The Nil Hecke Ring and Cohomology of $G/P$
for a Kac–Moody Group $G^*$

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INTRODUCTION

To any (not necessarily symmetrizable) generalized $l \times l$ Cartan matrix $A$, one associates a Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$ and group $G = G(A)$. If $A$ is a classical Cartan matrix, then $G$ is a finite dimensional semi-simple algebraic group over $\mathbb{C}$. We refer to this as the finite case. In general, one has subalgebras of $\mathfrak{g}$: $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}$, the Cartan subalgebra, Borel subalgebra, and a parabolic subalgebra, respectively. One also has the corresponding subgroups $H \subset B \leq P$. Let $W$ be the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ and let $\{r_i\}_{1 \leq i \leq l}$ denote the set of simple reflections. The group $W$ operates on $\mathfrak{h}$ (and hence on its dual space $\mathfrak{h}^*$).

$W$ parametrizes the Schubert cell decomposition of the generalized flag variety $G/B = \bigcup_{w \in W} V_w (= Bw^{-1}B/B)$. (A suitable subset $W^1 \subset W$ does the same for $G/P$.) Our principal concern is the cohomology ring $H(G/B)$ (more generally $H(G/P)$) and in fact the cohomology ring of arbitrary (left) $B$-stable closed subspaces of $G/P$.

Now besides having a ring structure and having a distinguished basis consisting of Schubert classes (given by the dual of the closures of Schubert cells, $H(G/B)$ is also a module for $W$. In addition, in the finite case, a ring

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of operators $A$ (with $C$-basis $\{ A_w \}_{w \in W}$) on $H(G/B)$ was introduced in [3], where $A_w$ ($1 \leq i \leq l$), although defined algebraically, correspond topologically to the integration on fiber for the fibration $G/B \to G/P_i$ ($P_i$ is the minimal parabolic containing $r_i$). Kac and Peterson have extended the definition of the ring of operators $A$ on $H(G/B)$ to the general case and they have used these operators to study the topology of $G$ (as well as $G/B$).

The problems, we wish to deal with, are to describe $H(G/B)$:

(1) as a ring, in particular the cup product of two arbitrary Schubert classes, and

(2) as a module for $W$ and $A$.

Our main result is that all these structures arise very naturally from a single ring $R$, which admits a simple and concrete definition, using only the Weyl group $W$ and its representation on $h^*$ and which has some rather remarkable properties. We refer to $R$ as the nil Hecke ring, corresponding to the pair $(W, h^*)$ (see (4.12)).

We would like to remark that there are a number of serious obstacles in trying to directly pass from the finite to the general infinite case, and as a consequence we have sought a new approach. Among the obstacles are (1) the characteristic homomorphism: $S(h^*) \to H(G/B)$ fails to be surjective in general, (2) the failure of complete reducibility of the $W$-modules $S(h^*)$ and $H(G/B)$, (3) the absence of "harmonics," and (4) the absence of the fundamental (top) cohomology class and as a consequence, the absence of Poincaré duality. An approach, which remains valid in the general case, was motivated from [3, Theorem 5.9] a result of the first author. This theorem arises from the correspondence of the Lie algebra cohomology $H(n)$ ($n$ is the nil-radical of $b$) and $H(G/B)$ established by the first author in [17] in the finite case and was established, in the general case, by the second author in [23].

It may be mentioned that different aspects of the topology of $G$, $G/B$ (and $G/P$) in the infinite case have been studied, among others, by Bott (who has done an extensive work on the topology of loop groups via Morse theory), Iwahori and Matsumoto [12] and more recently by Garland and Raghunathan [10], Tits [30], Kac and Peterson [21, 22], Gutkin and Slodowy [11], Kumar [23, 24], Kac [14], and Pressley and Segal [27].

We describe the contents of the paper in more detail. Section 1 is devoted to recalling some standard facts from Kac–Moody theory and setting up notations to be used throughout the paper. In Section 2 we establish a certain relationship between the cohomology of $G/B$ with $\text{End}_b H^*(n)$, where $n$ is the nil radical of the Borel subalgebra $b$. The main result of this chapter is Theorem (2.12) (see also Remark (2.13)(a)), which asserts that there is a graded algebra isomorphism from $H(G/B)$ with $\text{Gr } C \{ W \}$, where
\( \mathbb{C}\{W\} \) is the algebra (under pointwise addition and pointwise multiplication) of all the functions: \( W \to \mathbb{C} \) and \( \text{Gr} \mathbb{C}\{W\} \) denotes the associated graded algebra, with respect to some "natural" filtration of \( \mathbb{C}\{W\} \). Let us recall that \( \text{End}_H H^*(n) \) can be identified with \( \mathbb{C}\{W\} \) by a result of Garland and Lepowsky [9]. The filtration of \( \mathbb{C}\{W\} \) arises from a filtration of \( \text{End} C(n) \), which in turn is a "super" analog of the usual filtration of differential operators on a manifold. Our proof of this theorem is based upon the correspondence of the (Lie algebra) cohomology of \( n \) with \( H^*(g, h) \), as given by the "d, \partial-Hodge theory," proved by the first author in the finite case [17] and established by the second author, in the general case [23].

In Section 3 we construct [23] certain \( d, \partial \) harmonic forms \( \{\delta^w\}_{w \in W} \subset C(g, h) \), which are dual to (up to a positive real number depending upon \( w \)) the Schubert varieties \( \{\overline{P}_w\} \). It is further shown [23] that (properly defined) \( \int_{\nu \cdot s^w} = (-1)^{n^w \cdot 1/2} \int_{h \cdot u^{n^w \cdot 1} \cap U^{n^w \cdot 1}} \exp(2\pi i (\rho \cdot h^w \cdot s^w \cdot h^w)) d\nu \), where \( \rho = l(w) \) and \( U \) (resp. \( U^- \)) is the commutator subgroup of \( B \) (resp. the opposite Borel subgroup \( B^- \)). We explicitly compute the above integral in this chapter and show (Theorem 3.1) it to be \( (-1)^{n^w \cdot 1/2} (2\pi)^n \prod_{\nu \in l(w) \cap A_{\nu}, \sigma(\rho, \nu)^{1/2}} \). Our proof of Theorem (3.1) occupies the whole of Section 3 and proceeds via an induction on \( l(w) \). Interestingly, we use information about the cup product in \( H(G/B) \), as given in Corollary (3.12), to compute the integral.

Section 4 can be viewed as the main algebraic part of this paper. We denote by \( Q_w \) (resp. \( S_w \)), the smash product of the group ring \( \mathbb{C}[W] \) with the \( W \)-field \( Q = Q(h^*) \), the field of rational functions on \( h \) (resp. the \( W \)-ring \( S = S(h^*) \)). The ring \( Q_w \) admits an involutary anti-automorphism \( t \) (see (1.3.)). With the help of Proposition (4.2), we define certain elements \( \{x_w\}_{w \in W} \) in \( Q_w \), which form a right (as well as left) \( Q \)-basis (Corollary (4.5)). The basis \( x_w \) behaves like a "degenerate" Hecke basis (see Proposition (4.3)(a)). Further, \( Q \) has a canonical (left) \( Q_w \)-module structure, given by (1.3.3). We define (a basic ring) \( R \) as the subring of \( Q_w \), consisting of all those elements \( x \in Q_w \) such that \( x \) keeps \( S \) stable. We prove a crucial structure theorem (Theorem 4.6) for the ring \( R \). This asserts that \( R \) is a free (left as well as right) \( S \)-module, with the \( \{x_w\}_{w \in W} \) as basis and \( R \cap R' = S_w \). In the finite case, part (a) of this theorem admits a simple proof using the theory of "harmonics." We refer to \( R \) as the nil Hecke ring (4.12). We also put a co-product structure \( A \) on \( Q_w \) (4.14) and prove Proposition (4.15), which describes \( A \) in terms of the \( \{x_w\}_{w \in W} \) basis of \( Q_w \), where \( x_w = x'_w \).

We dualize the above objects and define \( \omega = \text{Hom}_Q(Q_w, Q) \) (considered as a right \( Q \)-module) and the \( S \)-subalgebra \( A = \{\psi \in \text{Hom}_Q(Q_w, Q) \mid \psi(R') \subset S \text{ and } \psi(x_w) = 0, \text{ for all but a finite number of } w \in W\} \). Since \( Q_w \) has \( \{\delta_w\}_{w \in W} \) as a right \( Q \)-basis, we can (and often will) view \( \omega \) as the space of
all the maps: $W \to Q$. Of course, $\Omega$ is an algebra under pointwise addition and pointwise multiplication (of maps: $W \to Q$). More subtly, since $Q_w$ has an involutory anti-automorphism $t$, we can also put the structure of a left $Q_w$-module on $\Omega$ defined by $(1.g)$. It is easy to see that $A$ is stable under (the left action of) $R$. In particular, the elements $\delta_w, x_w$ act on $A$. We refer to the action of the $\delta_w$ as the Weyl group action and the action of the $x_w$ is referred to as the Hecke operators, for reasons which will be clear in Section 5. As a consequence of (structure) Theorem (4.6)(a), we deduce (Proposition (4.20)) that $A$ is a free $S$-module with basis $\{\xi^w\}_{w \in W}$, where $\xi^w$ is defined by $\xi^w(v) = \delta_{v,w}$, for all $v \in W$. We collect various properties of $\{\xi^w\}$ in Proposition (4.24), and define a matrix $D = (d_{v,w})_{v,w \in W}$ by $d_{v,w} = \xi^v(w)$. The matrix $D$ is fundamental to our paper. In the finite case, it can be extracted from [3, Theorem 5.9]. We give an explicit formula (Proposition (4.32)) for the arbitrary product $\xi^u \cdot \xi^v$ (as well as $\delta_u \cdot \xi^v$) in the $\{\xi^w\}$ basis, purely in terms of the matrix $D$. We give a different formula for the arbitrary product $\xi^u \cdot \xi^v$ in Proposition (4.31), in terms of the action of the ring $R$ on $A$.

Section 5 synthesizes Section 2 and Section 4. Since it is basically an application of Section 2 and since in Section 2 we have assumed $A$ is symmetrizable we assumed the same in Section 5. However, using the algebraic results of Kac–Peterson the main theorem in Section 5 (as pointed out by Peterson) may be proved without the symmetrizability assumption. See Remark 5.7. We show that the (a priori very complicated) filtration of $C\{W\}$ (given in Sect. 2), obtained by purely geometrical considerations, also arises from a very explicit combinatorial construction, using only the Weyl group $W$ (associated to $g$) and its representation on the Cartan subalgebra $h$. By using $d, \partial$ harmonic forms $s^v_0 = s^v/\int_W s^v$ and the map $\eta$ (given in (2.7)), we define a map $\tilde{D}: W \times W \to \mathbb{C}$ in Section 5.1. The map $\tilde{D}$, in turn, gives rise to the filtration (of Theorem (2.12)) of $C\{W\}$. Moreover, the matrix $D$ can also be thought of as a map: $W \times W \to S$. By evaluating $D$ at $h(\rho)$ ($h(\rho)$ is defined in Proposition (5.2)), we get a map $D_{h(\rho)}: W \times W \to \mathbb{C}$. We further prove a characterization of the matrix $D_{h(\rho)}$ in Proposition (5.5), which roughly asserts that any $W \times W$ matrix $E$ (over $\mathbb{C}$), which is upper triangular, has the same diagonal values as that of $D_{h(\rho)}$, and satisfies a suitable “cup product formula” is the same as $D_{h(\rho)}$. Finally, we show (Proposition (5.2)) that $\tilde{D}$ does satisfy all the characterizing properties (given in (5.5)) of $D_{h(\rho)}$ and hence we have the Corollary (5.6); $\tilde{D} = D_{h(\rho)}$. (We should mention that our calculation of the integral, in Sect. 3, is crucially used to establish that the diagonal entries of

\footnote{Note added in proof. Our later applications to $H(G/B)$ involve a quotient ring of $\Lambda$. However, recently A. Arabia has proved that $\Lambda$ itself is isomorphic to the $T$-equivariant cohomology of $G/B = K/T$. (See [31].)}
D are precisely equal to those of \( D_{\text{he}(p)} \).

