An Appreciation of the Work of Jim Stasheff

by John McCleary

to Jim, on your 60th birthday

My very first meeting with Jim occured before I started my graduate studies. Seeking the advice of graduate students at Temple University about coming there to study, one of Jim's students pointed over his shoulder at Jim and whispered that we were in the presence of the "king of topology." Jim's warm handshake and interest in me as a prospective student was no small influence on my decision to go to Temple—also the possibility to study with the "king of topology."

When I searched MathSci for references to Jim's papers and to his work in reviews, I fished up about 70 papers, and another 100+ papers refering to the keyword "`Stasheff`." I can't do justice to such a broad array of contributions—to homotopy theory, algebra, differential topology, and mathematical physics. In this short paper, I want to touch upon some of my favorite parts of Jim's work and 'appreciate' it, as art critics might appreciate a work—for its beauty, its insight, and its influence.

§1. Homotopy invariant algebraic notions

In our collective memory of homotopy theory is the diagram:

It depicts one of the first true brushes with what homotopy theory can do—though the multiplication of loops is not associative, it is associative up to homotopy, and so one of the properties of a group holds for the fundamental group. The loop multiplication on the space of based loops, ΩX, gives it the strcture of an H-space with multiplication that is associative up to homotopy. Moore's version [?] of the space of based loops shows that there is a based loop space which is homotopic to the familiar one for which the loop multiplication is associative on the nose. The conclusion is deceptively simple, and open to further investigation—*associativity is not a homotopy invariant property.*

The foundations for a homotopy-invariant notion of associativity are built in Jim's thesis [**?**, **?**] (Part I from Princeton under Moore, Part II from Oxford under James). The basic question is one of recognition—how do we recognize a space Y with $Y \simeq \Omega X$ for some space X? The answer is provided by Milnor [**?**]: When Y has a classifying space, for then, $Y \simeq \Omega BY$. However, expediting this answer is no mean feat. Sugawara [**?**] and Dold and Lashof [**?**] set out conditions that reflect the building of the projective space of an H-space and its classifying space, if it exists. Jim begins by interpolating all the stages between H-space and group-like space.

Definition 1. *An* A_n**-structure** *on a space* X *consists of an* n*-tuple of maps* $(p_1, p_2, \ldots, p_n)$

$$X = E_1 @ >\subset>> E_2 @ >\subset>> \cdots @ >\subset>> E_n @Vp_1VV @Vp_2VV @. @Vp_nVV B_1 @ >\subset>> B_2 @ >\subset$$

such that $p_{i*}\, \pi_*(E_i, X) \to \pi_*(B_i)$ *is an isomorphism for* $1 \leq i \leq n$ *and there is a contracting homotopy* $h\, CE_{n-1} \subset E_n$ *such that* $h(CE_{i-1}) \subset E_i$ *for each* $i > 1$.

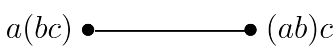
One can recognize the projective plane of an H-space as the second space B_2*, and an* A_∞*-structure gives a classifying space. The next step is to alter Sugawara's definition of a strongly homotopy multiplicative map to suit the case of associativity more precisely and naturally. This leads to a now familiar family of polyhedra.*

Definition 2. Let K_i *denote the CW-complex constructed inductively as follows:* $K_2 = *$*, a point. Let* K_i *be the cone* CL_i *where* L_i *is the union of copies of* $(K_r \times K_s)_k$ *of* $K_r \times K_s$*, where* $r + s = i + 1$*, and* k *corresponds to inserting a pair of parentheses into* i *symbols*

$$(1 \quad 2 \quad \cdots \quad k-1 \quad (k \quad k+1 \quad \cdots \quad k+s-1) \quad k+s \quad \cdots \quad i).$$

The intersection of copies corresponds to inserting two pairs of parentheses with no overlap or with one as subset of the other. Define $\partial_p(r,s)\, K_r \times K_s \to K_i$ *to be the inclusion of the copy indexed by* $(1\ 2\ \cdots\ (p\ p+1\ \cdots\ p+s-1)\ \cdots\ i)$.

