_i}^\Delta &= \bar{b}_i - \bar{a}_{ij}\Delta, \quad i = 1, \ldots, m. \\ \hat{x}_k^\Delta &= 0, \quad \text{otherwise.} \end{aligned}$$

Another way of looking at this is to say that $x^\Delta = \hat{x} + \Delta\hat{v}$, where $\hat{x}$ is the current BFS, and $\hat{v}$ has components

$$\begin{aligned} \hat{v}_j &= \quad 1, \\ \hat{v}_{B_i} &= -\bar{a}_{ij}, \ \ i = 1, \ldots, m. \\ \hat{v}_k &= \quad 0, \quad \text{otherwise.} \end{aligned}$$

Then clearly

$$\begin{aligned} A\hat{v} &= B\hat{v}_B + N\hat{v}_N = -B\bar{A}_{\cdot j} + A_{\cdot j} \cdot 1 \\ &= -B(B^{-1}A_j) + A_j = -A_j + A_j \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} c\hat{v} &= c_B\hat{v}_B + c_N\hat{v}_N = -c_B\bar{A}_{\cdot j} + c_j \\ &= -c_B(B^{-1}A_j) + c_j \\ &= \bar{c}_j \end{aligned}$$

and thus so long as $\bar{c}_j > 0$ this satisfies the properties of change vectors given above. We call $\hat{v}$ a **basic change vector** associated with basis B and nonbasic index j.

Basic Geometry of the Phase II Simplex Method

Consider a canonical max LP, written in the following inequality form:

$$
(P_I)\quad
\begin{array}{rrcrcrcrcl}
\max z & = & c_1x_1 & + & \ldots & + & c_nx_n & & & \text{(slack)} \\
 & & -x_1 & & & & & \leq & 0 & x_1 \\
 & & & & \ddots & & & & & \\
 & & & & & & -x_n & \leq & 0 & x_n \\
 & & a_{11}x_1 & + & \ldots & + & a_{1n}x_n & \leq & b_1 & x_{n+1} \\
 & & \vdots & & & & & \vdots & & \\
 & & a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & \leq & b_m & x_{n+m}
\end{array}
$$

and its standard equality form equivalent:

$$
(P_E)\quad
\begin{array}{rcrcccrcccccl}
\max z & = & c_1x_1 & + & \ldots & + & c_nx_n & & & & & & \\
 & & a_{11}x_1 & + & \ldots & + & a_{1n}x_n & + & x_{n+1} & & & = & b_1 \\
 & & \vdots & & & & \vdots & & & \ddots & & & \vdots \\
 & & a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & & & + & x_{n+m} & = & b_m \\
 & & x_1 \geq 0 & & \ldots & & x_n \geq 0 & & x_{n+1} \geq 0 & \ldots & x_{n+m} \geq 0 & &
\end{array}
$$

Now associate each of the $n+m$ **variables** in P_E to the appropriate one of the $n+m$ **constraints** in P_I.

Relationships Between Algebra and Geometry

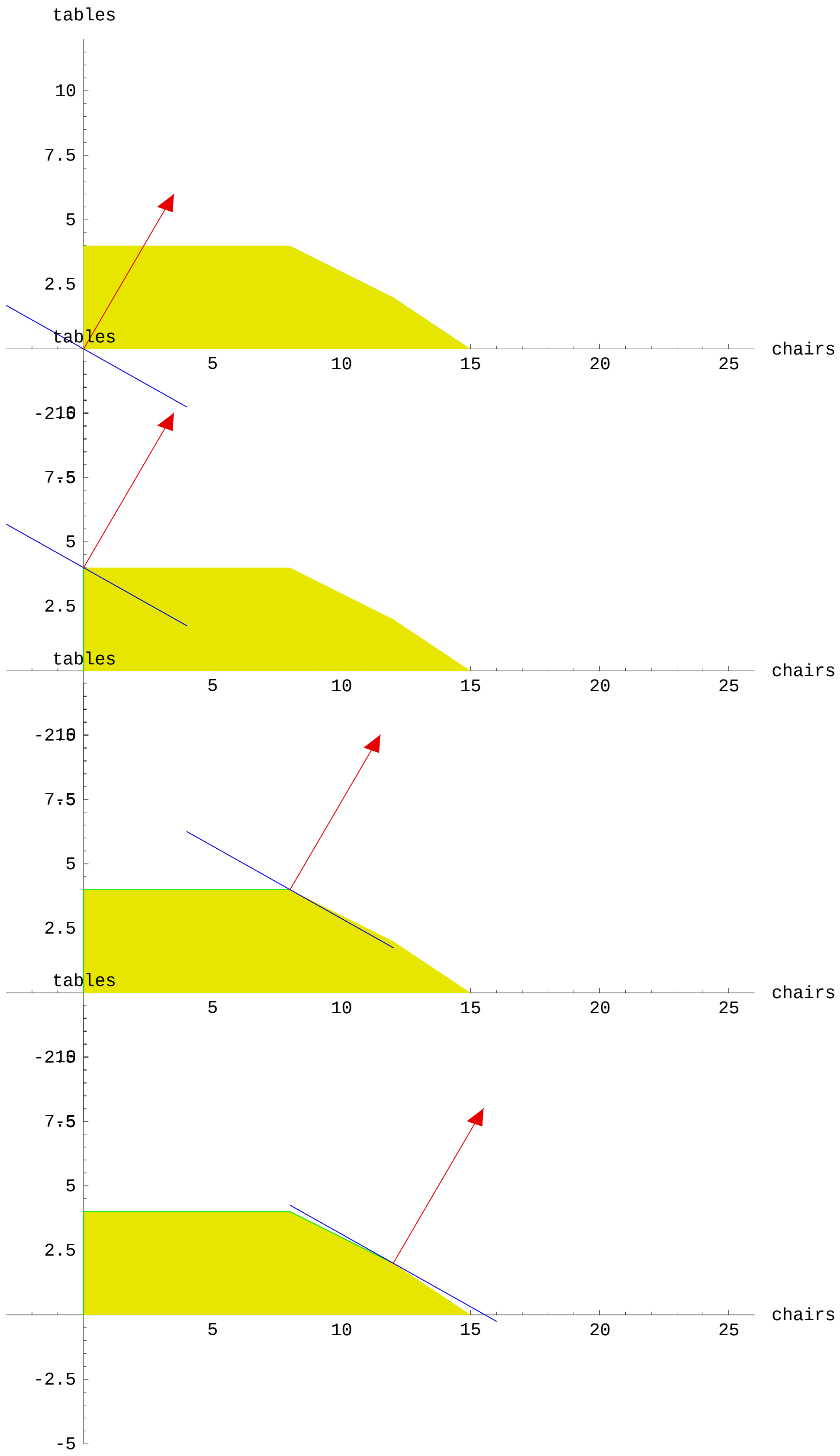
P_E	P_I
basic feasible solution: A solution obtained by setting n nonbasic variables to 0 and solving for the remaining m basic variables.	**corner-point solution:** A solution obtained by setting n constraints to equality and finding the corresponding solution.
adjacent basic feasible solutions: A pair of BFSs whose bases differ by exactly one element. The set x^Δ, $0 \leq \Delta \leq \Delta_*$ defines those solutions obtained by pivoting from one BFS to an adjacent one in the direction of the basic change vector.	**edge:** Set of feasible points satisfying exactly $n-1$ constraints at equality.
optimal BFS: A BFS whose associated tableau has all reduced costs nonnegative	**optimal corner point:** corner point whose associated level set contains the entire feasible region on the $\leq$ side.
unbounded tableau: A tableau having a column whose reduced cost is negative and whose associated basic change vector has all nonnegative components.	**extreme half-line (extreme ray):** an edge whose objective-function- increasing direction never leaves the feasible region