This leads to our main theorem (5.12), which asserts that \( H(G/B) \) is canonically isomorphic (as a graded algebra) with \( \mathbb{C}_0 \otimes_S A \). Moreover, under this isomorphism, the (Weyl group) action of \( w \in W \) and the Hecke operator \( A_w \) on \( H(G/B) \) correspond respectively to the action of \( \delta_w \) and \( x_w \) on \( \mathbb{C}_0 \otimes_S A \), where \( \mathbb{C}_0 \) is the quotient of \( S \) by the augmentation ideal \( S^+ \). This, in particular, gives (by Proposition (4.32)) an expression, by a fairly explicit formula, for the cup product of two arbitrary Schubert classes (as well as the Weyl group action of arbitrary \( w \)) on \( H(G/B) \), purely in terms of the matrix \( D \). Of course, as a particular case, it gives the cup product of two arbitrary cohomology classes of the based loop group \( \Omega \)\((K)\) for a finite dimensional compact S.S. simply connected group \( K \), in terms of the Schubert basis. One can easily generalize Theorem (5.12) so that an arbitrary parabolic subgroup \( P \) replaces the Borel subgroup \( B \), as done in Corollary (5.13)(c). Very interestingly, we can prove an analog of Theorem (5.12) for an arbitrary (left) \( B \)-stable closed subspace of \( G/P \). This is the content of our Theorem (5.16). Recently, Akyildiz, Carrell, and Lieberman [1] (see also [5]) have, quite independently and by a different method, proved an analog of Theorem (5.16) in the particular case of Schubert varieties \( G/B \), where \( G \) is finite. It is not clear if their proofs can be extended to the general (infinite) case.

In Section 6 we prove that in the finite case, the matrix \( D \) can be obtained from the \( W \)-Harmonic polynomials on \( h \), as shown in Theorem (6.3) (see also Remark (6.4)). This says that the matrix \( D \) is nothing but the “upper triangular part” of the matrix obtained by the \( W \)-translates of the harmonic polynomials.

The main results of this paper have been announced in [15].

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1. Preliminaries and Notations

(1.1) Definitions and Basic Properties [13, 25].

Let \( A = (a_{ij})_{1 \leq i, j \leq l} \) be any generalized Cartan matrix (i.e., \( a_{ii} = 2 \) and \( -a_{ij} \in \mathbb{Z}_+ \) for all \( i \neq j \)). \( A \) is called symmetrizable if \( DA \) is symmetric for some diagonal matrix \( D = \text{diag}(q_1, \ldots, q_l) \) with \( q_i > 0 \) and rational.

Choose a triple \((h, \pi, \pi^-)\), unique up to isomorphism, where \( h \) is a vector space over \( \mathbb{C} \) of \( \dim(l + \text{co-rank} \, A) \), \( \pi = \{\pi_i\}_{1 \leq i \leq l} \subseteq h^* \) and \( \pi^- = \{h_i\}_{1 \leq i \leq l} \subseteq h \) are linearly independent indexed sets satisfying \( \pi_i(h_j) = a_{ij} \). The Kac–Moody algebra \( g = g(A) \) is the Lie algebra over \( \mathbb{C} \), generated by \( h \) and symbols \( e_i \) and \( f_i \) (1 \( \leq i \leq l \)) with the defining relations \([h, h] = 0; \quad[h, e_i] = \pi_i(h) e_i, \quad[h, f_i] = -\pi_i(h) f_i \) for \( h \in h \) and all \( 1 \leq i \leq l; \)
\[ [e_i, f_j] = \delta_{ij} h_j \] for all \( 1 \leq i, j \leq l \); \( (\text{ad} \ e_i)^{1-a_{ij}} \text{ad} \ e_j = 0 = (\text{ad} \ f_i)^{1-a_{ij}} \text{ad} \ f_j \) for all \( 1 \leq i \neq j \leq l \). \( h \) is canonically embedded in \( g \) and is called the Cartan subalgebra of \( g \).

One has a root space decomposition \( g = h \oplus \sum_{\alpha \in \Delta} (g_{\alpha} \oplus g_{-\alpha}) \), where \( g_{\alpha} = \{ x \in g : [h, x] = \alpha(h) x, \text{ for all } h \in h \} \) and \( \Delta = \{ \alpha \in \sum_{i=1}^{l} \mathbb{Z} \alpha_i : \alpha \neq 0 \} \). Define \( \Delta = \Delta_+ \cup \Delta_- \) (\( \Delta_- = -\Delta_+ \)).

We fix a subset \( X \) (including \( X = \emptyset \)) of \( \{ 1, \ldots, l \} \). Put \( \Delta^+_X = \Delta_+ \cap \{ \sum_{i \in X} \mathbb{Z} \alpha_i \} \) and define the following Lie subalgebras:

\[ n = \sum_{\alpha \in \Delta} g_{\alpha}, \quad n = \sum_{\alpha \in \Delta} g_{-\alpha}; \]

\[ u = u_X = \sum_{\alpha \in \Delta^+_X} g_{\alpha}, \quad u = u_X = \sum_{\alpha \in \Delta^+_X} g_{-\alpha}; \]

\[ r = r_X = h \oplus \sum_{\gamma \in \Delta^+_X} (g_{\gamma} \oplus g_{-\gamma}); \]

\[ B = h \oplus n \text{ and } p = p_X = r \oplus u. \]

In the case when \( X \) is of finite type (i.e., \( r_X \) is finite dimensional) \( r_X \) is a reductive subalgebra and since \([r_X, u_X] \subset u_X \) (resp. \([r_X, u_X] \subset u_X \)), \( r_X \) acts on \( u_X \) (resp. \( u_X \)).

There is a Weyl group \( W \subset \text{Aut}(h^*) \) generated by the "simple" reflections \( \{ r_i \}_{1 \leq i \leq l} \) \( r_i(\chi) = \chi - \alpha_i(h_i) \alpha_i \), for any \( \chi \in h^* \), associated to the Lie algebra \( g \). \( (W, \{ r_i \}_{1 \leq i \leq l}) \) is a Coxeter group, and hence we can speak of the Bruhat ordering \( \leq \) and lengths of elements of \( W \). We denote the length of \( w \) by \( l(w) \). \( W \) preserves \( \Delta \). \( \Delta_{re} \) is defined to be \( W \cdot \pi \) and \( \Delta_{im} = \Delta \setminus \Delta_{re} \). For \( \alpha \in \Delta_{re}, \dim g_{\alpha} = 1 \). We set \( \Delta^+_X = \Delta^+_X \cap \Delta_+ \), similarly \( \Delta^-_X = \Delta^-_X \cap \Delta_- \). By dualizing, we get a representation of \( W \) in \( h \). Explicitly, \( r_i(h) = -h, \alpha_i(h), \) \( h \in h \) and any \( 1 \leq i \leq l \).

For any \( X \subseteq \{ 1, \ldots, l \} \), let \( W_X \) be the subgroup of \( W \) generated by \( \{ r_i \}_{i \in X} \) and define a subset \( W_X^1 \) of the Weyl group \( W \), by \( W_X^1 = \{ w \in W : \Delta_+ \cap w \Delta_- \subset \Delta_+ \setminus \Delta_X^1 \} \). \( W_X^1 \) can be characterized as the set of elements of minimal length in the cosets \( W_Xw(w \in W) \) (each such coset contains a unique element of minimal length).

There is a (\( \mathbb{C} \)-linear) involution \( \omega \) of \( g \) defined (uniquely) by \( \omega(f_i) = -e_i \) for all \( 1 \leq i \leq l \) and \( \omega(h) = -h \), for all \( h \in h \). It is easy to see that \( \omega \) leaves \( g(\mathbb{R}) \) (= "real points" of \( g \)) stable. Let \( \omega_\sigma \) be the conjugate linear involution of \( g \), which coincides with \( \omega \) on \( g(\mathbb{R}) \). In the case when \( g \) is symmetrizable, there is a nondegenerate, \( W \)-invariant, symmetric \( \mathbb{C} \)-bilinear form \( \sigma \) on \( h^* \). \( \sigma \) gives rise to a nondegenerate, \( g \)-invariant, symmetric \( \mathbb{C} \)-bilinear form (called the Killing form) \( \langle , \rangle \) on \( g \) (see [13]).

We fix, once and for all, one such \( \sigma \) (and hence \( \langle , \rangle \)). The symmetric form \( \langle , \rangle \) on \( g \) gives rise to a Hermitian form \( \{ , \} \) on \( g \), defined by
{x, y} = − ⟨x, ω_o(y)⟩, for x, y ∈ g. The Hermitian form { , } is positive definite on n− (and n) (see [8, Sect. 12] and [21, Remark IV, p. 1782].)

(1.2) Algebraic Group Associated to a Kac–Moody Lie algebra g [30, 21, 22].

A g^1 (=the commutator subalgebra [g, g]) module (V, π) (π: g^1 → End V) is called integrable, if π(x) is locally nilpotent whenever x ∈ g_z, for x ∈ A^rc. Let G* be the free product of the additive groups \{g_\alpha\}_\alpha \in A^\text{rc}, with canonical inclusions \iota_\alpha: g_\alpha → G*. For any integrable g^1-module (V, π), define a homomorphism \pi*: G* → Aut_\mathbb{C} V by π*(\iota_\alpha(x)) = exp(π(x)), for x ∈ g_\alpha. Let N* be the intersection of all Ker π*. Put G = G*/N*. Let q be the canonical homomorphism: G* → G. For x ∈ g_\alpha (x ∈ A^rc), put exp(x) = q(\iota_x), so that U_\alpha = exp g_\alpha is an additive one parameter subgroup of G. Denote by U (resp. U+) the subgroup (of G) generated by the U_\alpha's with \alpha ∈ A^\text{rc} (resp. \alpha ∈ A^rc). We put a topology on G as given in [22, 4(G)]. G may be viewed as, possibly infinite dimensional, affine algebraic group in the sense of Šafarevič with Lie-algebra g^1. For a proof, see [22, Sect. 4]. We call G the group associated to the Kac–Moody Lie algebra g.

The conjugate linear involution ω_o of g, on "integration," gives rise to an involution \tilde{\omega}_o of G. Let K denote the fixed point set of this involution. K is called the standard real form of G.

For each 1 ≤ i ≤ l, there exists a unique homomorphism \beta_i: SL_2(\mathbb{C}) → G, satisfying \beta_i(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \exp(z e_i) and \beta_i(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}) = \exp(z f_i) (for all z ∈ \mathbb{C}), where e_i, f_i is as in (1.1). Define H_i = \beta_i(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), z ∈ \mathbb{C}^*; H^+_i = \beta_i(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}); z is real and >0; G_i = \beta_i(SL_2(\mathbb{C})); N_i = \text{Normalizer of } H_i in G_i; H (resp. H^+) = the subgroup (of G) generated by all H_i (resp. H^+_i); N = the subgroup (of G) generated by all N_i. There is an isomorphism \tau: W ≃ N/H, such that \tau(r_i) is the coset N_i H \setminus H mod H [21, Sect. 2]. We would, sometimes, identify W with N/H under \tau.