The first few examples of these polyhedra are pictured:



We first observe that these objects generalize Sugawara's definitions.

Proposition 3. K_i *is an* $(i-2)$*-cell.*

Definition 4. An A_n**-space** $(X; M_1, \ldots, M_n)$ *consists of a space* X *and a family of maps defined* $M_i\, K_i \times X^{\times i} \to X$, $i \le n$ *such that*

M_2 *is a multiplication with unit.*

For $\rho \in K_r$ *and* $\sigma \in K_s$,

$$M_i(\partial_k(r,s)(\rho,\sigma), x_1, \ldots, x_i) = M_r(\rho, x_1, \ldots, x_{k-1}, M_s(\sigma, x_k, \ldots, x_{k+s-1}), x_{k+s}, \ldots, x_i).$$

For $\tau \in K_i$, $i > 2$, we have

$$M_i(\tau, x_1, \ldots, x_{j-1}, e, x_j, \ldots, x_i) = M_{i-1}(s_j(\tau), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i)$$

where the maps $s_j\ K_i \to K_{i-1}$ are degeneracies. If the M_i exist and satisfy these conditions for all $i \geq 2$ we speak of $(X; M_i)$ as an A_∞-space.

Out of this combinatorial tour-de-force come many consequences:

I. An A_n-space Y has a projective n-space $YP(n)$. An A_∞-space Y has the homotopy type of a loop space, that is, $Y \simeq \Omega X$, for some X.

II. There are H-spaces that are A_n, but not A_{n+1} for certain n. This opened up a new schema that contributed to an on-going question of how much H-spaces were like Lie groups up to homotopy.

III. When X is an A_n-space, then $C_*(X)$ enjoys extra algebraic structure.

Definition 5. Let k be a field. An $A(n)$-**algebra** *is an $n+1$-tuple $(A, m_1, m_2, \ldots, m_n)$ where A is a graded k-module, $A = \bigoplus_i A_i$, and k-linear maps $m_i\ A^{\otimes i} \to A$ satisfying the following properties:*

m_i raises degree by $i - 2$, that is, $m_i([A^{\otimes i}]_q) \subset A_{q+i-2}$, for all q.

If $u = u_1 \otimes \cdots \otimes u_i \in A^{\otimes i}$, then

$$\sum_{r+s=i+1, 1\leq p\leq r} \pm m_r(u_1 \otimes \cdots \otimes m_s(u_p \otimes \cdots \otimes u_{p+s-1}) \otimes \cdots \otimes u_i) = 0,$$

where $\pm$ is determined by $(-1)^\epsilon$ where $\epsilon = (s+1)p + s\left(i + \sum_{j=1}^{p-1} \dim u_j\right)$. An $A(\infty)$-**algebra** *consists of an augmented k-module A and maps $m_i\ A^{\otimes i} \to A$ satisfying the conditions above for all $i \geq 1$.*

This definition generalizes the notion of a chain complex (an $A(1)$-algebra with $m_1 = \partial$) and a differential graded algebra (an $A(2)$-algebra with $m_2 =$ the product on A). If (A, ∂, μ) is an associative differential graded algebra, then $(A, \partial, \mu, 0, 0, \ldots)$ is an $A(\infty)$-algebra structure on A.

IV. If $\overline{A} = \ker(\eta\ A \to k)$ is the kernel of the augmentation, then the bigraded k-module associated to an $A(n)$-algebra

$$\tilde{B}(A) = \bigoplus_{i=0}^{n} \overline{A}^{\otimes i}$$

has a differential built out of the m_s and determines a spectral sequence useful in studying A_n-spaces. This is called the **tilde bar construction** *and it has the remarkable property that, for an A_∞-space, it makes the algebra and topology coincide, that is, there is a filtration-preserving mapping*

$$\theta\ \tilde{B}(C_*(X;k))@>>> C_*(BX;k)$$

that is a chain equivalence. Inside the tilde bar construction there are Massey products with respect to the Pontryagin multiplication (called Yessam products by Jim, a name that never caught on) which are controlled by the A_n-structure, and determine the differentials in the associated spectral sequence.