Phase II for Woody's Problem

$$\begin{array}{rrcrcr}
\max\ z = & 35x_1 & + & 60x_2 & & \\
 & 8x_1 & + & 12x_2 & \le & 120 \\
 & & & 15x_2 & \le & 60 \\
 & 3x_1 & + & 6x_2 & \le & 48 \\
 & x_1 \ge 0 & & x_2 \ge 0 & &
\end{array}$$

basis	z	x_1	x_2	x_3	x_4	x_5	rhs
x_3	0	8	12	1	0	0	120
x_4	0	0	$\boxed{15}$	0	1	0	60
x_5	0	3	6	0	0	1	48
z	1	−35	−60	0	0	0	0
x_3	0	8	0	1	−4/5	0	72
x_2	0	0	1	0	1/15	0	4
x_5	0	$\boxed{3}$	0	0	−2/5	1	24
z	1	−35	0	0	4	0	240
x_3	0	0	0	1	$\boxed{4/15}$	−8/3	8
x_2	0	0	1	0	1/15	0	4
x_1	0	1	0	0	−2/15	1/3	8
z	1	0	0	0	−2/3	35/3	520
x_4	0	0	0	15/4	1	−10	30
x_2	0	0	1	−1/4	0	2/3	2
x_1	0	1	0	1/2	0	−1	12
z	1	0	0	5/2	0	5	540

Simplex Path for Woody's Problem



Phase II for 3-Variable Woody's

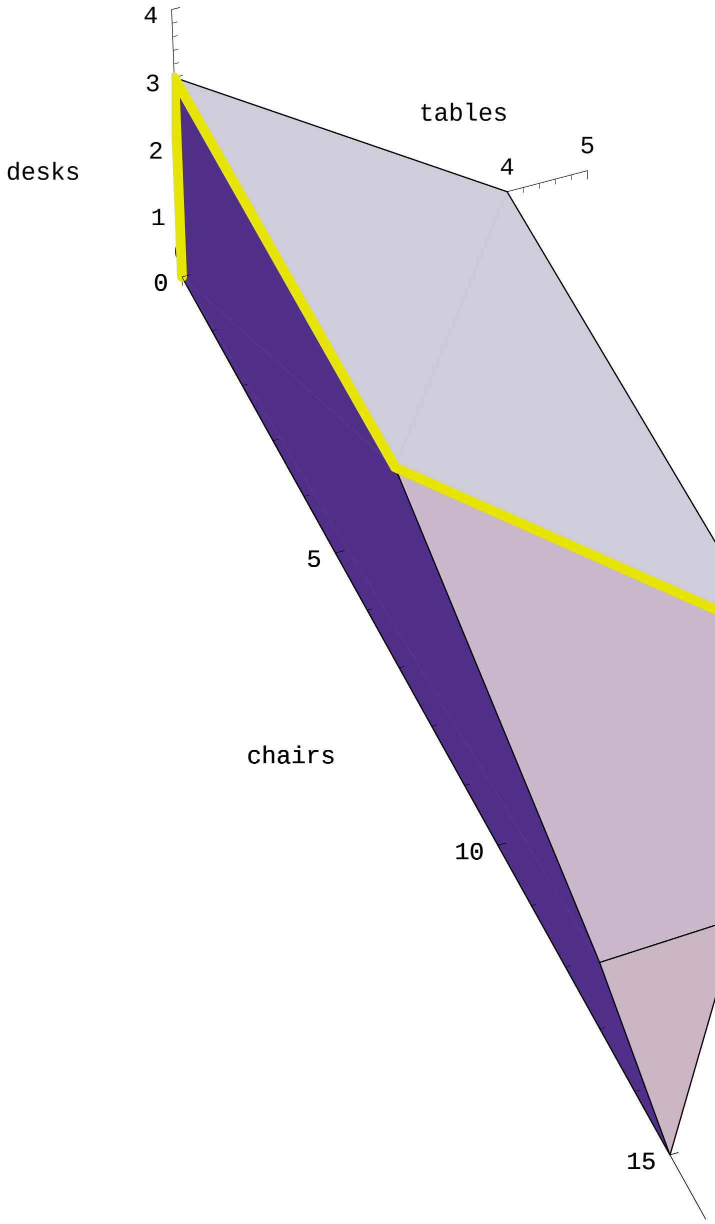
$$\begin{array}{rcrcrcrcr} \max\ z & = & 35x_1 & + & 60x_2 & + & 75x_3 & & \\ & & 8x_1 & + & 12x_2 & + & 16x_3 & \le & 120 \\ & & & & 15x_2 & + & 20x_3 & \le & 60 \\ & & 3x_1 & + & 6x_2 & + & 9x_3 & \le & 48 \end{array}$$

