Discontinuous Galerkin Methods for Spectral Wave/Circulation Modeling

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Outline

Action Balance Equation

- The governing equation is the **Action Balance Equation**, which is

$$\frac{\partial}{\partial t}N + \nabla \cdot \mathbf{c}N = \frac{S}{\sigma}$$

where

$N(x, y, \sigma, \theta, t)$ is the action density
(x, y) are the Cartesian coordinates
$\sigma = 2\pi f$ is the relative frequency
θ is the direction of wave propagation
t is time

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \sigma}, \frac{\partial}{\partial \theta}\right)$$

$\mathbf{c} = (c_x, c_y, c_\sigma, c_\theta)$ are the propagation velocities
S is the source term

Propagation Velocities - Geographical Space

- The propagation velocities are

$$c_x = c_{g,x} + u_x \qquad c_\sigma = \frac{\partial \sigma}{\partial d}\left(\frac{\partial H}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}} H\right) - c_g \mathbf{k}\cdot\frac{\partial \mathbf{u}}{\partial s}$$

$$c_y = c_{g,y} + u_y \qquad c_\theta = -\frac{1}{k}\left(\frac{\partial \sigma}{\partial H}\frac{\partial H}{\partial m} + \mathbf{k}\cdot\frac{\partial \mathbf{u}}{\partial m}\right)$$

where

$\mathbf{u} = (u_x, u_y)$ is the velocity of the current

c_g is the group velocity

H is the total water depth

$\mathbf{k}$ is the wave number

(s, m) are spatial coordinates with s being normal to the wave direction θ and m being perpendicular to the wave direction θ.

Discontinuous Galerkin Method

- We define the domain Ω to be a tensor product of the geographic domain Ξ and the spectral domain κ.
- An element $\Omega_e = \Xi_g \times \kappa_s$ where Ξ_g is a triangular element and $\kappa_s = (\theta_m, \theta_{m+1}) \times (\sigma_\ell, \sigma_{\ell+1})$.
- Define

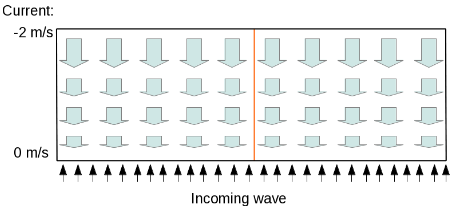
$$V_h = \{v \in L^2(\Omega) : v|_{\Omega_e} = \Phi|_{\Xi_g} * \Psi|_{\kappa_s} \in P^p(\Xi_g) * P^q(\kappa_s) \forall \Omega_e\}$$

- Approximate $N \approx N_h \in V_h$ and then obtain

$$\left(\frac{\partial N_h}{\partial t}, v_h\right)_{\Omega_e} - (\mathbf{c}N_h, \nabla v_h)_{\Omega_e} + \langle \mathbf{c}\hat{N}_h \cdot \mathbf{n}, v_h\rangle_{\partial\Omega_e} = \left(\frac{S}{\sigma}, v_h\right)_{\Omega_e}$$

- Upwind fluxes are taken to determine the value of $\mathbf{c}\hat{N}_h \cdot \mathbf{n}$ on the edges.
- The Dubiner basis functions are used in geographic space and products of Legendre polynomials are used in spectral space.
- We use Runge-Kutta for time-stepping.

Opposing Current



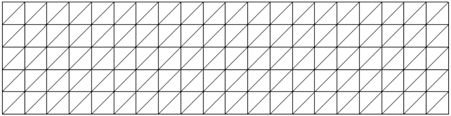
- Deep water with constant depth (10,000 m)
- Incoming waves have $H_s = 1$m, a Gaussian-shaped frequency spectrum with $f_{peak} = 0.1$ Hz and standard deviation 0.01 Hz, and a $\cos^{500}(\theta)$ directional distribution with a main direction of 90°.
- Geographic domain is $16,000m \times 4,000m$
- We are interested in the steady state solution along the orange line of the significant wave height,