The notion of A_n-spaces and the clarity they provide for the recognition problem for topological groups became the basis for the development of homotopy invariant algebraic structures. In particular, the recognition problem for infinite loop spaces and the simultaneous interest in coherence properties in categories led to the idea of an **operad** *([**?**] and [**?**]). These ideas are enjoying a renaissance [**?**] in recent years in which new families of polyhedra, new phenomena for recognition, and new associated structures are being found in this formalism.*

Continuing in this vein, Jim flexed his topological muscles (quasi-fibrations, higher homotopy structures) to produce some further remarkable results:

*I. The theory of characteristic classes for principal bundles with fibre a Lie group was a model example in its thorough and far-reaching structure. The work of Steenrod [**?**], Milnor [**?**], and Borel [**?**] presented a beautiful edifice to emulate in the realm of homotopy theory. If H-spaces are like Lie groups, then this classification theory ought to have a topological analogue. This is provided in the following result of [**?**].*

Theorem 6 (Stasheff Classification Theorem. Let H denote the topological monoid of self-equivalences of a finite CW-complex F. Let $LF(X)$ denote the set of fibre homotopy equivalence classes of Hurewicz fibrations $p\, E \to X$ with fibres of the homotopy type of F. Then the functors, $[\ \ ,BH]$ and $LF(\ \ )$ are naturally equivalent.

Of some interest is the use of terminology from the theory of differential equations in this paper. It foresees other analogues to the smooth theory.

II. In the algebraic case, always consequent from Jim's topological constructions, there is a notion of homotopy and hence a notion of strongly homotopy *multiplicative maps. In [?] Jim and Steve Halperin turned this notion on the problem of computing the cohomology of homogeneous spaces. Here the input couldn't be nicer—$H^*(BG;k)$ and $H^*(BH;k)$ are polynomial algebras, that is, free commutative. One knows that the Eilenberg-Moore spectral sequence for the fibration $G/H \to BH@ > Bi >> BG$ converges to $H^*(G/H;k)$. The problem is to go from $Tor_{H^*(BG;k)}(k, H^*(BH;k))$ to $H^*(G/H;k)$, which is isomorphic to $Tor_{C^*(BG;k)}(k, C^*(BH;k))$. The line of argument, used by Cartan [?] in the case $k = \mathbb{R}$ where there are the deRham cochains for computing cohomology, is to compare $H^*(BG;k)$ and $C^*(BG;k)$ directly. Since one algebra is free commutative, we could guess a morphism by choosing representatives. However, the other algebra is only commutative up to homotopy, and so the most we could hope for is a strongly homotopy multiplicative map—a hope predicated on the right viewpoint. This program eventually led to a powerful collapse result for the Eilenberg-Moore spectral sequence proved by H.J. Munkholm [?].*

§2. "How algebraic is algebraic topology?"

In the 1970's there was considerable development of a line of ideas pioneered by D. Sullivan [?] and D. Quillen [?]. The main idea was that of localization, at a prime p, at a set of primes, over the rationals, an idea introduced by Serre [?]. The tools of localization gave new access to the fracturing of homotopy into its separate components indexed over the primes. Zabrodsky [?] showed how to apply these ideas backwards to H-spaces and "mix" homotopy types to produce new H-spaces. The Hilton-Roitberg criminal [?] is one such H-space, spoiling the homotopy theoretic version of Hilbert's Fifth Problem—does an H-space that is also a manifold have to be a Lie group? Jim added significantly to this picture [?] by analyzing the family of possible H-spaces given by the Hilton-Roitberg construction and determining which were, in fact, topological groups. His main result includes the following statement of the failure of the homotopy version of Hilbert's Fifth Problem.

Theorem 7. There is a topological group G of the homotopy type of a compact manifold, which is not of the homotopy type of any Lie group.

Concern with rational homotopy equivalences plays a considerable role in localization theory and as Sullivan and Quillen showed, this is a matter for the algebraic side of algebraic topology. Sullivan introduced rational deRham theory drawing attention to the homotopy theory of commutative differential graded algebras over $\mathbb{Q}$ as equivalent to the rational homotopy theory of spaces.