Put B = HU and P = P_\chi = BW_\chi B. B is called the standard Borel subgroup and P_\chi the standard parabolic subgroup of G, associated to the subset X. Denote by K_\chi the subgroup K \cap P_\chi. We denote by T = K \cap B, the "maximal torus" of K. It is easy to see that the canonical inclusion K/K_\chi → G/P_\chi is a (surjective) homeomorphism. Use [22, Theorem 4(d)]. (K ⊂ G is given the subspace topology.)

(1.3) Bruhat decomposition [12, 30, 21, 22].

G can be written as disjoint union G = \bigcup_{w ∈ W_\chi^+} (\bigcup w^{-1} P_\chi), so that G/P_\chi = \bigcup_{w ∈ W_\chi^+} (\bigcup w^{-1} P_\chi/P_\chi). G/P_\chi is a C–W complex with cells \{V_w = \bigcup w^{-1} P_\chi/P_\chi\}_{w ∈ W_\chi^+} and dim_w V_\chi = 2l(w).

(1.4) Notations. Unless otherwise stated, vector spaces will be over \mathbb{C} and linear maps will be \mathbb{C}-linear maps. For a vector space V, V* denotes
For a Lie algebra pair \((\mathfrak{g}, \mathfrak{r})\), \(C^{e}(\mathfrak{g}, \mathfrak{r})\) (resp. \(\wedge(\mathfrak{g}, \mathfrak{r})\)) denotes the standard co-chain (resp. chain) complex associated to the pair \((\mathfrak{g}, \mathfrak{r})\). \(p\), as usual, means any element \(\mathfrak{h}^{*}\) satisfying \(\rho(h) = 1\), for all \(1 \leq i \leq l\). We fix elements \(\chi_{i}\) (\(1 \leq i \leq l\)) in \(\mathfrak{h}^{*}\) satisfying \(\chi_{i}(h) = \delta_{i,j}\), for all \(1 \leq j \leq l\).

Complex manifolds are oriented by its complex structure. Explicitly, on \(\mathbb{C}^{n}\) (with complex coordinates \(z_{1} = x_{1} + \sqrt{-1} y_{1}, \ldots, z_{n} = x_{n} + \sqrt{-1} y_{n}\)), \(\{\partial/\partial x_{1}, \partial/\partial y_{1}, \ldots, \partial/\partial x_{n}, \partial/\partial y_{n}\}\) will be declared as positively oriented basis of \(T(\mathbb{C}^{n})\).

### 2. Identification of the Cohomology Algebra \(H^{*}(G/P, \mathbb{C})\) with \(\text{Gr } \mathbb{C}\{W_{X}^{i}\}\)

We recall the following well-known

\[(2.1) \text{DEFINITION. A differential graded algebra } \mathcal{A}/\mathbb{C} \text{ (abbreviated as DGA) is a graded associative algebra (over } \mathbb{C}, \text{ having identity, } \mathcal{A} = \sum_{i \geq 0} \mathcal{A}^{i}, \text{ with a differential } d: \mathcal{A} \rightarrow \mathcal{A} \text{ of degree } +1, \text{ such that}
\]

\[(1) \mathcal{A} \text{ is graded commutative, i.e., } \alpha \cdot \beta = (-1)^{i} \beta \cdot \alpha, \text{ for } \alpha \in \mathcal{A}^{i} \text{ and } \beta \in \mathcal{A}^{j}\]

\[(2) d \text{ is a derivation, i.e., } d(\alpha \cdot \beta) = d\alpha \cdot \beta + (-1)^{i} \alpha \cdot d\beta, \text{ for } \alpha \in \mathcal{A}^{i}.
\]

Let \(\mathcal{A}\) be a DGA with differential \(d\). Denote by

\[\text{End}(\mathcal{A}) = \text{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{A}) \text{ (all the } \mathbb{C}\text{-linear maps from } \mathcal{A} \text{ into itself)}\]

\[\text{End}^{-p,q}(\mathcal{A}) = \text{Hom}_{\mathbb{C}}(\mathcal{A}^{p}, \mathcal{A}^{q}), \text{ for all } p, q \geq 0, \text{ and}\]

\[\text{End}^{i}(\mathcal{A}) = \bigcap_{p \geq 0} \text{End}^{-p, p+i}(\mathcal{A}), \text{ for } i \in \mathbb{Z}. \text{ We clearly have}
\]

\[\bigcap_{i \in \mathbb{Z}} \text{End}^{i}(\mathcal{A}) \supset \text{End}(\mathcal{A}) \supset \bigsum_{i \in \mathbb{Z}} \text{End}^{i}(\mathcal{A}).\]

\(d\) induces a derivation (of degree \(+1\)) \(\delta\) in \(\text{End}(\mathcal{A})\) (\(\text{End}(\mathcal{A})\) is viewed as an algebra, with product as composition of maps) defined by

\[\delta \tau = d\tau - (-1)^{i} \tau d \quad \text{for } \tau \in \text{End}^{i}(\mathcal{A}).\]

From the definition, it is easy to check that \(\delta^{2} = 0\). \((I _{1})\)
Further, $\tau \in \text{End}(\mathcal{A})$, with $\delta \tau = 0$, takes $Z(\mathcal{A})$ ($= \{ a \in \mathcal{A}: da = 0 \}$) into itself and the same is true with $Z(\mathcal{A})$ replaced by $d(\mathcal{A})$. Hence, there is a canonical homomorphism of $\mathbb{Z}$-graded algebras (obtained by restriction)

$$\gamma: H(\text{End}(\mathcal{A}), \delta) \to \text{End}(H(\mathcal{A})).$$

A proof of the following proposition can be seen, e.g., in [4, Theorem 31(a), p. 114].

(2.2) Proposition. The map $\gamma$, defined above, is an isomorphism of graded algebras.

(2.3) Let $g = g(A_l)$ be any symmetrizable Kac–Moody Lie algebra. Fix a subset $X \subseteq \{ 1, ..., l \}$ of finite type. Recall the Lie algebras $u_X, u_X', r_X$ defined in Section 1.1. We would often abbreviate $u_X, u_X', r_X$ (respectively) by $u, u', r$.

Let $C(g, r) = \sum_{i \geq 0} \text{Hom}_\mathbb{C}(\wedge^i (u \oplus u^-), \mathbb{C})$ denote the standard co-chain complex associated to the Lie algebra pair $(g, r)$ and $C(u) = \sum_{i \geq 0} \text{Hom}(A^i(u), \mathbb{C})$ is the DGA associated to the Lie algebra $u$.

As in [23, Sect. 3], we put the topology of pointwise convergence on $C(g, r)$ and $C(u)$. We also put the topology of pointwise convergence on $\text{End} C(u)$, i.e., $\{ \tau_n \} \subseteq \text{End} C(u)$ converges to $\tau \in \text{End} C(u)$ if and only if $\tau_n(x) \to \tau(x)$ (in the topology of $C(u)$), for all $x \in C(u)$.

$\delta: \text{End} C(u) \to \text{End} C(u)$ is continuous under this topology. Further, it can be easily seen that $\delta$ commutes with the canonical $r$ action on $\text{End} C(u)$. We denote by $\delta_0$, the restriction of $\delta$ to $\text{End}_r C(u)$.

The map $\eta$ (defined below) is basic to this section.

(2.4) Lemma. There exists a (unique) continuous map $\eta: C(g, r) \to \text{End}_r C(u)$, such that

$$\eta \left( \sum_{n=1}^k a^n \otimes e(a^n) \right) = \left( \frac{2\pi i}{\sqrt{-1}} \right)^q \sum_n e(a^n) i(a^n)$$

for $a^n \in C(u)$ and $a^n \in \wedge^q(u)$ (with $\sum a^n \otimes e(a^n) \in C(g, r)$), where $e: C(u) \to \text{End} C(u)$ is exterior multiplication, $i: A(u) \to \text{End} C(u)$ is interior multiplication, and $e: \wedge(u) \to C(u^-)$ is induced from the Killing form. Moreover $\eta$ is injective. $\square$

It may be remarked that, though $\eta$ is defined on a dense subspace of $\sum_{i \geq 0} \text{Hom}_\mathbb{C}(\wedge^i (u \oplus u^-), \mathbb{C})$, it does not extend to a continuous map on $\sum_{i \geq 0} \text{Hom}_\mathbb{C}(\wedge^i (u \oplus u^-), \mathbb{C})$, in general. Further, $\eta$ does not commute with the differentials. In fact, we have
(2.5) **Lemma.** \( \eta d_0 = \delta_0 \eta \), where the operator \( d_0 \) on \( C(\mathfrak{g}, \mathfrak{r}) \) is defined to be \( d' + (2\pi/\sqrt{-1}) \partial'' \). \( d' \) and \( \partial'' \) are defined in [23, Sect. 3]. One has \( d_0^2 = 0 \).

**Proof.** For \( \mathbf{a} = a_1 \wedge \cdots \wedge a_q \mathcal{E} \wedge^q(\mathfrak{u}) \), we have

\[ d_u i(\mathbf{a}) + (-1)^{q+1} i(\mathbf{a}) d_u = \sum_{k=1}^q (-1)^{k+1} \text{ad}(a_k) \circ i(a_1) \circ \cdots \circ i(a_k) \circ \cdots \circ i(a_q) \\
+ i(\partial_u \mathbf{a}) \quad \text{as operators on} \ C(\mathfrak{u}) \quad (I_3) \]

where \( \partial_u \) (resp. \( d_u \)) denotes the chain (resp. co-chain) map of the standard complex \( \wedge(\mathfrak{u}) \) (resp. \( C(\mathfrak{u}) \)), associated to the Lie algebra \( \mathfrak{u} \) and \( \text{ad}: \mathfrak{u} \to \text{End} C(\mathfrak{u}) \) is induced from the adjoint representation.

The identity \((I_3)\) can be easily proved by induction on \( q \), using the well-known identities \((I_4)\) and \((I_5)\): For any \( a_1, a_2 \in \mathfrak{u} \),

\[ d_u i(a_1) + i(a_1) d_u = \text{ad}(a_1) \quad (I_4) \]
\[ i(a_1) \text{ad}(a_2) - \text{ad}(a_2) i(a_1) = i[a_1, a_2] \quad (I_5) \]

as operators on \( C(\mathfrak{u}) \). Further, for any \( a \in \mathfrak{u} \),

\[ \text{ad}(a) = \sum_{\phi \in I} \varepsilon(a^*_\phi) \ i[a_\phi, a], \quad (I_6) \]

where \( \{a^*_\phi\}_{\phi \in I} \) is an orthonormal basis of \( \mathfrak{u} \), consisting of weight vectors, and \( a^*_\phi \in \text{Hom}(\mathfrak{u}, \mathbb{C}) \) is the element satisfying \( a^*_\phi(a^*_\phi') = \delta_{\phi, \phi'} \) for all \( \phi', \phi \in I \).

To prove identity \((I_6)\), observe that both the sides of \((I_6)\) are derivations and, moreover, both are continuous maps: \( C(\mathfrak{u}) \to C(\mathfrak{u}) \). So it suffices to prove that \( \text{ad}(a) \alpha = \sum_{\phi \in I} \varepsilon(a^*_\phi) \ i[a_\phi, a] \alpha, \) for \( \alpha \in C^1(\mathfrak{u}) \), which is easy to verify.