$$H_s = 4\sqrt{\iint N\sigma d\sigma d\theta}.$$

Opposing Current

Coarse grid

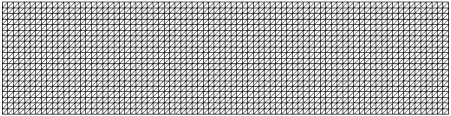
- Geographic space



- Spectral space
  - 31 directional elements and 20 frequency elements

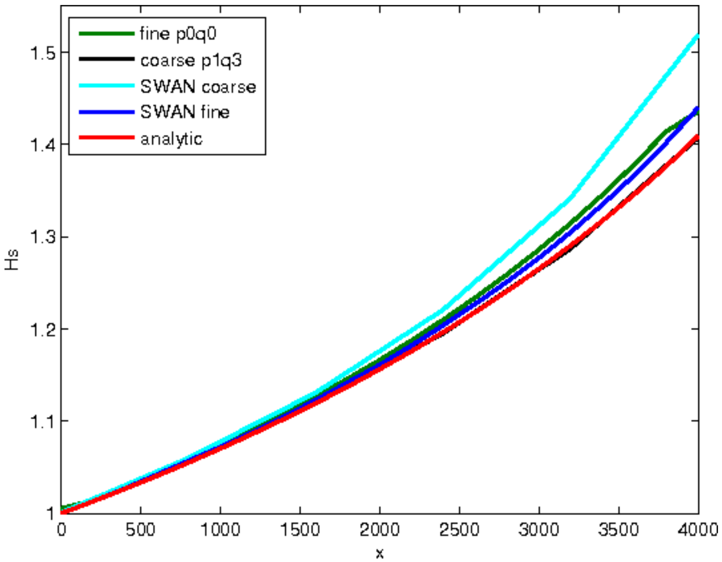
Fine grid

- Geographic space

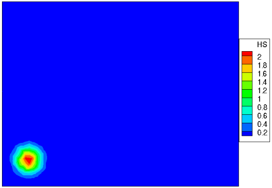


- Spectral space
  - 60 directional elements and 40 frequency elements

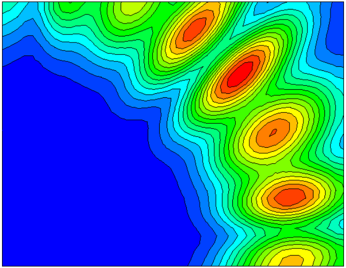
Opposing Current



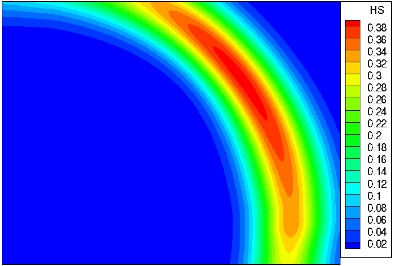
Garden Sprinkler Effect



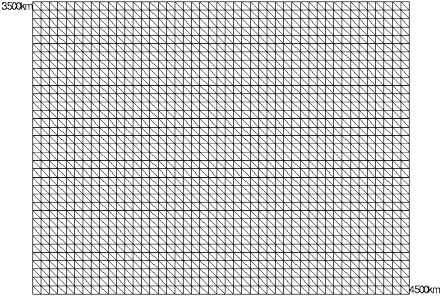
Initial Condition



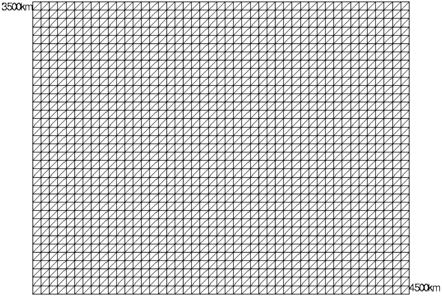
"Garden Sprinkler Effect"