$$x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 \ge 0$$

The Simplex Tableaus:

basis	z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
x_4	0	8	12	16	1	0	0	120
x_5	0	0	15	$\boxed{20}$	0	1	0	60
x_6	0	3	6	9	0	0	1	48
z	1	-35	-60	-75	0	0	0	0
x_4	0	8	0	0	1	$-4/5$	0	72
x_3	0	0	$3/4$	1	0	$1/20$	0	3
x_6	0	$\boxed{3}$	$-3/4$	0	0	$-9/20$	1	21
z	1	-35	$-15/4$	0	0	$15/4$	0	225
x_4	0	0	2	0	1	$2/5$	$-8/3$	16
x_3	0	0	$\boxed{3/4}$	1	0	$1/20$	0	3
x_1	0	1	$-1/4$	0	0	$-3/20$	$1/3$	7
z	1	0	$-25/2$	0	0	$-3/2$	$35/3$	470
x_4	0	0	0	$-8/3$	1	$\boxed{4/15}$	$-8/3$	8
x_2	0	0	1	$4/3$	0	$1/15$	0	4
x_1	0	1	0	$1/3$	0	$-2/15$	$1/3$	8
z	1	0	0	$50/3$	0	$-2/3$	$35/3$	520
x_5	0	0	0	-10	$15/4$	1	-10	30
x_2	0	0	1	2	$-1/4$	0	$2/3$	2
x_1	0	1	0	-1	$1/2$	0	-1	12
z	1	0	0	10	$5/2$	0	5	540

Simplex Path for Woody's 3D Problem



An Unbounded LP

$$\begin{array}{rrcrcl} \max\ z = & 35x_1 & + & 60x_2 & & \\ & -8x_1 & + & 12x_2 & \leq & 120 \\ & -20x_1 & + & 15x_2 & \leq & 60 \\ & 3x_1 & - & 6x_2 & \leq & 48 \\ & x_1 \geq 0 & & x_2 \geq 0 & & \end{array}$$

The tableaus:

basis	z	x_1	x_2	x_3	x_4	x_5	rhs
x_3	0	-8	12	1	0	0	120
x_4	0	-20	$\boxed{15}$	0	1	0	60
x_5	0	3	-6	0	0	1	48
z	1	-35	-60	0	0	0	0
x_3	0	$\boxed{8}$	0	1	$-4/5$	0	72
x_2	0	$-4/3$	1	0	$1/15$	0	4
x_5	0	-5	0	0	$2/5$	1	72
z	1	-115	0	0	4	0	240
x_1	0	1	0	$1/8$	$-1/10$	0	9
x_2	0	0	1	$1/6$	$-1/15$	0	16
x_5	0	0	0	$5/8$	$-1/10$	1	117
z	1	0	0	$115/8$	$-15/8$	0	1275

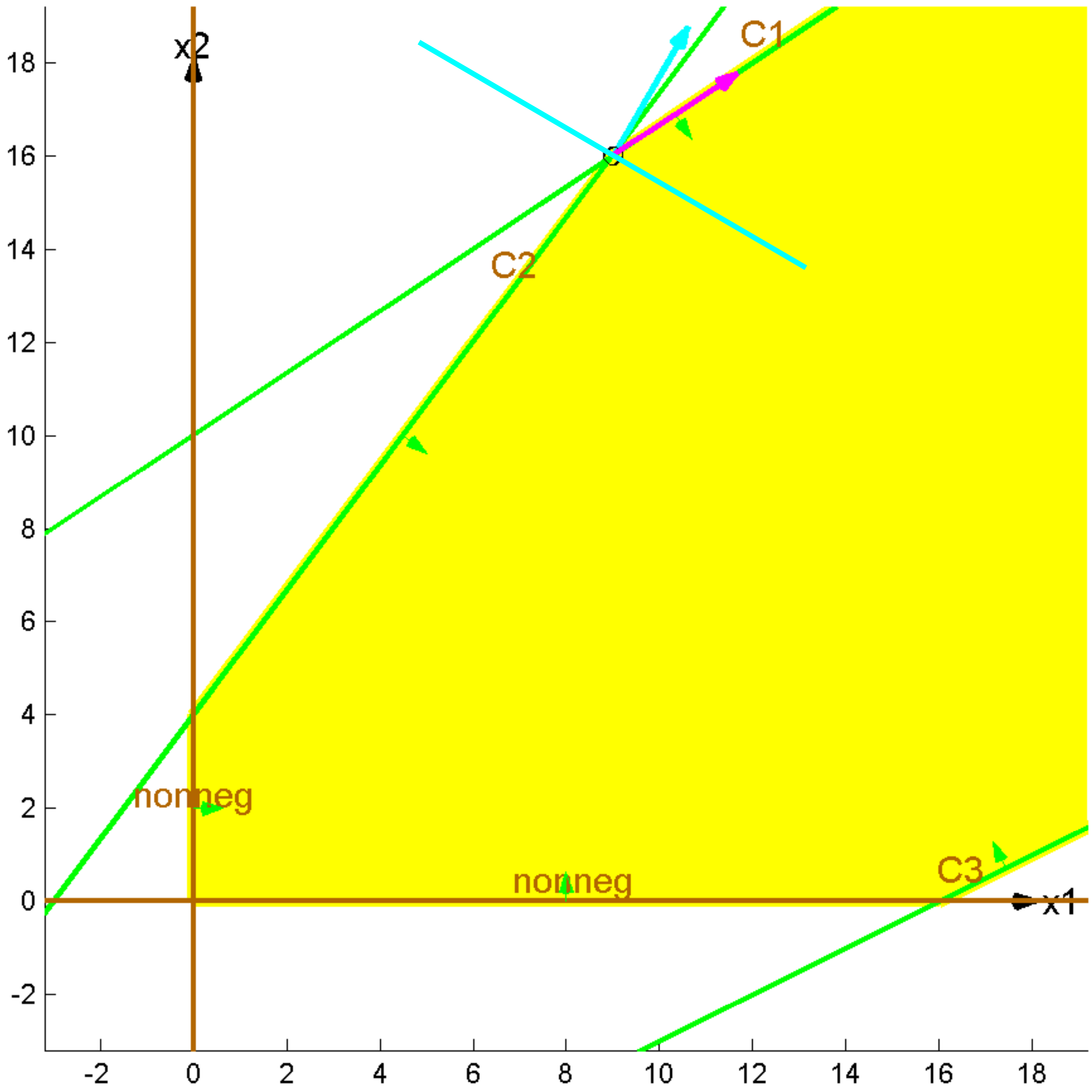
The change vector for the final tableau is