We are ready to prove the lemma. Since all the maps \( \eta, \delta_0, d_0 \) are continuous, it suffices to prove that \( \eta d_0 = \delta_0 \eta \) on \( e(\bigwedge(\mathfrak{u}^-) \otimes \bigwedge(\mathfrak{u})^*) \). Fix an element \( \sum_n b^n \otimes a^n \in [\bigwedge(\mathfrak{u}^-) \otimes \bigwedge(\mathfrak{u})]^* \), where \( b^n \in \bigwedge^p(\mathfrak{u}^-) \) and \( a^n = a^n_1 \wedge \cdots \wedge a^n_q \in \bigwedge^q(\mathfrak{u}) \). Then

\[
\left( \frac{\sqrt{-1}}{2\pi} \right)^q \delta_0 \eta (e \sum b^n \otimes a^n) = \sum_n [d_u \varepsilon(e b^n) i(a^n) - (-1)^{p-q} \varepsilon(e b^n) i(a^n) d_u] \]

\[
= \sum_n [\varepsilon(d_u(e b^n)) i(a^n) + (-1)^p \varepsilon(e b^n) d_u i(a^n) - (-1)^{p-q} \varepsilon(e b^n) i(a^n) d_u] \]
\[\sum_n \left[ \varepsilon(d_u(e^b_n)) i(a^n) + (-1)^n \varepsilon(e^b_n) \left\{ \sum_{k=1}^q (-1)^{k+1} \text{ad}(a^a_k) \right\} \right]\]

\[\times \sum_{k=1}^q (-1)^{k+1} \sum_{\phi \in I} \varepsilon(a^a_\phi) i[i(a^a_1) \circ \cdots \circ i(a^a_q) + \hat{i}(e_u a^n)]\]

using (I\(_3\))

\[
\left(\frac{-1}{2\pi}\right)^q \delta_0 \eta \left(e \sum_n b^n \otimes a^n\right) = \sum_n \left[ \varepsilon(d_u(e^b_n)) i(a^n) + (-1)^n \varepsilon(e^b_n) i(\hat{\varepsilon}_u a^n) \right. \\
+ (-1)^n \sum_{\phi \in I} \varepsilon(e^b_n) \varepsilon(a^a_\phi) i[\text{ad} a^a_\phi a^n] \left. \right] \quad (I_7)
\]

Also,

\[
\eta d_0 \left(e \sum_n b^n \otimes a^n\right) \\
= \eta \left( \sum_n \left[ (d_u(e^b_n)) \otimes (e^a_n) + (-1)^n \sum_{\phi \in I} e(b^a) \cdot a^a_\phi \otimes e(\text{ad} a^a_\phi a^n) \right. \\
+ (-1)^n \left(\frac{2\pi}{\sqrt{-1}}\right) (e^b_n) \otimes e(\hat{\varepsilon}_u a^n) \left. \right] \right)
\]

(See the proof of [23, Lemma (3.1)].) So, we have

\[
\left[ \frac{-1}{2\pi}\right]^q \eta d_0 \left(e \sum_n b^n \otimes a^n\right) \\
= \sum_n \left[ \varepsilon(d_u(e^b_n)) i(a^n) + (-1)^n \sum_{\phi \in I} \varepsilon(e(b^a) a^a_\phi) i(\text{ad} a^a_\phi a^n) \right. \\
+ (-1)^n \varepsilon(e^b_n) i(\hat{\varepsilon}_u a^n) \left. \right] \quad (I_8)
\]

Comparing (I\(_7\)) and (I\(_8\)), we get the lemma. \[\square\]

Recall the definition of the operator \(S = d\partial + \partial d\), acting on \(C(g, r)\), from [23, Sect. 3]. From [23, Lemma (3.5), Theorem 3.13, and Remark 3.14] we have, \(\text{Ker } S = \text{Ker } d' \cap \text{Ker } \partial' \cap \text{Ker } d'' \cap \text{Ker } \partial''\). As an immediate corollary of lemma (2.5), we get
(2.6) Lemma. \( \eta(\text{Ker } S) \subset \text{Ker } \delta_0 \).

By virtue of this lemma, \( \eta \) gives rise to a map \( \tilde{\eta}: \text{Ker } S \to H(\text{End}_r C(u), \delta_0) \). Recall the definition of the map \( \gamma \) from Proposition (2.2). Obviously, \( \gamma \) gives rise to a map \( \gamma_0: H(\text{End}_r C(u), \delta_0) \to \text{End}_r H^*(u) \), where \( H^*(u) = \sum_{i \geq 0} H^i(u) \) is the Lie algebra cohomology of \( u \) (with trivial coefficients \( C \)).

Let us recall the structure of \( H^*(u) \), as an \( r \)-module, from [9].

(2.7) Theorem. [9, Theorem 8.6]. As \( r \)-modules,

\[
H^i(u) \sim \sum_{\substack{w \in W^+ \text{ with } \ell(w) = i}} L(wp - \rho),
\]

where \( L(wp - \rho) \) is an irreducible \( r \)-module with highest weight \( wp - \rho \). In particular, any irreducible \( r \)-module occurs with multiplicity at the most one in \( H^*(u) \).

So, we get

\[
\text{End}_r H^*(u) \cong \prod_{i \geq 0} \text{End}_r H^i(u) \cong \prod_{i \geq 0}\prod_{w \in W^+} \text{End}_r L(wp - \rho).
\]

\( L(wp - \rho) \) being irreducible, \( \text{End}_r L(wp - \rho) \) is 1-dimensional with a canonical generator \( 1_w \) (= the identity map of \( L(wp - \rho) \)). This identifies \( \text{End}_r H^*(u) \) with \( \prod_{w \in W_+} \mathbb{C} 1_w \). The space \( \prod_{w \in W_+} \mathbb{C} 1_w \) can (and will) also be thought of as \( \mathbb{C} \{W^+_\chi\} \) (= the vector space of all the functions from \( W^+ \chi \) to \( \mathbb{C} \)). The product in \( \mathbb{C} \{W^+_\chi\} \), inherited from \( \text{End}_r H^*(u) \), is nothing but the pointwise multiplication. Let \( \tilde{\eta}: \text{Ker } S \to \mathbb{C} \{W^+_\chi\} \) be the composite map

\[
\text{Ker } S \xrightarrow{\tilde{\eta}} H(\text{End}_r C(u), \delta_0) \xrightarrow{\gamma_0} \text{End}_r H^*(u) \cong \mathbb{C} \{W^+_\chi\}.
\]

(2.8) A filtration of \( C(\mathfrak{g}, \mathfrak{r}) \) and \( \mathbb{C} \{W^+_\chi\} \). Define a decreasing filtration \( \mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}} \) (\( \mathbb{Z} \) is the set of non-positive integers) by \( \mathcal{F}_p = \sum_{s \leq k \leq -p} C^*^{s,k}(\mathfrak{g}, \mathfrak{r}) \), where \( C^{s,k}(\mathfrak{g}, \mathfrak{r}) = \text{Hom}_r(\Lambda^s(\mathfrak{u}) \otimes \Lambda^k(\mathfrak{u}^\vee), \mathbb{C}) \). Clearly \( \mathcal{F}_p \) is \( d' \), \( d'' \) (and hence \( d_0 \)) stable. This gives rise to a filtration \( \mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}} \) of \( \text{End}_r C(u) \) by defining \( \mathcal{F}_p = \eta(\mathcal{F}_p) \). (Caution! We do not claim that \( \bigcup_{p \in \mathbb{Z}} \mathcal{F}_p = \text{End}_r C(u) \).) This also gives rise to a filtration \( \{\mathcal{J}_p\}_{p \in \mathbb{Z}} \) of \( \text{End}_r H^*(u) \cong \mathbb{C} \{W^+\chi\} \), by taking the image of the induced filtration of \( H(\text{End}_r C(u), \delta_0) \) under the canonical map \( \gamma_0 \), defined in (2.6). Here again, we do not claim that \( \bigcup_{p \in \mathbb{Z}} \mathcal{J}_p = \mathbb{C} \{W^+\chi\} \). In fact, it is true if and only if \( \mathfrak{g} \) is a finite dimensional Lie algebra.

(2.9) Remark. Let \( \mathcal{A} \) be any finite dimensional DGA. Then the com-
plex \text{End}(\mathcal{A}) \) comes equipped with a natural filtration \( \{ F_p \}_{p \in \mathbb{Z}_-} \) defined as follows: \( F_0 = \varepsilon(\mathcal{A}) \) (recall that \( \varepsilon: \mathcal{A} \rightarrow \text{End}(\mathcal{A}) \) is exterior multiplication). \( F_{-1} = \varepsilon(\mathcal{A}) + \text{Der}(\mathcal{A}) \), where \( \text{Der}(\mathcal{A}) = \sum_i \text{Der}^i(\mathcal{A}) \) and \( \text{Der}^i(\mathcal{A}) \) is the set of all the degree \( i \) derivations of \( \mathcal{A} \). Finally (for \( p \geq 0 \)) \( F_{-p} \) is the \( \mathbb{C} \)-span of all those elements in \( \text{End}(\mathcal{A}) \) which can be written as products of \( \leq p \) operators in \( F_{-1} \). (\( F_{-1} \) can be thought of as the set of first-order differential operators on \( \mathcal{A} \).)

If we take \( \mathcal{A} = C(\mathfrak{u}) \) then, in the case when \( \mathfrak{g} \) is finite dimensional, the filtration \( \{ F_p \}_{p \in \mathbb{Z}_-} \) (described above in (2.8)) coincides with \( \{ F_p \cap \text{End}(C(\mathfrak{u})) \}_{p \in \mathbb{Z}_-} \), as can be easily seen. In the general case \( F_p \) is some appropriate "partial completion" of \( F_p \). So, our filtration \( \{ F_p \} \) arises as a "super" analog of the usual filtration of differential operators on a manifold.

The behavior of \( \eta \) under products is given by

\[ \eta(\lambda \lambda') = \eta(\lambda) \eta(\lambda') \mod F_{-p - p' + 1}. \] (2.10) \text{LEMMA.} For \( \lambda \in C^* \mathfrak{g}, \mathfrak{r} \) and \( \lambda' \in C^* \mathfrak{g}, \mathfrak{r} \),

\[ \eta(\lambda \lambda') = \eta(\lambda) \eta(\lambda') \mod F_{-p - p' + 1}. \]

\text{Proof.} For \( \alpha \in C^k(\mathfrak{u}) \) and \( a \in \mathfrak{u} \), we have

\[ i(a) \varepsilon(\alpha) - (-1)^k \varepsilon(\alpha) i(a) = \varepsilon(i(a) \alpha) \] (1.0)

as elements of \( \text{End}(C(\mathfrak{u})) \).

From the above relation, by induction on \( p \), we get (for \( \beta \in C(\mathfrak{u}), \alpha \in C^k(\mathfrak{u}), a \in \mathfrak{u}, \mathfrak{u}, \) and \( b \in \mathfrak{u} \))

\[ \varepsilon(\beta) i(a) \varepsilon(\alpha) i(b) - (-1)^p \varepsilon(\beta a) i(a \wedge b) \in F_{-p - p' + 1} \] (1.10)

which proves the lemma. \( \blacksquare \)

As an immediate corollary, we have

\[ \varepsilon(\beta) i(a) \varepsilon(\alpha) i(b) - (-1)^p \varepsilon(\beta a) i(a \wedge b) \in F_{-p - p' + 1} \]

which proves the lemma. \( \blacksquare \)

As usual, let us define \( \text{Gr} C \{ W_\mathfrak{X}^X \} = \sum_{p \geq 0} \text{Gr}_p \), where \( \text{Gr}_p = \mathcal{J}_p \mathcal{J}_{-p + 1} \).