Fine Grid Solution

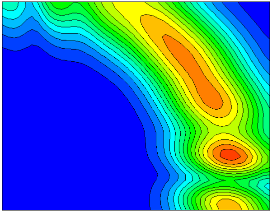


Mesh 1

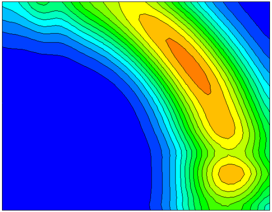


Mesh 2

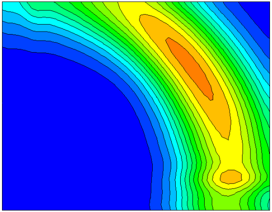
Garden Sprinkler Case



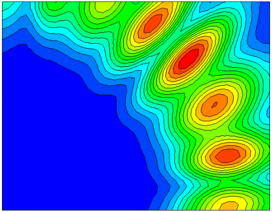
Mesh 1: p=0, q=0



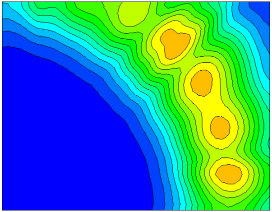
Mesh 1: p=0, q=1



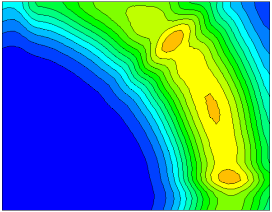
Mesh 1: p=0, q=2



Mesh 2: p=0, q=0

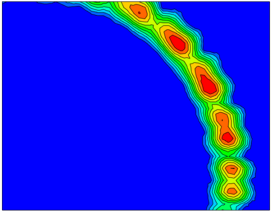


Mesh 2: p=0, q=1

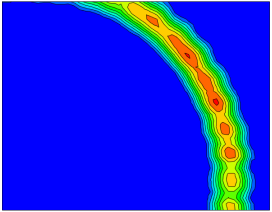


Mesh 2: p=0, q=2

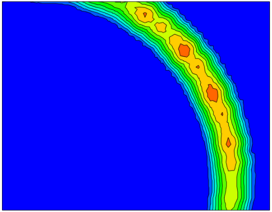
Figure: The significant wave height is shown with contours at 0.02 m.



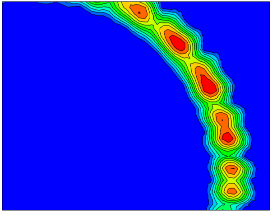
Mesh 1: p=1, q=1



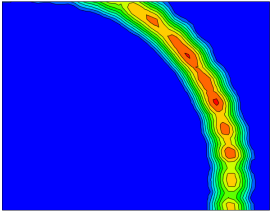
Mesh 1: p=1, q=2



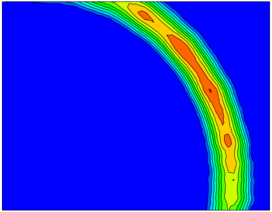
Mesh 1: p=1, q=3



Mesh 2: p=1, q=1



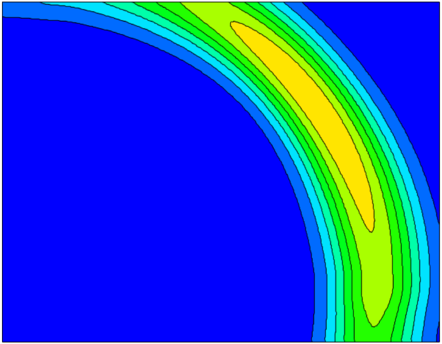
Mesh 2: p=1, q=2



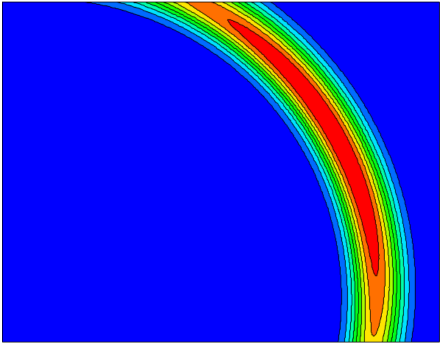
Mesh 2: p=1, q=3

Figure: The significant wave height is shown with contours at 0.05 m.