$$\hat{v} = (\frac{1}{10}, \frac{1}{15}, 0, 1, \frac{1}{10}) \geq 0$$

with

$$c\hat{v} = \frac{15}{8} > 0.$$

The Picture



with the unbounded arrow in the direction $(\frac{1}{10}, \frac{1}{15})$.

Geometry of the Phase I Simplex Method

The Phase I Simplex Method simply tries to satisfy the set of inequalities one inequality at a time, by moving in the direction of the feasible half of each inequality, while continuing to stay on the feasible side of all of the previous inequalities. The stopping rules can be interpreted:

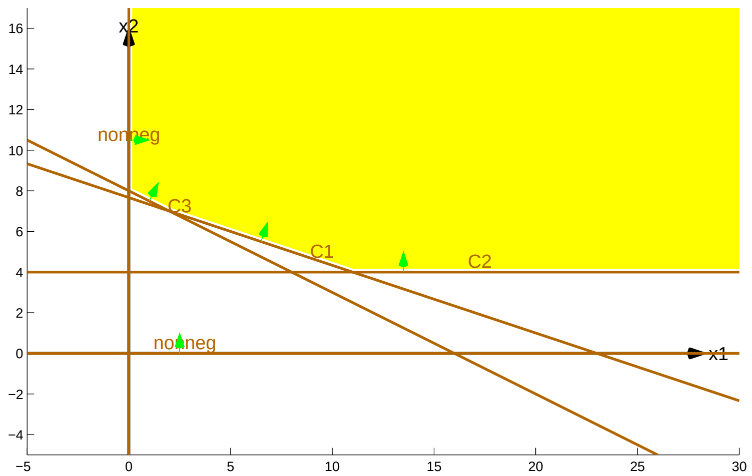
$\bar{b}_r$ **becomes nonnegative:** You have passed across the inequality into its feasible side.

Step P1 fails: You have reached a point as close to the inequality as you can get, and therefore can never pass into the feasible region.

Step P2 fails: The pivot on $\bar{a}_{rj}$ jumps you right onto the equality, so that it is immediately satisfied.

Example: Consider the following system:

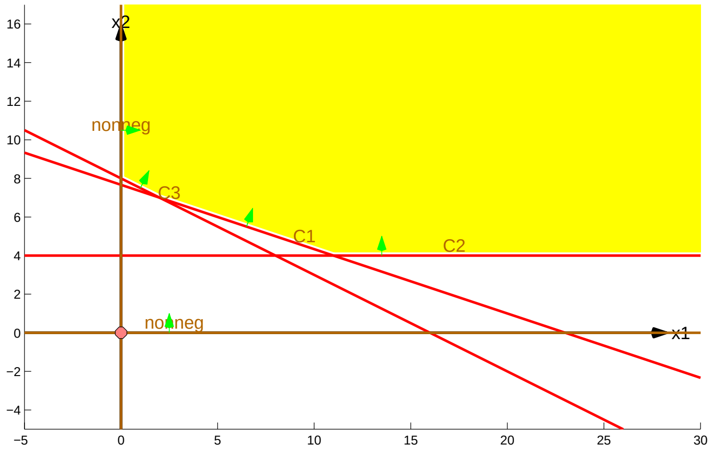
$$\begin{array}{rcrcl} 4x_1 & + & 12x_2 & \geq & 92 \\ & & 15x_2 & \geq & 60 \\ 3x_1 & + & 6x_2 & \geq & 48 \\ x_1 \geq 0 & & x_2 \geq 0 & & \end{array}$$



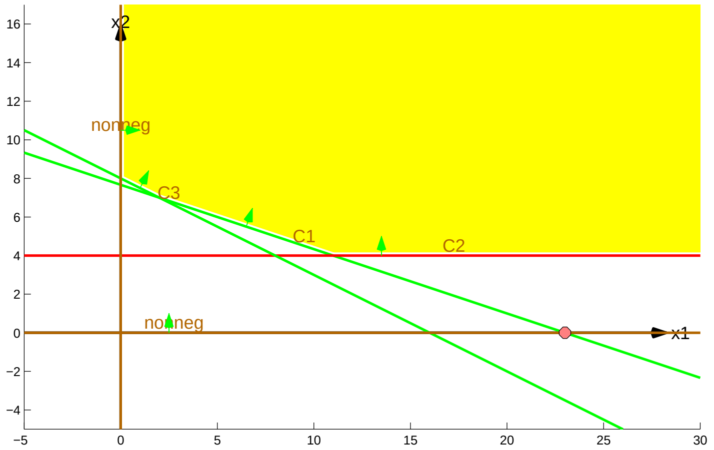
The Phase I Tableaus

basis	z	x_1	x_2	x_3	x_4	x_5	rhs	
x_3	0	[-4]	-12	1	0	0	-92	(P2 fails)
x_4	0	0	-15	0	1	0	-60	
x_5	0	-3	-6	0	0	1	-48	
basis	z	x_1	x_2	x_3	x_4	x_5	rhs	
x_1	0	1	[3]	$-1/4$	0	0	23	
x_4	0	0	-15	0	1	0	-60	(pivot, go to
x_5	0	0	3	$-3/4$	0	1	21	next row)
basis	z	x_1	x_2	x_3	x_4	x_5	rhs	
x_2	0	$1/3$	1	$-1/12$	0	0	$23/3$	
x_4	0	[5]	0	$-5/4$	1	0	55	
x_5	0	-1	0	$-1/2$	0	1	-2	(pivot)
basis	z	x_1	x_2	x_3	x_4	x_5	rhs	
x_2	0	0	1	0	$-1/15$	0	4	
x_1	0	1	0	$-1/4$	$1/5$	0	11	(feasible)
x_5	0	0	0	$-3/4$	$1/5$	1	9	