In view of Lemma (2.11), \( \text{Gr} C \{ W_\mathfrak{X}^X \} \) inherits a canonical algebra structure from \( C \{ W_\mathfrak{X}^X \} \). Denote \( \text{Ker} S \cap \text{C}_p \mathfrak{g} \mathfrak{r} \) by \( \text{Ker}^p \mathfrak{g} \mathfrak{r} \). By the definition of \( \eta, \eta(\text{Ker}^p \mathfrak{g} \mathfrak{r} S) \subset \mathcal{F}_{-p} \). Hence \( \eta(\text{Ker}^p \mathfrak{g} \mathfrak{r} S) \subset \mathcal{F}_{-p} \) (\( \eta \) is defined in Sect. 2.7). Denote the composition \( \text{Ker}^p \mathfrak{g} \mathfrak{r} S \rightarrow \eta \mathcal{J}_{-p} \mathcal{J}_{-p + 1} \) by \( \eta (\mathfrak{g} \mathfrak{r} S) \) and let \( \text{Gr}(\tilde{\eta}) \) be the map: \( \sum \text{Ker}^p \mathfrak{g} \mathfrak{r} S \rightarrow \text{Gr} C \{ W_\mathfrak{X}^X \} \), such that \( \text{Gr}(\tilde{\eta}) |_{\text{Ker}^p \mathfrak{g} \mathfrak{r} S} = \text{Gr}^p(\tilde{\eta}) \). Consider the inclusion \( \text{Ker} S \subset C(\mathfrak{g}, \mathfrak{r}) \). By [23, Theorem 3.13] it induces isomorphism (of vector spaces) \( \psi_{d.S} : \text{Ker} S \cong H^*(\mathfrak{g}, \mathfrak{r}) \) and, moreover [23, Sect. 3.], \( \text{Ker} S = \sum_{p \geq 0} \text{Ker}^p \mathfrak{g} \mathfrak{r} S \).

Now we are in a position to state the main theorem of this section.
(2.12) Theorem. Let $g = g(A_i)$ be a symmetrizable Kac–Moody Lie algebra and let $X \subseteq \{1, \ldots, l\}$ be a subset of finite type. Let $r = r_X$ be the reductive subalgebra of $g$ as defined in (1.1). Then the map $\text{Gr}(\tilde{\eta}) \circ \psi_{d,c}^{-1}: H^*(g, r) \to \text{Gr} \mathbb{C}\{W_X\}$ (defined above) is an isomorphism of graded algebras. (Of course, under this isomorphism, $H^{2p}(g, r)$ corresponds with $\text{Gr}^p$.)

(2.13) Remarks. (a) By [24, Theorem 1.6], $H^*(g, r_X)$ is isomorphic (as a graded algebra) with $H^*(G/P_X, \mathbb{C})$ under a suitably defined integration map. This, in particular, gives a graded algebra isomorphism of $H^*(G/P_X, \mathbb{C})$ with $\text{Gr} \mathbb{C}\{W\}$. (b) In the case when $X = \emptyset$, we will show (Theorem 5.12) that the isomorphism of $H^*(G/B, \mathbb{C})$ with $\text{Gr} \mathbb{C}\{W\}$ is $W$-equivariant, where $W$ acts on $\mathbb{C}\{W\}$ by the left regular representation.

Proof of the Theorem (2.12). Recall the definition of the filtration $\mathcal{G}$ of $C(g, r)$ and the filtration $\mathcal{F}$ of $\text{End}_C(u)$, see (2.8). By Lemma (2.5), $\mathcal{F}_p$ is $\delta_0$ stable. The corresponding spectral sequence has $E_r^{p,q}(\mathcal{F}) = H_{d_0}^{p+q}(\mathcal{F}_r/\mathcal{F}_{r+1}) = H_{d_0}^{p+q}(\mathcal{G}_r/\mathcal{G}_{r+1}) = H_{d}^{p+q}(C^*\mathcal{g}, r)$, since the differentials on $\mathcal{G}_r/\mathcal{G}_{r+1}$ induced by $d_0$ and $d'$ are the same. ($H_d(C^*\mathcal{g}, r)$ denotes the cohomology of the complex $C^*\mathcal{g}, r$ with respect to the differential $d'$.) By (a subsequent) Lemma (2.15), $\psi_{d,c}^{-1}: H^*(g, r) \to H^*(g, r)$ is an algebra isomorphism. But, by [23, Theorem 3.15], $H_d^i(C(g, r)) = 0$ for odd values of $i$ and for even $i$, $\dim C_i H_d^i(C(g, r))$ is the number of elements of length $i/2$ in $W_X$. In particular, $E_0^{p,q}(\mathcal{F}) = 0$ unless $p + q = 0$. Then all the differentials $d_i$ (for $i \geq 1$), of the spectral sequence $E_r(\mathcal{F})$, are zero. So, we have $E_r^{p,q}(\mathcal{F}) \approx E_{\infty}^{p,q}(\mathcal{F})$ and (of course) this isomorphism is an algebra isomorphism.

Further, the map $\tilde{\eta}: \text{Ker} S \to \mathbb{C}\{W_X\}$ is injective. To prove this, let $0 \neq s = x, v, w, z \in \text{Ker} S$ such that $\eta(\sum z^{s}) = 0$, for some constants $z^{s} \in \mathbb{C}$. ($\{s^{n}\}_{n \in \mathbb{N}}$ is a $C$-basis of $\text{Ker} S$ and is defined in the next section.) Let $w_0$ be an element of least possible length such that $z^{w_0} \neq 0$. In particular, evaluating at $w_0$, we have $\eta(\sum z^{s})(w_0) = 0$. But by using (a subsequent) Proposition 5.2(a) and (b), we get $\tilde{\eta}(z^{v_0})(w_0) = 0$ and $\tilde{\eta}(s^{v_0})(w_0) = 0$. A contradiction! This proves that $\tilde{\eta}$ is injective.

Furthermore, the canonical map $\gamma_0: H(\text{End}_r C(u), \delta_0) \to \text{End}_r H^*(u)$ is an isomorphism. To prove this, denote $\overline{C}(u) = \text{Hom}_C(\wedge(u), \mathbb{C})$. We decompose $\wedge(u) = \bigoplus_{\theta \in \tilde{t}} V_{\theta}$, where $V_{\theta}$ is the isotypical component corresponding to the irreducible representation $\theta$ of $r$. Of course, for any fixed $\theta$, $V_{\theta}$ is finite dimensional (use the fact that root spaces are finite dimensional). So, we have

$$\overline{C}(u) = \prod_{\theta \in \tilde{t}} V_{\theta}^* (V_{\theta}^* = \text{Hom}_C(V_{\theta}, \mathbb{C})) \quad \text{and} \quad \text{End}_r \overline{C}(u) = \prod_{\theta \in \tilde{t}} \text{End}_r(V_{\theta}^*).$$
From the finite dimensionality of $V_\theta$'s, it is easy to see that $\text{End}_r C(u) = \text{End}_r \bar{C}(u)$. Hence

$$H(\text{End}_r C(u)) = H(\text{End}_r \bar{C}(u))$$

$$= \prod_{\theta \in \mathfrak{t}} H(\text{End}_r (V_\theta^*))$$

$$\approx \prod_{\theta \in \mathfrak{t}} \text{End}_r H(V_\theta^*) \quad \text{(from the complete reducibility of the $r$-module $\text{End} V_\theta^*$ and Proposition (2.2))}$$

$$= \text{End}_r \left( \prod_{\theta \in \mathfrak{t}} H(V_\theta^*) \right)$$

$$= \text{End}_r H(\bar{C}(u)).$$

Now, consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{K} & \xrightarrow{i_*} & \text{End}_r H^*(u) \\
\downarrow & & \downarrow \\
\text{Ker} S & \xrightarrow{\eta} & \text{End}_r H^*(u) \\
\end{array}$$

where $i_*$ is induced by the inclusion $\text{Ker} S \subseteq C(g, r)$ ($d_0 |_{\text{Ker} S} \equiv 0$). Defining a filtration $\mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}}$, where $\mathcal{F}_p = \sum_{0 \leq k \leq -p} \text{Ker}^k(S)$, of $\text{Ker} S$, we get that $E^p_{p,q}(\mathcal{F}) \approx \text{Ker}^p(S)$ and $E^p_{p,q}(\mathcal{F}) = 0$ for $p \neq q$. So $E^p_{p,q}(\mathcal{F}) \rightarrow E^p_{p,q}(\mathcal{F})$ for all $p$ and $q$. Hence $i_*$ is an isomorphism. This, using the injectivity of $\eta$, proves that $H(\eta)$ is injective.

Finally, using [4, Chap. XV, Sect. 1] and Lemma (2.10), we get that $\text{Gr} C \{ W^1 \}$ is isomorphic with $E_{\mathbb{Z}}(\mathcal{F})$ as graded algebras. Now (a subsequent) Lemma (2.15) completes the proof. 

As an immediate consequence of the theorem, we get

\begin{equation}
(2.14) \text{COROLLARY.} \quad \text{Recall the definition of the filtration} \ \{ \mathcal{J}_p \}_{p \in \mathbb{Z}} \ \text{of} \ \text{C} \{ W^1 \} \ \text{from} \ (2.8). \ \text{For any} \ p \geq 0, \ \text{we have} \ \mathcal{J}_{-p} = \sum_{k=0}^p \Gamma_k, \ \text{where} \ \Gamma_k \ \text{is the image of} \ \text{Ker}^k(S) \ \text{in} \ \text{C} \{ W^1 \} \ \text{under} \ \eta.
\end{equation}

We will give another "combinatorial" description of $\mathcal{J}_{-p}$ in Section 5.

The inclusion of $\text{Ker} S \subseteq C(g, r)$ induces the map $\psi_{d.S}: \text{Ker} S \rightarrow H^*_d(g, r)$ and $\psi_{d.S}: \text{Ker} S \rightarrow H^*_d(g, r)$. By [23, Theorem 3.13 and Remark 3.14] $\psi_{d.S}$ and $\psi_{d.S}$ are both vector space isomorphisms. We have
(2.15) Lemma. The map $\psi_{d',S} \circ \psi_{d,S}^{-1}: H^*_d(\mathfrak{g}, \mathfrak{r}) \to H^*_d(\mathfrak{g}, \mathfrak{r})$ is an algebra isomorphism.