*A jumping off point for [**?**] is the question titling this section, "how algebraic is algebraic topology?" Sullivan claimed that the vanishing of Massey products in rational cohomology implied that the rational cohomology algebra of such a space characterized its rational homotopy type, for example, its homotopy groups tensored with* $\mathbb{Q}$ *are calculable from the cohomology. Starting with a commutative graded algebra* H $(= H^*(X;\mathbb{Q}))$*, one can try to recover as much of the rational homotopy theory as is possible from* H*. Let* $(A^*_{PL}(X), d_X)$ *denote the Sullivan-deRham cochains on* X*. We have* $H(X;\mathbb{Q}) \cong H(A^*_{PL}(X), d_X)$*. Sullivan mimics the Postnikov tower construction algebraically to produce a minimal model of* X *which is a free graded-commutative differential algebra* $(\Lambda Z, d)$ *and a morphism of DG-algebras* $\rho(\Lambda Z, d) @ >>> (A^*_{PL}(X), d_X)$ *that induces a homology isomorphism. When you carry out the construction of a minimal model for* $(H, 0)$*, you take a page out of the book of commutative algebra and you have produced a Tate-Jozefiak resolution of the commutative graded algebra* H*.*

*In [**?**] Steve and Jim ask what you can do with the model* $(\Lambda Z, d)$ *associated to* $(H^*(X;\mathbb{Q}), 0)$ *to recover information about the homotopy type of the space* X*. Experience with small homological models leads to the notion of a perturbation of the differential d and the following theorem.*

Theorem 8. For a space X *with* $H = H^*(X;\mathbb{Q})$*, there is a differential* D *on a Tate-Jozefiak resolution* $(\Lambda Z, d)$ *associated to* H *such that* $D - d$ *lowers the resolution degree by at least two and there is a homology equivalence*

$$(\Lambda Z, D) @ > \pi >> (A^*_{PL}(X), d_X).$$

Though the new model $(\Lambda Z, D)$ *need not be minimal in the sense of Sullivan, it is a homotopy invariant of* X *and this leads to a toe-hold from which to climb higher:*

1. One can construct an obstruction theory for resolving the question of whether one can realize a fixed algebraic isomorphism $H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$ *via a rational homotopy equivalence* $Y \to X$*. This is the key to the question of formality, that is, whether a space is determined rationally by its cohomology.*

2. One can study the homotopy types over $\mathbb{Q}$ *with a fixed cohomology algebra by studying the perturbations of the free commutative filtered graded model* $(\Lambda Z, d)$*. This analysis was carried out by Jim with Mike Schlessinger in [**?**]. The perturbed differential,* D *may be written*

$$D = d + d_2 + d_3 + \cdots,$$

where d_i lowers the resolution degree by i. Since $D^2 = 0$, d_2 is a cocycle in the complex $Der(\Lambda Z)$ of derivations on ΛZ with differential $\theta \mapsto [d, \theta]$. Now $Der(\Lambda Z)$ is a differential graded Lie algebra over $\mathbb{Q}$ studied by Quillen [?]. In particular, Quillen associated a free commutative DG-coalgebra to $(Der(\Lambda Z), \theta \mapsto [d, \theta])$, denoted $(C(Der(\Lambda Z, d), D)$ with D a coderivation and

$$D(\theta) = [d, \theta], \quad D(\theta \wedge \phi) = [d, \theta] \wedge \phi \pm \theta \wedge [d, \phi] + \frac{1}{2}[\theta, \phi]$$

Then, the algebraic cohomology of this DG-Lie algebra is calculable from Quillen's model:

$$H^{dgLie}(Der(\Lambda Z, d)) \equiv H(C(Der(\Lambda Z, d), D).$$

What this can tell us about rational homotopy types is contained in the main result of [?]:

Theorem 9. Perturbations D of $(\Lambda Z, d)$ are in one-to-one correspondence with path components of $C(Der^{-}(\Lambda Z, d), D)$ where $Der^{-}(\Lambda Z, d)$ consists of weight (= resolution degree + internal degree) decreasing derivations.