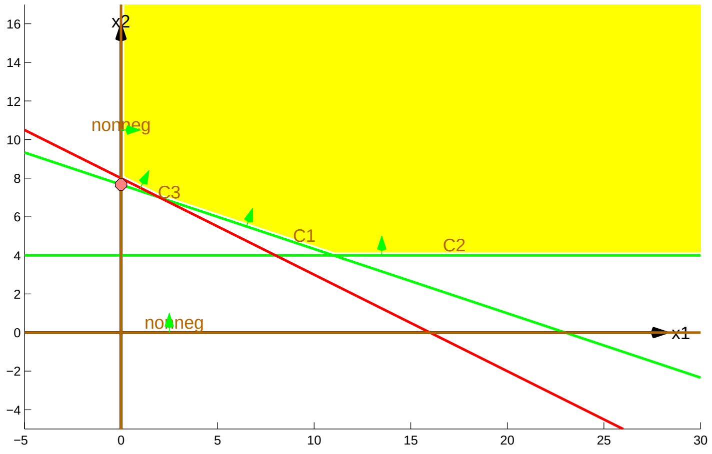
The Figures



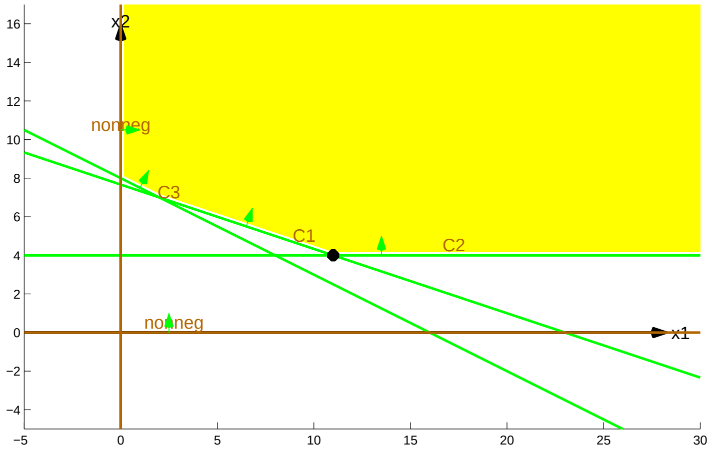
x2
nonneg
C3
C1
C2
nonneg
x1



x2
nonneg
C3
C1
C2
nonneg
x1



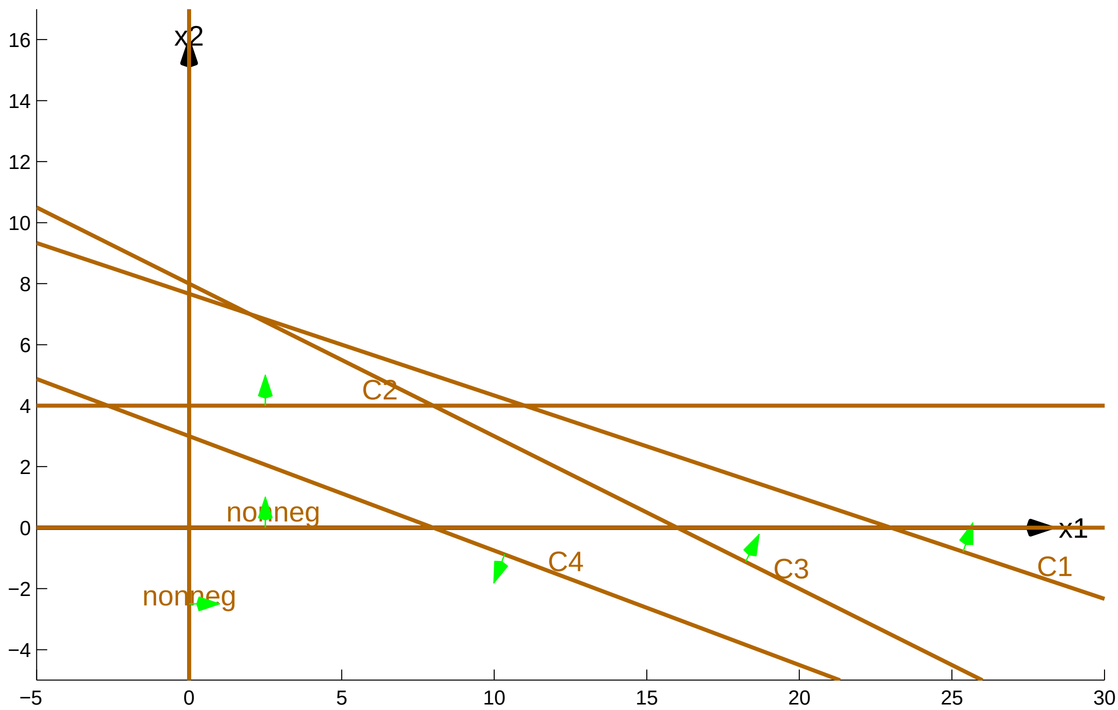
x2
nonneg
C3
C1
C2
nonneg
x1



x2
nonneg
C3
C1
C2
nonneg
x1

On the other hand, if we add a fourth constraint to the above system:

$$\begin{array}{rcrcr} 4x_1 & + & 12x_2 & \geq & 92 \\ & & 15x_2 & \geq & 60 \\ 3x_1 & + & 6x_2 & \geq & 48 \\ 3x_1 & + & 8x_2 & \leq & 24 \\ x_1 \geq 0 & & x_2 \geq 0 & & \end{array}$$