Proof. Let $s_1, s_2 \in \text{Ker } S$. There exists $s_3 \in \text{Ker } S$ such that $s_1 \cdot s_2 - s_3 \in \text{Image } d$. By [23, Lemma 3.8 and Theorem 3.13], $s_1 \cdot s_2 - s_3 \in \text{Im } S$. Defining $S' = d'\partial + \partial' d'$, we have $\frac{1}{2}S = S'$ by [23, Lemma 3.5] and hence there exists $\lambda \in C(\mathfrak{g}, \mathfrak{r})$ such that

$$s_1 \cdot s_2 - s_3 = (d'\partial' + \partial' d') \lambda.$$

Thus, on taking $d'$, we get $d'(s_1 \cdot s_2) - d's_3 = d'\partial' d' \lambda$, i.e., $(d's_1) s_2 + s_1 \cdot d's_2 - d's_3 = d'\partial' d' \lambda$. Since $s_1, s_2, s_3 \in \text{Ker } S \subset \text{Ker } d'$, we get $d'\partial' d' \lambda = 0$. By the disjointness of $d'$ and $\partial'$ [23, Proposition 3.7] $\partial' d' \lambda = 0$. Hence $s_1 \cdot s_2 - s_3 \in \text{Im } d'$. This proves the lemma. \[\square\]

3. Determination of the Integral

In this section $\mathfrak{g} = \mathfrak{g}(A_1)$ will denote a symmetrizable Kac–Moody Lie algebra.

Let $X \subset \{1, \ldots, l\}$ be a subset of finite type. In [17] and recently in the general case in [23], we constructed “$d, \partial$ harmonic” forms $\{s^w\}_{w \in W_\chi} \subset C(\mathfrak{g}, \mathfrak{r}_X)$ which are dual (up to a nonzero scalar multiple) to the Schubert cells $\{V_w = Bw^{-1}P_X/P_\chi \}_{w \in W_\chi}$. More precisely [23, Theorem 4.5], we have, for $w, w' \in W_\chi$ of equal length,

(a) $\int_{U_\chi} s^w \cdot s^{w'} = 0$, unless $w = w'$ and

(b) $\int_{U_\chi} s^w = (-1)^{l(p)} 2^{2p} \int_{U_\chi} \exp(2(w \rho - \rho) h(g)) \, dg$, if $l(w) = p$.

(Where $U_w = wUw^{-1} \cap U$ and, for any $g \in G$, $h(g)$ denotes the projection of $g$ on the $H^+$ factor under the inverse of the Iwasawa decomposition: $K \times H^+ \times U \to G$, defined by $(k, h, u) \to khu$. See (1.2), for various notations.) The additional sign factor $(-1)^{l(p-1)/2}$, in the expression of $\int_{U_\chi} s^w$, is due to the fact that we have taken a different orientation on complex manifolds (see (1.4)) than the one used in [23, Theorem 4.5].

In this section, we explicitly compute this integral. This, in particular, would give expression for the $d, \partial$ harmonic forms $\{s^w_0 = s^w/\int_{U_\chi} s^w \}_{w \in W_\chi}$, which are exactly dual to the Schubert cells. The main result of this section is

(3.1) Theorem. With the notations as above, for any $w \in W$, we have

$$2^{2(l(w))} \int_{U_\chi} \exp(2(w \rho - \rho) h(g)) \, dg = (4\pi)^{l(w)} \prod_{v \in W^{-1}A_\chi \cap A_\chi} \sigma(\rho, v)^{-1}.$$
Proof. We prove the theorem by induction on $l(w)$. In the case when $l(w) = 1$, the integral can be explicitly computed. The details are given in [24, Sect. 6.16].

So, we come to the general case. Write $w = r_j v$, where $r_j$ is a simple reflection such that $v < w$. From the Hodge-type decomposition of $C(g, h)$ [23, Theorem 3.13], we can express

$$s^v \cdot s^i = \sum_{h(w) = h(w')} z^w s^w + \lambda,$$

for some $d$-exact form $\lambda \in C(g, h)$ and some (uniquely determined) constants $z^w \in \mathbb{C}$.

The next few lemmas are devoted to finding the value of $z^w$. Define an operator $\bar{R} = -(S - L) \bar{M} : C(g, h) \to C(g, h)$, where $L$, $S$ and $\bar{M}$ are defined in [23, Sect. 3]. As in the proof of [23, Lemma 3.8], one can see that $\sum_{n \geq 0} \bar{R}^n \mu$ converges in $C(g, h)$, for all $\mu \in C(g, h)$. (Topology on $C(g, h)$ is the one, described in (2.3)). We have

(3.2) Lemma. Let $s_1, s_2 \in \text{Ker } S$. Write $P \sum_{n \geq 0} \bar{R}^n (s_1 \cdot s_2) = \sum_{w \in \mathcal{W}} z_n^w h^w$, for some (unique) constants $z_n^w$. ($h^w$ is defined in (1.15) and $P : C(g, h) \to C(g, h) = \text{Ker } L \oplus \text{Im } L$ is projection onto $\text{Ker } L$). Then

$$s_1 \cdot s_2 = \sum_{w \in \mathcal{W}} z_n^w s^w \in \text{Im } d.$$

Proof. First of all, $s_1 \cdot s_2 = d \partial \bar{M} (\sum_{n \geq 0} \bar{R}^n) (s_1 \cdot s_2) \in \text{Ker } S$. To prove this:

$$S(s_1 \cdot s_2) = S d \partial \bar{M} \left( \sum_{n \geq 0} \bar{R}^n \right) (s_1 \cdot s_2)
= S(s_1 \cdot s_2) - d \partial (S - L) \bar{M} \left( \sum_{n \geq 0} \bar{R}^n \right) (s_1 \cdot s_2) - d \partial L \bar{M} \left( \sum_{n \geq 0} \bar{R}^n \right) (s_1 \cdot s_2)
= S(s_1 \cdot s_2) - d \partial \sum_{n \geq 0} \bar{R}^{n+1} (s_1 \cdot s_2) - d \partial \left( \sum_{n \geq 0} \bar{R}^n \right) (s_1 \cdot s_2)
= S(s_1 \cdot s_2) - d \partial (s_1 \cdot s_2)
= 0 \quad (\text{since } d(s_1 \cdot s_2) = 0).$$
Furthermore,

\[-d\partial\vec{M}\left(\sum_{n \geq 0} \vec{R}^n\right)(s_1 \cdot s_2) = \left(-S + \hat{d}\right) \vec{M}\left(\sum_{n \geq 0} \vec{R}^n\right)(s_1 \cdot s_2)\]

\[= \left(-L - (S - L) + \hat{d}\right) \vec{M}\left(\sum_{n \geq 0} \vec{R}^n\right)(s_1 \cdot s_2)\]

\[= \sum_{n \geq 0} \vec{R}^{n+1}(s_1 \cdot s_2) + L(\mu)\]

for some $\mu \in C(g, h)$ (since $\text{Im} \, \hat{d} \subset \text{Im} \, L$). So

\[s_1 \cdot s_2 - d\partial \vec{M}\left(\sum_{n \geq 0} \vec{R}^n\right)(s_1 \cdot s_2) = \left(\sum_{n \geq 0} \vec{R}^n\right)(s_1 \cdot s_2) \mod \text{Im} \, L. \quad (I_{12})\]

But, since the left side of $(I_{12})$ belongs to $\text{Ker} \, S$, and $s^w - h^w \in \text{Im} \, L$ (by [23, Proposition (3.17)]) the lemma follows.

(3.3) Remark. The above lemma is essentially due to Koch. See [20, Theorem 4.3].

(3.4) Lemma. Let $w \in W$ be expressed as $w = r_i v$, with $r_i$ a simple reflection and $v < w$. Then $\langle \Phi_w \rangle = \langle \Phi_v \rangle + k_0 \alpha_i$, for some $k_0 > 0$ and, moreover, $k_0$ is equal to $\rho(v^{-1}h_i)$ (where $\langle \Phi_w \rangle$ denotes $\sum_{x \in \Phi_v} \alpha$ and $\Phi_w = A_+ \cap wA_-$).

Proof. By [26, Sect. 2], $\langle \Phi_w \rangle = r_i \langle \Phi_v \rangle + \alpha_i = \langle \Phi_v \rangle - \langle \Phi_v \rangle (h_i) \alpha_i = \langle \Phi_v \rangle + [1 - (\rho - v\rho) h_i] \alpha_i$. Furthermore, $(\rho - v\rho)(h_i) = 1 - 2\sigma(\rho, v^{-1}\alpha_i) / \sigma(\alpha_i, \alpha_i)$. Also, $\rho - v\rho$ being sum of roots, $(\rho - v\rho)(h_i) \in \mathbb{Z}$ and $\sigma(\rho, v^{-1}\alpha_i) > 0$, since $v^{-1}\alpha_i \in A_+$ (otherwise $\alpha_i \in \Phi_v$). Hence $(\rho - v\rho)(h_i) \leq 0$. This proves the lemma.

(3.5) Let $\tilde{A}_+$ be an index set for an orthonormal basis of root vectors of $n$ and for each $\Phi \in \tilde{A}_+$ let $b_\Phi$ be the corresponding root vector and let $-\Phi' \in \tilde{A}_-$ be the corresponding root. Since real roots have multiplicity 1 we may regard $\tilde{A}_+ \subseteq \tilde{A}_+$ and $\Phi' = \Phi$ for $\Phi \in \tilde{A}_+^{\text{Re}}$. Define $a_\Phi = -\omega_0 b_\Phi$. Clearly $\{a_\Phi\}_{\Phi \in \tilde{A}_+}$ is an orthonormal basis of $n$.

For a sequence $\Phi = (\Phi_1, \ldots, \Phi_p)$ with $\Phi_k \in \tilde{A}_+$, we define the operator $R_\Phi$, acting on $C(g, h)$, by

\[R_\Phi(e(b \otimes a)) = e(\text{ad} \, b_{\Phi_1} \circ \cdots \circ \text{ad} \, b_{\Phi_p} \, b \otimes \text{ad} \, a_{\Phi_1} \circ \cdots \circ \text{ad} \, a_{\Phi_p} \, a) \quad (I_{13})\]

for $b \otimes a \in [\wedge (n^{-}) \otimes \wedge (n)]^h$. (Recall that $e$ is defined in (2.4).)

It is easy to see that the operator $R_\Phi$, which is defined by $(I_{13})$ on a dense subspace $e([\wedge (n^{-}) \otimes \wedge (n)]^h)$ of $C(g, h)$, extends (uniquely) to a continuous operator (again denoted by) $R_\Phi : C(g, h) \to C(g, h)$. 

By [23, I_{24}, I_{34}, and I_{35}], we have

\[ S - L = -2 \sum_{\varphi \in \mathcal{A}} R_{\varphi} \quad \text{as operators on } C(\mathfrak{g}, \mathfrak{h}). \quad (I_{14}) \]

Fix \( v \in \mathcal{W} \). Let \( \Phi_v = \{ \beta_1, \ldots, \beta_p \} \ (p = l(v)) \). The element \( h_{\beta_i} \in \mathfrak{g}_{-\beta_i} \) (\( \beta_i \) being real, \( \mathfrak{g}_{-\beta_i} \) is 1-dimensional) is of unit norm. Define the element

\[ h^v = e \left[ (2 \sqrt{-1})^p (b_{\beta_1} \wedge \cdots \wedge b_{\beta_p} \otimes a_{\beta_1} \wedge \cdots \wedge a_{\beta_p}) \right]. \quad (I_{15}) \]

By [23, Sect. 4.2], for the special case \( X = \emptyset \), \( h^v \in \text{Ker } L \) and further, by [23, Proposition 3.17],

\[ s^v = \sum_{j \geq 0} R'(h^v) \quad (R \text{ is defined to be } -\tilde{M}(S - L)). \quad (I_{16}) \]

Specializing \( v = r_i \), we get

\[ s^{r_i} = 2 \sqrt{-1} \sum_{j \geq 0} R'(e(h_{r_i} a_{r_i})). \quad (I_{16}') \]

Let \( w \in \mathcal{W} \) be written as \( w = r_i v \), with \( v < w \). Recall the identity \((I_{11})\). We are interested in finding the value of \( z^w \). By Lemma (3.2), (I_{16}), and (I_{16}'), we have \( z^w = \text{coefficient of } h^w \text{ in } (2 \sqrt{-1} \sum_{n, n_1, n_2 \geq 0} \tilde{R}^n(R^n(h^w) \cdot (R^n(e(h_{r_i} a_{r_i})))) \). We have

(3.6) Lemma. With the notations as above,

\[ z^w = (k_0 c \sqrt{-1})^{2^{k_0}} \left( \prod_{k=1}^{k_0} [\sigma(v \rho - k \alpha_i, v \rho - k \alpha_i) - \sigma(\rho, \rho)] \right)^{-1} \]

where \( k_0 = \rho(v^{-1} h_i) \) and \( c \) is the coefficient of \( h^w \) in \( R_{x_i} (h^v) \cdot (R^n(e(h_{r_i} a_{r_i}))) \).