Two beautiful facts emerge from this study:

1. Algebro-geometric methods may be applied to describe the path components and hence give a moduli space of the rational homotopy types of spaces with a given cohomology algebra.

2. The homotopy theory of perturbations of DG-algebras was advanced with the use of DG-Lie algebra methods and the homotopy theory of DG-Lie algebras was established.

Perturbation theory in homological algebra emerged from the work of Cartan, Moore, Ed Brown, G. Hirsch, Ron Brown, and W. Shih in the 50's. Victor Gugenheim [?], with Milgram [?] and with May [?], gave it an elegant setting that leads to a homotopy theory of differential graded algebras based on A_∞-maps. Jim's work with Gugenheim [?] and Lambe [?] have codified the study of perturbations in the homotopy theory of differential graded algebras. Just as the idea of strong homotopy multiplicative mappings is fundamental to the study of homotopy invariants, so also is perturbation theory fundamental to the applications of homological algebra.

Example: *Gugenheim and Stasheff [?] extended G. Hirsch's work [?] to make it multiplicative, that is, for a fibration, $F \to E \to B$, there is a* *perturbation of the product differential on $C^*(B) \otimes H^*(F)$, call it D, with $H(C^*(B) \otimes H^*(F), D) \cong H^*(E)$, as algebras.*

§3. Natural structures and nature

One of Jim's long-term interests is physics. I remember his participation in the relativity seminar at Temple in the 70's. It is no surprise that his latest activity is mathematical physics. However, it may be a surprise that the structures of interest in physics admit the kind of manipulation topologists are used to.

*In [**?**] Jim studied Poisson algebras, which are commutative, associative algebras equipped with an anticommutative bracket product $\{\ ,\ \}$ which satisfies the Leibniz rule with respect to the commutative product:*

$$\{f, gh\} = \{f, g\}h + f\{g, h\}.$$

If W (a phase space) is a symplectic manifold with symplectic 2-form ω, then a Poisson bracket can be defined as follows: For $f, g \in C^\infty(W)$

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f)$$

where $\omega(X_f, -) = df$ determines the vector field X_f. Then

$(C^\infty(W), \cdot)$ is a commutative algebra.

$(C^\infty(W), \{\ ,\ \})$ is a Lie algebra.

$\{f,\ \}\ C^\infty(W) \to C^\infty(W)$ is a derivation with respect to the product.

Suppose there is a submanifold $V \subset W$ of "constraints" on which a solution to a Hamiltonian system is required to lie. The submanifold V may be taken to be the zero set of some set of functions $\phi_\alpha\ W \to \mathbb{R}$ called the constraints. Let I denote the ideal generated by $\{\phi_\alpha\}$.

If I is closed under the bracket, the ϕ_α are said to be **first class** *constraints (following Dirac). Then the vector fields X_{ϕ_α} are tangent to V and give rise to a foliation, F of V. The algebra $C^\infty(V/F)$ is identified with observables of the system and with an I-invariant subset of $C^\infty(W)/I$. Expand the observables to $\Omega^*(V, F)$, the deRham forms on vertical vector fields. We want to model this DG-algebra in some sensible way.*

*Following the yoga that the right perturbation on a small thing of interest gives more control of what you seek, Jim introduced an intermediate complex à la Rinehart [**?**] and Tate to prove the following result.*

Theorem 10. There is a DG-Poisson algebra model

$$\pi\left((\Lambda\Psi)^* \otimes C^\infty(W) \otimes \Lambda\Psi, \partial\right) \to (\Omega^*(V,F), d)$$

where ∂ is $\delta + d+$ "terms of higher order," and δ is defined as in the construction of the Koszul complex, d is the differential due to Rinehart.