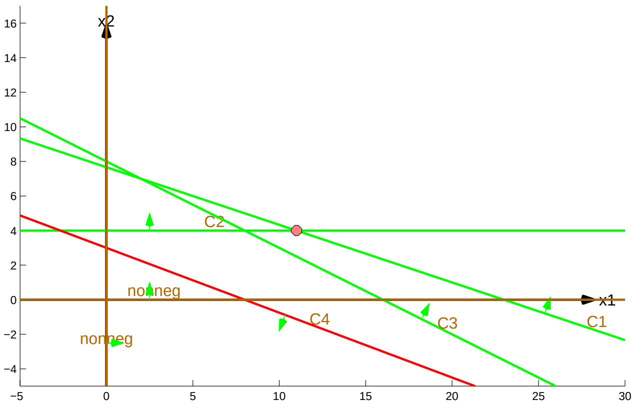
We take the same first three pivots

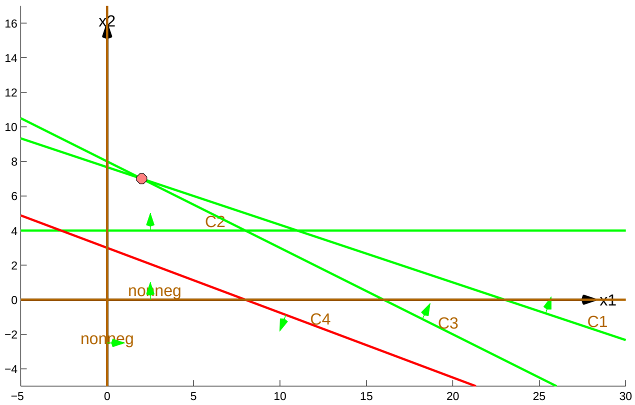
basis	$-z$	$x1$	$x2$	$s1$	$s2$	$s3$	$s4$	rhs
$x2$	0	0	1	0	$-1/15$	0	0	4
$x1$	0	1	0	$-1/4$	$1/5$	0	0	11
$s3$	0	0	0	$-3/4$	$1/5$	1	0	9
$s4$	0	0	0	$3/4$	$-1/15$	0	1	-41

but now one more pivot gives

basis	$-z$	$x1$	$x2$	$s1$	$s2$	$s3$	$s4$	rhs
$x2$	0	0	1	$-1/4$	0	$1/3$	0	7
$x1$	0	1	0	$1/2$	0	-1	0	2
$s2$	0	0	0	$-15/4$	1	5	0	45
$s4$	0	0	0	$1/2$	0	$1/3$	1	-38

for which P1 fails, and the problem is declared infeasible.





The Farkas Lemma solution associated with this turns out to be

$$\hat{y} = (1/2, 0, 1/3, -1)$$

This has an interesting interpretation when applied to the original inequality system. Multiplying the first constraint by $1/2$ and the third constraint by $1/3$, and then adding the two ($\geq$) constraints, we get implied *surrogate constraint*

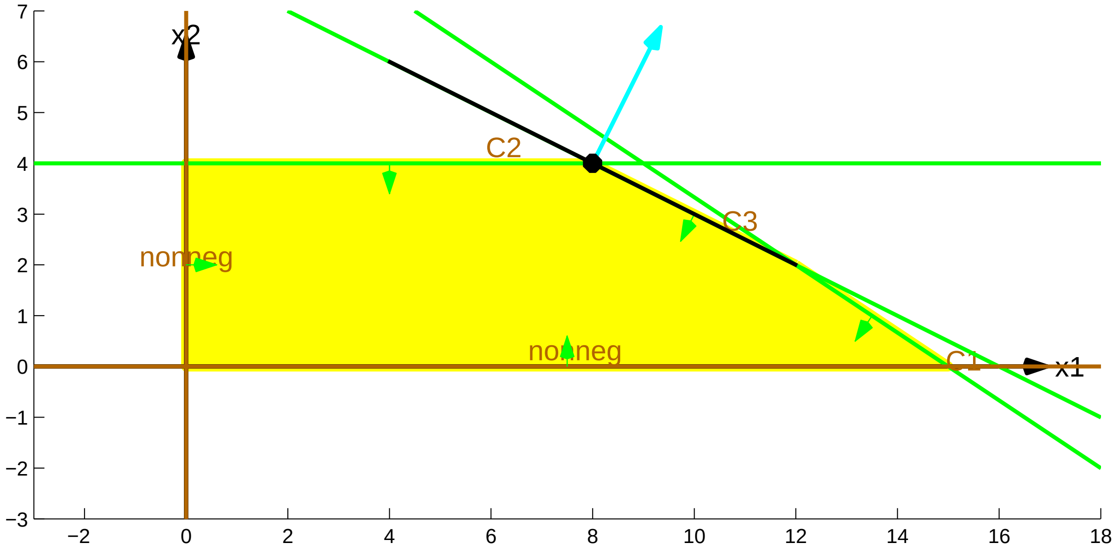
$$3x_1 + 8x_2 \geq 62$$

which immediately shows the fourth constraint can never be satisfied. More on surrogate constraints later.

Multiple Optimal Solutions

Suppose in Woody's problem we changed the cost of the second variable from -60 to -70.

$$\begin{array}{rcrcrcr} \max\ z & = & 35x_1 & + & 70x_2 & & \\ & & 8x_1 & + & 12x_2 & \leq & 120 \\ & & & & 15x_2 & \leq & 60 \\ & & 3x_1 & + & 6x_2 & \leq & 48 \\ & & x_1 \geq 0, & & x_2 \geq 0, & & \end{array}$$



If we perform the Phase II simplex method on this system, after the same first two pivots as for Woody's original problem we arrive at tableau

basis	z	x_1	x_2	x_3	x_4	x_5	**rhs**
x_3	0	0	0	1	$4/15$	$-8/3$	8
x_2	0	0	1	0	$1/15$	0	4
x_1	0	1	0	0	$-2/15$	$1/3$	8
z	1	0	0	0	0	$35/3$	560

This is an optimal tableau, with optimal solution $x_1 = 8$, $x_2 = 4$, $x_3 = 8$, $x_4 = 0$, $x_5 = 0$, and objective value $z = 560$, but now we have a *nonbasic* variable, x_4, whose *reduced cost* is 0. What this means is that if we bring x_4 into the basis by pivoting on $\bar{a}_{14}$, we arrive at tableau

basis	z	x_1	x_2	x_3	x_4	x_5	**rhs**
x_4	0	0	0	$15/4$	1	-10	30
x_2	0	0	1	$-1/4$	0	$2/3$	2
x_1	0	1	0	$1/2$	0	-1	12
z	1	0	0	0	0	$35/3$	560