(By Lemma (3.4), \( k_0 \) is a positive integer.)

Proof. By Lemma (3.4) and (I_{14}), the only terms contributing to the coefficient \( z^w \) (of \( h^w \) in \( 2 \sqrt{-1} \sum_{n, n_1, n_2 \geq 0} \tilde{R}^n(R^n(h^w) \cdot R^n(e(h_{r_i} a_{r_i}))) \)) are

\[ (2 \sqrt{-1})^{2^{k_0} - 1} \sum_{n_1 = 0}^{k_0 - 1} (\tilde{M} R_{x_i})^{k_0 - 1} n_1 [R_{x_i} (h^v) \cdot (R^n(e(h_{r_i} a_{r_i})))]. \quad (E) \]

By \([23, (I_{32}), (I_{36}), (I_{37})]\), we have \( L(e(b \otimes a)) = [\sigma(\rho - \beta, \rho - \beta) - \sigma(\rho, \rho)] e(b \otimes a) \), for a weight vector \( b \in \wedge (n^-) \) of weight \( -\beta \) and \( a \in \wedge (n) \) such that \( b \otimes a \in [\wedge (n^-) \otimes \wedge (n)]^b \).
Further $\sigma(\rho, \rho) \neq \sigma(\rho - \langle \Phi_r \rangle - n_1 \alpha_i, \rho - \langle \Phi_r \rangle - n_1 \alpha_i)$, for any $1 \leq n_1 \leq k_0 - 1$, since

$$
\sigma(v\rho - n_1 \alpha_i, v\rho - n_1 \alpha_i) = \sigma(\rho, \rho) + n_1^2 \sigma(\alpha_i, \alpha_i) - 2n_1 \sigma(v\rho, \alpha_i)
$$

$$
= \sigma(\rho, \rho) + n_1^2 \sigma(\alpha_i, \alpha_i) - n_1 \sigma(\alpha_i, \alpha_i)(v\rho(h_i))
$$

$$
= \sigma(\rho, \rho) + (n_1^2 - k_0 n_1) \sigma(\alpha_i, \alpha_i)
$$

$$
\neq \sigma(\rho, \rho).
$$

So the expression (E) reduces to the sum

$$
(2\sqrt{-1}) \sum_{n_1=0}^{k_0-1} \left( \prod_{k=1}^{k_0-1} \left[ \sigma(v\rho - k\alpha_i, v\rho - k\alpha_i) - \sigma(\rho, \rho) \right]^{-1} \right)
$$

$$
R_{\alpha_i}^{k_0-1}(1)(h^r) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i}).
$$

But $R_{\alpha_i}^{k_0-1}(1)(h^r) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})$ = $R_{\alpha_i}^{k_0-1}(1)(h^r) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})$. Hence the expression (E) equals

$$
2\sqrt{-1} \sum_{n_1=0}^{k_0-1} \left( \prod_{k=1}^{k_0-1} \left[ \sigma(v\rho - k\alpha_i, v\rho - k\alpha_i) - \sigma(\rho, \rho) \right]^{-1} \right)
$$

$$
R_{\alpha_i}^{k_0-1}(1)(h^r) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i}). \quad \Box
$$

The following lemma gives the coefficient of $h^w$ in $R^{k_0-1}_{\alpha_i}(h^r) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})$.

(3.7) LEMMA. Let $w = r_i v$ be such that $v < w$. Then

$$
(ad b_{\alpha_i})^{k_0-1} b_{\Phi_r} \otimes (ad a_{\alpha_i})^{k_0-1} a_{\Phi_r} = (-1)^{k_0-1}(k_0 - 1)!^2 \left( \frac{\sigma(\alpha_i, \alpha_i)}{2} \right)^{k_0-1} b_{\Phi_r} \otimes a_{\Phi_r}
$$

in $\wedge(g)$, where $\Phi = r_i(\Phi_r) \subset \Delta_+$, $k_0 > 0$ is the same as in Lemma (3.4) and the notation $b_{\Phi_r} \otimes a_{\Phi_r}$, for any $\Phi = \{ \Phi_1, ..., \Phi_p \} \subset \Delta_+$ consisting of real roots, means $b_{\Phi_1} \wedge ... \wedge b_{\Phi_p} \otimes a_{\Phi_1} \wedge ... \wedge a_{\Phi_p} \in \wedge(g)$.

Proof. Denote $l(v) = p$. Fix any positive roots $\{ \Phi_1, ..., \Phi_p \}$ (not necessarily real and not necessarily distinct) such that $\sum_{n} \Phi_n = \langle \Phi_r \rangle$. If there exist $b_n \in g - \Phi_n$ satisfying $b_1 \wedge ... \wedge b_p \neq 0$ in $\wedge^r(n)$ and none of the $\Phi_n$ equals $\alpha_i$, then $\{ \Phi_1, ..., \Phi_p \} = \Phi$. To prove this; consider the element $0 \neq b_1 \wedge ... \wedge b_p \wedge b_{\alpha_i} \in \wedge^{r+1}(n)$. Since $\Phi_1 + ... + \Phi_p + \alpha_i = \langle \Phi \rangle + \alpha_i = \langle \Phi_r \rangle$ (see the proof of Lemma (3.4)), by [9, Theorem 8.5], $b_1 \wedge ... \wedge b_p \wedge b_{\alpha_i} = b_{\Phi_r}$ (up to a scalar multiple). Hence the assertion, that $\{ \Phi_1, ..., \Phi_p \} = \Phi$, follows.
Since \( \alpha_i \notin \Phi_r \), we get
\[
(\text{ad } h_{x_i})^{k_0-1} b_{\alpha_r} \otimes (\text{ad } a_{x_i})^{k_0-1} a_{\alpha_r} = z b_{\alpha_r} \otimes a_{\alpha_r} \quad (I_{17})
\]
for some constant \( z \) (since, by Lemma (3.4), \( \langle \Phi \rangle = \langle \Phi_{\psi} \rangle - \alpha_i = \langle \Phi_r \rangle + (k_0 - 1) \alpha_i \)).

Consider the \( sl(2) \) spanned by \( \{e_i, f_i, h_i\} \). Let this \( sl(2) \) act on \( \wedge (g) \) by the adjoint representation. We claim that
\[
(\text{ad } e_i) b_{h_i} = 0 \quad (I_{18})
\]
\[
(\text{ad } h_i) b_{e_i} = (k_0 - 1) h_{e_i} \quad (I_{19})
\]
To prove \( (I_{18}) \), it suffices (since \( \alpha_i \notin \Phi_r \)) to observe that, for \( \beta \in \Phi_r \), if \( \beta - \alpha_i \) is a root then \( \beta - \alpha_i \in \Phi_r \). (If \( \alpha, \beta \in \Phi \cap \Phi_r \) such that \( \alpha + \beta \) is again a root, then \( \alpha + \beta \in \Phi \cap \Phi_r \).) Of course, \( (I_{18}) \) follows from the definition of \( k_0 \).

\( (I_{18}) \) and \( (I_{19}) \) in conjunction with [28, Theorem 1, p. IV-4] yield
\[
(\text{ad } e_i)^{k_0-1} (\text{ad } f_i)^{k_0-1} (b_{\alpha_r}) = (k_0 - 1)!^2 b_{\alpha_r} \quad (I_{20})
\]
Consider the pairing \( \langle \ , \rangle : \wedge \rho(n^-) \otimes \wedge \rho(n^-) \rightarrow \mathbb{C} \) (defined by the Killing form) \( b \otimes a \mapsto \langle b, a \rangle \). By \( (I_{17}) \), we get
\[
\langle (\text{ad } h_{x_i})^{k_0-1} b_{\alpha_r}, (\text{ad } a_{x_i})^{k_0-1} a_{\alpha_r} \rangle = z \langle b_{\alpha_r}, a_{\alpha_r} \rangle.
\]
Using the invariance of \( \langle \ , \rangle \), this yields
\[
(-1)^{k_0-1} \langle (\text{ad } a_{x_i})^{k_0-1} (\text{ad } h_{x_i})^{k_0-1} b_{\alpha_r}, a_{\alpha_r} \rangle = z
\]
(since \( \langle b_{\alpha_r}, a_{\alpha_r} \rangle = 1 \)). \( (I_{21}) \)

Further, \( \{e_i, e_i\} = \{f_i, f_i\} = \langle e_i, f_i \rangle = 2/\sigma(\alpha_i, \alpha_i) \) and hence \( a_{x_i} \) (resp. \( b_{x_i} \)) can be taken to be \( (\sigma(\alpha_i, \alpha_i)/2) e_i \) (resp. \( (\sigma(\alpha_i, \alpha_i)/2) f_i \)). Now \( (I_{20}) \) and \( (I_{21}) \) give \( (-1)^{k_0-1}(k_0 - 1)!^2 (\sigma(\alpha_i, \alpha_i)/2)^{k_0-1} = z \) (since \( \langle b_{\alpha_r}, a_{\alpha_r} \rangle = 1 \)).

Putting together Lemmas (3.6) and (3.7), we get
\[
(I_{22}) \quad \text{Corollary. } z^n = (-1)^{h_0+1} k_0, \text{ where } k_0 \text{ is as defined in Lemma (3.4)}.
\]

\textbf{Proof.} Use the relation \( \sigma(v \rho - k \alpha_i, v \rho - k \alpha_i) - \sigma(\rho, \rho) = k(k - k_0) \sigma(\alpha_i, \alpha_i) \) (see the proof of Lemma (3.4)).

\textbf{(3.9) Proof of Theorem (3.1).} Denote by \( d^n \) the integral
2^{|w|} \int_{U_w} \exp(2(w\rho - \rho) h(g)) \, dg. \text{ Write } w = r_v v \text{ such that } v < w. \text{ By (a subsequent) Corollary (3.13), we have (for some } n^w \text{)}

\frac{s^v}{d^v} \frac{s^{r_v}}{d^{r_v}} - (-1)^{|r_v|} \frac{s^w}{d^w} + \sum_{n^w \neq n^w \text{ and } l(n^w) = l(n)} n^w s^{r_v} \in \text{Im } d.