Garden Sprinkler Case



Fine Mesh Solution: p=0, q=1

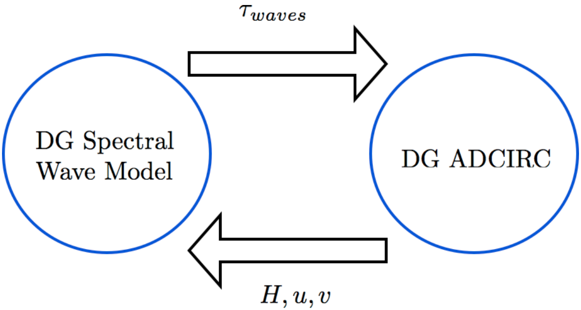


Fine Mesh Solution: p=1, q=1

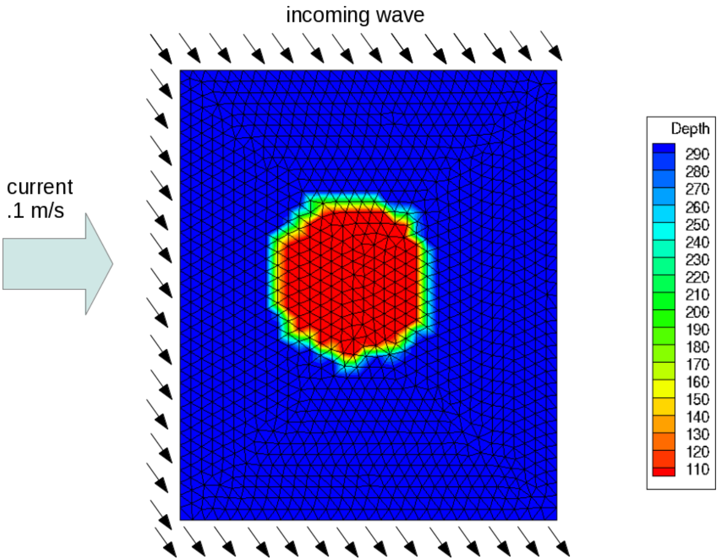
Figure: The significant wave height is shown with contours at 0.05 m.

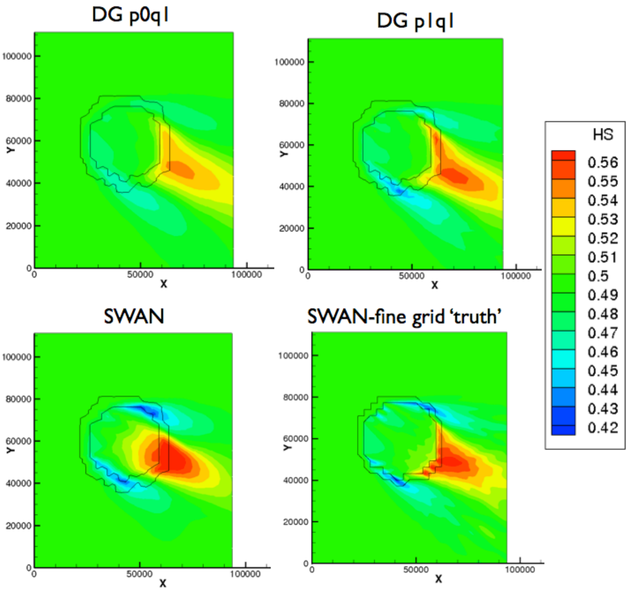
Coupling Between Wave and Circulation Models

- Each model will be run on the same geographic mesh.
- Information will be passed between the models every N time steps.



Coupled Model: Near Circular Shoal




DG p0q1
DG p1q1
SWAN
SWAN-fine grid 'truth'
HS
0.56
0.55
0.54
0.53
0.52
0.51
0.5
0.49
0.48
0.47
0.46
0.45
0.44
0.43
0.42
100000
80000
60000
40000
20000
0
Y
50000
100000
X

Future Work

Future Work

- Validating the model in realistic cases with source terms
- Optimizing the wave model
- Adaptivity for the wave model
- Tightly couple the models