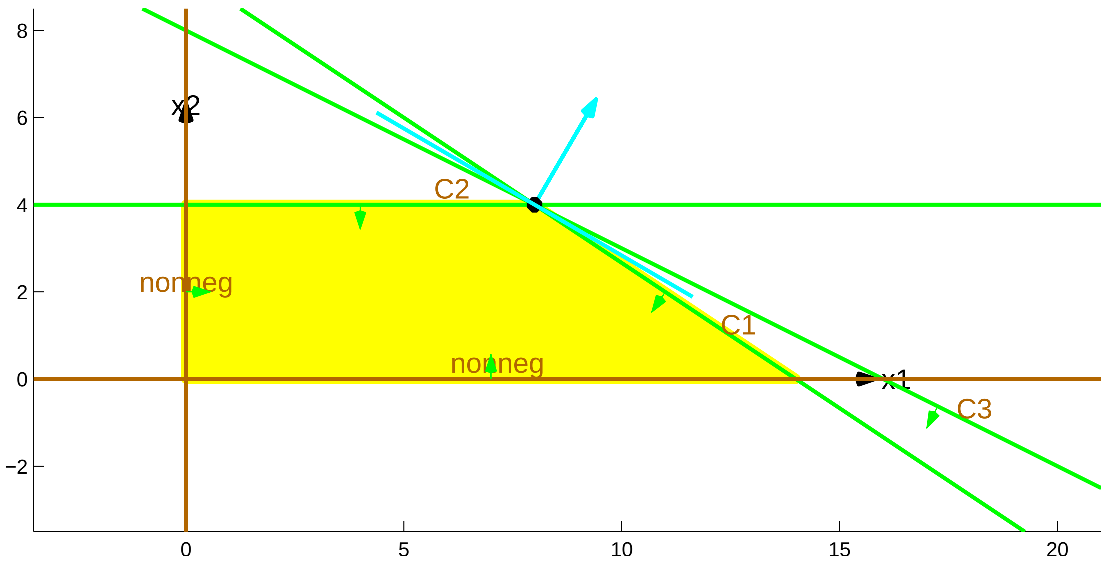
This is *also* an optimal tableau, with *different* solution $x_1 = 12$, $x_2 = 2$, $x_3 = 0$, $x_4 = 30$, $x_5 = 0$, which is *also optimal.*

Fact: If the optimal tableau has a nonbasic column with reduced cost 0, then a pivot on that column will result in an **alternate optimal solution**.

Tie Breaking and Degeneracy in the Simplex Method

Usually tie-breaking can be done arbitrarily in the simplex method. This can lead to odd situations. For example, suppose we consider Woody's problem, with the pine resource changed to 112:

$$
\begin{array}{rrcrcrcr}
\max\ z & = & 35x_1 & + & 60x_2 & & \\
 & & 8x_1 & + & 12x_2 & \leq & 112 \\
 & & & & 15x_2 & \leq & 60 \\
 & & 3x_1 & + & 6x_2 & \leq & 48 \\
 & & \multicolumn{5}{l}{x_1 \geq 0, \quad x_2 \geq 0,}
\end{array}
$$



After the first pivot we obtain the following tableau:

basis	z	x_1	x_2	x_3	x_4	x_5	**rhs**
x_3	0	8	0	1	$-4/5$	0	64
x_2	0	0	1	0	$1/15$	0	4
x_5	0	3	0	0	$-2/5$	1	24
z	1	-35	0	0	4	0	240

and when we do the ratio test on the x_1 column we get **two** candidates for blocking variable, x_3 and x_5. Suppose we arbitrarily choose x_3 to enter the basis. The next tableau is:

basis	z	x_1	x_2	x_3	x_4	x_5	**rhs**
x_3	0	0	0	1	$4/15$	$-8/3$	0
x_2	0	0	1	0	$1/15$	0	4
x_1	0	1	0	0	$-2/15$	$1/3$	8
z	1	0	0	0	$-2/3$	$35/3$	520

with associated solution $x_1 = 8$, $x_2 = 4$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, and objective function value $z = 520$. This is called a **degenerate tableau**, since a basic variable (x_3) also has value 0. A pivot is indicated in the x_4 column, but when we perform the ratio test, we find that $\Delta_* = 0$. What this means is that x_3 is the blocking variable, and a pivot on entry $\bar{a}_{41}$ give the tableau

basis	z	x_1	x_2	x_3	x_4	x_5	**rhs**
x_4	0	0	0	$15/4$	1	-10	0
x_2	0	0	1	$-1/4$	0	$2/3$	4
x_1	0	1	0	$1/2$	0	-1	8
z	1	0	0	$5/2$	0	5	520

which has the *same* solution and objective value, but can now be verified to be optimal. What has occurred is a **degenerate pivot**.

Finiteness of the Simplex Method

Theorem 4.1 *If the simplex method performs no degenerate pivots, then it will always terminate in a finite number of steps.*

Proof: If a degenerate pivot does not occur, then the blocking ratio Δ_* will always be positive, and so the succeeding solution x^{Δ_*} will have objective value $z = \bar{z}_0 + \bar{c}_j\Delta_* > \bar{z}_0$. Therefore the sequence of BFSs will have *decreasing objective values* and so we can never repeat a BFS. Now BFSs are determined by their basis set, and since there are only a finite (at most $\binom{n}{m}$) such sets, then a stopping tableau must be found in a finite number of pivots.

Unfortunately, this is not enough to guarantee the finiteness of the simplex method in the presence of degenerate pivots. In fact, the simplex method *is not guaranteed to terminate* under fairly reasonable tie-breaking rules. Consider the following degenerate tableau:

basis	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**rhs**
x_1	0	1	0	0	1	1	1	1	1
x_2	0	0	1	0	1/2	$-11/2$	$-5/2$	9	0
x_3	0	0	0	1	1/2	$-3/2$	$-1/2$	1	0
z	1	0	0	0	-1	7	1	2	0

If we solve this using the standard simplex method entering/leaving rules (most negative reduced cost/minimum ration), breaking ties by choosing the leftmost or topmost among the candidate pivot columns and rows, we get the following sequence of bases:

$$[x_1\ x_2\ x_3],\ [x_1\ x_4\ x_3],\ [x_1\ x_4\ x_5],\ [x_1\ x_6\ x_5],$$
$$[x_1\ x_6\ x_7],\ [x_1\ x_2\ x_7],\ [x_1\ x_2\ x_3],\ \ldots$$

The simplex method, using these rules, will **never terminate**. Such a sequence of bases is called a **cycle**.