The vector space Ψ is negatively graded and the degree in the free graded-commutative algebra $\Lambda\Psi$ gives the so-called **ghost** *degree. In the manner of syzsygies, the resolution degrees explain ghosts, ghosts-of-ghosts, and so on. Furthermore, $\partial = \{Q, \ \}$, where Q is the BRST generator [**?**].*

There is a lot more to say and parts of Jim's work that I have not touched on. The papers in this volume represent a sample of how Jim's work has inspired many of us. In fact, his influence cannot simply be measured in these terms. The breadth of topics represented by the titles of PhD theses written under Jim attest to his instincts far afield of the list one might compile from his bibliography. I am no longer surprised when I read a paper and find an acknowledgement to Jim. All of us have benefited from Jim's generosity with ideas—the e-mail network alone attests to it.

I want to end with a bit of prose from Jim's thesis:

> *To study spaces which admit A_n-structures, we can work directly with the maps.... In this case of a topological group, this amounts to working only with the classifying bundle and never mentioning group operations. This would be an exercise in rectitude of thought of which it would be pointless to countenance this austerity, for not only would it eliminate a useful perspective on the subject, but, by disguising its own main point, it would place the reader behind a cloud of unknowing. A similar remark can be made about A_n-spaces.Jim has a habit of bringing his readers,*
>
> *students, and coworkers out from "behind a cloud of unknowing." We wish him many more years of the same.*

Cited Papers

*[**?**] Borel, A., Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts, Ann. of Math.* **57** *(1953), 115–207.*

*[**?**] Cartan, H., Notions d'algèbre differéntielle; applications aux groupes de Lie et aux variétés où opère un groupes de Lie, Colloque de topologie, CBRM Bruxelles (1950), 15–27.*

[?], Principal quasi-fibrations and etc., Ill. J. Math., **3***(1959), 285–305.*

[?] Gugenheim, V.K.A.M., On the chain complex of a fibration, Ill. J. Math. **16***(1972), 353–*

[?] Gugenheim, V.K.A.M. and Milgram, R.J., On successive approximations in homological algebra, TAMS **150***(1970), 157–182.*

[?] Gugenheim, V.K.A.M. and May, J.P., On the theory and applications of differential torsion products, Memoir AMS **142***(1974).*

[?] Henneaux, M. and Teitelboim, C., Quantization of Gauge Systems, Princeton University Press, NJ, 1992.

[?] Hilton, P.J. and Roitberg, J., On principal S^3-bundles over spheres, Ann. of Math., **90***(1969), 91–107.*

[?] Hirsch, G., Sur les groupes d'homologie des espaces fibrés, Bull. Soc. Math. Belg. **6***(1954), 79–96.*

[?] Loday, J.L., La renaissance des opérades, Astérique **237***(1996), 47–74.*

[?] Mac Lane, S., Categorical algebra, BAMS **75***(1965) 40–106.*

[?] May, J.P., The Geometry of Iterated Loop Spaces, LNM **271***(1972).*

[?] Milnor, J.W., Construction of universal bundles I, II, Ann. of Math. **63***(1956), 272–284, 430–436.*

[?] Moore, J.C., The double suspension and p-primary components of homotopy groups of spheres, Bol. Soc. Mat. Mexicana **1***(1956), 28–37.*

[?] Munkholm, H.-J., The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps, JPAA **5***(1974), 1–50.*

[?] Quillen, D.G., Rational homotopy theory, Ann. of Math. **90***(1969), 205–295.*

[?] Rinehart, G., Differential forms for general commutative algebras, TAMS **108***(1963), 195–222.*

[?] Serre, J.-P., Groupes d'homotopie et classes de groupes abéliens, Ann. of Math., **58***(1953), 258–294.*

[?] Steenrod, N., Topology of Fibre Bundles, Princeton U. Press, 1951.

[?] Sugawara, M., On the homotopy-commutativity of groups and loop spaces, Mem. Coll. Sci, Univ. Kyoto Ser. A Math. **33***(1960), 257–269.*

[?] Sugawara, M., On a condition that a space is an H-space, Math. J. Okayama Univ., **6***(1956-7), 109–129.*

*[**?**] Sullivan, D., Geometric Topology, part I: Localization, Periodicity, and Galois Symmetry, MIT notes, 1970.*

*[**?**] Zabrodsky, A., Homotopy associativity and finite CW-complexes, Topology* **9***(1970), 121–128.*