Corollary: *If no cycle occurs in the simplex method, then the simplex method will terminate in a finite number of steps.*

Bland's Rule

Cycles are extremely rare (if not altogether nonexistent) in real-world problems. Nevertheless, they pose theoretical problems for any results that depend upon the simplex method for a proof. Luckily, there is a simple pivot rule — called **Bland's Rule** — that avoids cycles:

Bland's Rule: Use the following rules to determine the entering and leaving variables for a simplex pivot:

- Among all of the variables eligible to enter the basis choose the one with **smallest index**.
- Among all of the variables eligible to leave the basis, again choose the one with the **smallest index**.

In the cycling example given above, for the basis $[x_1\ x_6\ x_5]$ instead of choosing the variable x_7 with the most negative $\bar{c}_j$ value, Bland's Rule would choose the variable x_2 with the smallest index among those with $-\bar{c}_j < 0$. This gives an optimal tableau.

Theorem 4.2 *If Bland's Rule is used to choose the entering and leaving variables in the simplex method, then the simplex method will never cycle.*

Proof: Suppose we obtain cycle $B^0, B^1, \ldots, B^k = B^0$ of basis sets using Bland's Rule. Define a **active** variable to be one which *enters* the basis during a pivot in this cycle (and hence will necessarily *leave* the basis during another pivot in the cycle). Note that all active variables will have value 0 throughout the cycle, whether they are basic or nonbasic. Let t be the *largest index* of a active variable.

We concentrate on two specific tableaus in the cycle: tableau $\tilde{T}$, with associated basis set $\tilde{B}$, where x_t *enters* the basis, and tableau $\hat{T}$, with associated basis set $\hat{B}$, where x_t *leaves* the basis. In tableau $\hat{T}$, suppose $t = \hat{B}_r$, and let x_s be the entering variable, that is, the pivot in tableau $\hat{T}$ occurs on entry (r, s)

There are two key points to notice about these tableaus in relation to Bland's Rule:

Tableau $\tilde{T}$: Since x_t enters the basis, then $\tilde{c}_t > 0$, and $\tilde{c}_j \leq 0$ for every other active variable x_j.

Tableau $\hat{T}$: Since x_s enters the basis, then $\hat{c}_s > 0$, and since $x_t = x_{\hat{B}_r}$ leaves the basis, then $\hat{a}_{rs} > 0$, and $\hat{a}_{is} \leq 0$ for every other active variable $x_j = x_{\hat{B}_i}$.

Now consider the basic change vector associated with the pivot performed with respect to tableau $\hat{T}$:

$$\begin{array}{rcll} \hat{v}_s & = & 1 & \\ \hat{v}_{\hat{B}_i} & = & -\hat{a}_{is} & i = 1, \ldots, m \\ \hat{v}_j & = & 0 & j \in \hat{N} \setminus \{s\} \end{array}$$

We know from the basic properties of change vectors that

$$A\hat{v} = 0 \text{ and } c\hat{v} = \hat{c}_s > 0.$$

We compute the value of $\tilde{c}\hat{v}$ two ways. First note that

$$\tilde{c}\hat{v} = (c - \tilde{y}A)\hat{v} = c\hat{v} - \tilde{y}(A\hat{v}) = c\hat{v} - \tilde{y} \cdot 0 = c\hat{v} = \hat{c}_s > 0$$

where $\tilde{y} = \tilde{c}_{\tilde{B}}\tilde{B}^{-1}$. If we write out the vector product, however, we get

$$\tilde{c}\hat{v} = \sum_{\substack{j \in \hat{B} \cap \tilde{N} \cup \{s\} \\ (\Rightarrow x_j \text{ active})}} \tilde{c}_j \hat{v}_j \qquad \begin{pmatrix} \text{every other index has} \\ \text{either } \tilde{c}_j = 0 \text{ or } \hat{v}_j = 0 \end{pmatrix}$$

$$= \underset{(\tilde{c}_t > 0,\, \hat{a}_{rs} > 0)}{-\tilde{c}_t \hat{a}_{rs}} - \underset{(\tilde{c}_j \le 0,\, \hat{a}_{is} \le 0)}{\sum_{j = \hat{B}_i \in \hat{B} \cap \tilde{N} \setminus \{t\}} \tilde{c}_j \hat{a}_{is}} + \underset{(\tilde{c}_s \le 0)}{\tilde{c}_s \cdot 1} < 0$$

a contradiction. It follows that no such cycle can exist.

Conclusions

Theorem 4.3 *The Phase I/Phase II Simplex Method, using Bland's anticycling rule, will always terminate in a finite number of steps.*

Corollary: *Any LP is either infeasible, unbounded, or has an optimal solution.*

Corollary: *For any equality form LP (P):*

- *If (P) has a feasible solution, then (P) has a* basic *feasible solution.*
- *If (P) has an optimal solution, then (P) has a* basic feasible *optimal solution.*

Computational Complexity of the Simplex Method

How many simplex pivots are required to guarantee that a solution will always be reached?

- There are $\binom{n}{m}$ possible bases in an $m \times n$ equality LP system. This number can grow as fast as 2^m.

- Again, there are "laboratory" examples of classes of LPs that exhibit exponential growth in the number of Phase II pivots, using almost any "local" pivot rules.

- **In practice,** the number of pivots to optimality seems to grow proportional to (linear in) m.

Goal: To construct a **polynomial-time algorithm** for solving LPs. The **interior point method** is one such algorithm. (More on this method later in the course.)

Big Open Question: Is there an implementation of the simplex method (effective choice of pivots) that will make it a polynomial-time algorithm?