Assume, by induction, that \( d^v = (4\pi)^{|r_v|} \prod_{v \in \mathfrak{g}^{-1} \cap A_+} \sigma(\rho, v)^{-1} \). Of course, \( d'^v = (4\pi)^{|r_v|} \prod_{v \in \mathfrak{g}^{-1} \cap A_+} \sigma(\rho, v)^{-1} \). By (1.11) and Corollary (3.8), \( d^v d'^v = (-1)^{|r_v|} d^v z^w = d^v \cdot k_0 \). So,

\( d^v = \frac{(4\pi)^{|r_v|} + 1}{k_0} \left( \prod_{v \in \mathfrak{g}^{-1} \cap A_+} \sigma(\rho, v)^{-1} \right) \cdot \sigma(\rho, \chi_i)^{-1} \)

(since \( k_0 = 2 \frac{\sigma(\rho, v^{-1} \chi_i)}{\sigma(\chi_i, \chi_i)} \) and \( \sigma(\rho, \chi_i) = \frac{1}{2} \sigma(\chi_i, \chi_i) \)),

\( = (4\pi)^{|r_v|} \prod_{v \in \mathfrak{g}^{-1} \cap A_+} \sigma(\rho, v)^{-1} \)

(since \( w^{-1} A_+ \cap A_+ = (v^{-1} A_+ \cap A_+) \cup \{v^{-1} \chi_i\} \)).

Hence the proof of the theorem is complete modulo the following proposition due to Bernstein, Gelfand, and Gelfand [3, Theorem (3.17)]. (Though their proof is in the finite dimensional situation, the proof goes through in the infinite dim case without any change.)

Let \( \varepsilon^v \) denote the cohomology class of \( G/B \) (where \( G \) is the group associated to any, not necessarily symmetrizable, Kac-Moody algebra \( g(A_i) \) and \( B \) is standard Borel subgroup of \( G \) as defined in (1.2)) which is dual to the closure of the Schubert cell \( B u_1 B/B \). (Of course, the cell, being complex, is oriented. See (1.4)).

(3.10) Proposition. Recall, from (1.4), that \( \chi_i \) \((1 \leq i \leq l)\) is any element of \( h^* \) satisfying \( \chi_i(h_j) = \delta_i, j \). For any simple reflection \( r_i \) and any element \( v \in W \),

\( \varepsilon^v \cdot \varepsilon^w = \sum_{w \rightarrow v} \chi_i(v^w) \varepsilon^w. \)

(As in [3], the notation \( v \rightarrow v \) means that \( v \in A_+^\circ \) with \( \sigma_v v = w \), and \( l(w) = l(v) + 1 \), where \( \sigma_v \in W \) denotes the reflection \( \sigma_v(\chi) = \chi - \chi(v^v) \) \( v \), for all \( \chi \in h^* \). Of course, \( v^v \in h \) denotes the co-root defined explicitly by \( v^v = u h_j \), if \( u \alpha_j = v \) for any \( u \in W. \))
(3.11) Remark. In the notation of [3], our $v''$ is the same as $P_{u^{-1}}$ and that is why we have $v''$ instead of $v^{-1}v''$ as in [3].

As an immediate consequence of the proposition, we have

(3.12) Corollary. Let $r, v = w \in W$ be such that $l(w) = l(v) + 1$, then the coefficient of $v''$ in $v''v'$ is 1.

We get the following as a corollary of the above corollary.

(3.13) Corollary. With $r, v = w$ as above (i.e., $l(w) = l(v) + 1$), the coefficient of $s''/s''$ in $(s''/s')(s''/s'')$ is $(-1)^{l(v)}$.

Proof. By [23, Theorem 4.5] and [24, Theorem 1.6], there is an algebra isomorphism $[\cdot] : H^\ast(g, h) \to H^\ast(G/B, C)$, such that the cohomology class $(s''/s'') \in H^\ast(g, h)$ maps onto $(-1)^{p(r-1)/2}v''$, where $p = l(w)$. (The sign $(-1)^{p(r-1)/2}$ appears because of a different orientation convention, on complex manifolds, in this paper.)

4. The Nil Hecke Ring $R$ and Its "Dual" $A$

Throughout this section $g = g(A)$ denotes an arbitrary (not necessarily symmetrizable) Kac–Moody Lie algebra, associated to a $l \times l$ generalized Cartan matrix $A$, with its Cartan subalgebra $h$ and Weyl group $W$ (1.1).

Let $Q = Q(h^\ast)$ denote the quotient field of the polynomial algebra $S = S(h^\ast)$, i.e., $Q$ is the field of all the rational functions on $h$.

(4.1) Ring structure on $Q$. The group $W$ operates as a group of automorphisms on the field $Q$. Let $Q_{\cdot\cdot}$ be the smash product of $Q$ with the group algebra $C[W]$. More specifically, $Q_{\cdot\cdot}$ is a right $Q$-module (under right multiplication) with a (free) basis $\{ \delta_w \}_{w \in W}$ and the multiplicative structure is given by

$$(\delta_w q_v) \cdot (\delta_w q_u) = \delta_w (w^{-1}q_v) q_u \quad \text{for} \quad v, w \in W \text{ and } q_v, q_u \in Q. \quad (I_{22})$$

Observe that though $Q_{\cdot\cdot}$ is an associative ring (with unity $\delta_e$), it is not an algebra over $Q$, since $\delta_w q = Q \delta_w$ is not central in $Q_{\cdot\cdot}$.

Let $S_{\cdot\cdot} \subset Q_{\cdot\cdot}$ be defined in the same way as $Q_{\cdot\cdot}$ with $S$ replacing $Q$. The ring $Q_{\cdot\cdot}$ (and $S_{\cdot\cdot}$) admits an involutary anti-automorphism $t$, defined by

$$(\delta_w q)^t = \delta_{w^{-1}}(w q) \quad \text{for} \quad w \in W \text{ and } q \in Q. \quad (I_{23})$$

Of course, $t$ is not right $Q$-linear. Clearly the left action of $W$ on $Q_{\cdot\cdot}$ (given as the left multiplication by $\delta_w$, for $w \in W$) is (right) $Q$-linear but the right action is not.
Now, for $i = 1, \ldots, l$, consider the elements
\[ x_i = x_r = - (\delta_r + \delta_c) \frac{1}{\alpha_i} = \frac{1}{\alpha_i} (\delta_r - \delta_c) \in Q_w \]  
(1.24)

where $r_i \in W$ is a simple reflection and $\alpha_i$ is the corresponding simple root.

Inspired by [3], we have

(4.2) PROPOSITION. Let $w \in W$ and let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression. Then the element $x_{i_1} \cdots x_{i_n} \in Q_w$ does not depend upon the choice of reduced expression of $w$. We define $x_{w}$ as $x_{i_1} \cdots x_{i_n}$. We denote by $\tilde{x}_w = (x_{w-1})'$.

Proof. For any $\chi \in h^*$, we have
\[ \chi x_i = x_i (\chi(r_i \chi)) - \frac{1}{\alpha_i} \delta_c (h_i). \]  
(1.25)

We assume, by induction on $n$, that for any $v$ with $l(v) < n$ and any two reduced expressions $v = r_{i_1} \cdots r_{i_p} = r_{j_1} \cdots r_{j_p}$ ($p = l(v)$), we have $x_{i_1} \cdots x_{i_p} = x_{j_1} \cdots x_{j_p}$ (which we denote by $x_v$) and
\[ \chi x_v = x_v (v^{-1} \chi) - \sum_{u' \rightarrow v} x_u \chi(v^u) \quad \text{for any} \quad \chi \in h^*. \]  
(1.26)

(See Proposition (3.10), for the notations $u \rightarrow v$ and $v^u$.)

Fix $w = vr_j$ with $v < w$. Now
\[ \chi x_v x_i = \left[ x_v (v^{-1} \chi) - \sum_{u' \rightarrow v} x_u \chi(v^u) \right] x_i \quad \text{(by 1.26)}}. \]
\[ \chi x_v x_i = x_v x_i (r_i n^{-1} \chi) - x_i (v^{-1} \chi(h_i)) - \sum_{u' \rightarrow v} x_u x_i \chi(v^u) \quad \text{(by 1.25)}}. \]  
(1.27)

Further, by [6, Theorem 1.1],
\[ \{u': u' \rightarrow v \text{ and } u' r_j > u'\} \rightarrow \{u: u \rightarrow w \text{ and } u \neq v\}. \]
under the map $u' \mapsto u'r_i$, and, under this correspondence, $v' \mapsto v'$. Hence we get (by $I_{27}$), exchange condition [13, Lemma 3.11(c)] and the fact that $x_i^2 = 0$)

$$\chi x_i x_i = x_i x_i (w^{-1} \chi) - \sum_{u \mapsto w} x_u \chi(v') .$$  \hspace{1cm} (I_{28})

In particular, if $w = vr_i = v'r_i$ with $l(v) = l(v') < l(w)$, we have

$$\chi x_i x_i - x_i x_i (w^{-1} \chi) = \chi x_i x_i - x_i x_i (w^{-1} \chi).$$  \hspace{1cm} (I_{29})

By (a subsequent) Proposition (4.3)(c), we can write

$$x_i x_i = \sum_{w' < w} \delta_{w'} q_{w'} + \delta_w \prod_{v \in A} v^{-1}$$  \hspace{1cm} (I_{30})

and

$$x_i x_i = \sum_{w' < w} \delta_{w'} \tilde{q}_{w'} + \delta_w \prod_{v \in A} v^{-1}$$  \hspace{1cm} (I_{31})

for some $q_{w'}$ and $\tilde{q}_{w'} \in Q$. Substituting (I_{30}) and (I_{31}) in (I_{29}), we get

$$\sum_{w' < w} \delta_{w'} (w'^{-1} \chi)(q_{w'} - \tilde{q}_{w'}) = \sum_{w' < w} \delta_w (w^{-1} \chi)(q_{w'} - \tilde{q}_{w'}).$$

In particular, for any $w' < w$ and any $\chi \in h^*$, we have $w'^{-1} \chi - w^{-1} \chi(q_{w'} - \tilde{q}_{w'}) = 0$. But $W \to \text{Aut}(h^*)$ being a faithful representation, we get $q_{w'} = \tilde{q}_{w'}$ for all $w' < w$. This proves, by (I_{30}) and (I_{31}), that $x_i x_i = x_i x_i$ and hence, by (I_{28}), the induction is complete.  

As a corollary, we get

(4.3) PROPOSITION. \hspace{1cm} (a) $x_i x_i = x_i x_i$, if $l(vw) = l(v) + l(w)$ and $x_i x_i = 0$ otherwise.

(b) $x_i x_i = x_i (w^{-1} \chi) - \sum_{w' \mapsto w} x_u \chi(v')$, for any $\chi \in h^*$ and $w \in W$.

(c) Write

$$x_i = \sum_{w} c_{r,w} \delta_{w}$$  \hspace{1cm} (I_{32})

Then (c_1) $c_{r,w} = 0$, unless $w \leq v$, and (c_2) $c_{r,v} = \prod_{v \in A} v^{-1}$, $v = 1$. In particular, $c_{r,v} \neq 0$.

Proof. \hspace{1cm} (a) Follows from Proposition (4.2) and the exchange condition [13, Lemma 3.11(c)] together with the fact that $x_i^2 = 0$. 


(b) is nothing but (I_{26}).

(c) follows from [6, Theorem 1.11]. Let \( v = r_{i_1} \cdots r_{i_n} \) be a reduced expression. To prove (c), observe that by the definition of \( x_{e_{i_{-1}}} \), \( c_{e,v} = [x_{i_n}(r_{i_n}x_{i_{n-1}}) \cdots (r_{i_2}x_{i_1})]^{-1} \). But then, by [26, Sect. 2], (c) follows.

(4.4) Remark. Observe that (c) does not depend upon Proposition (4.2), in fact we use it to prove Proposition (4.2).

The elements \( \{ \delta_w \}_{w \in W} \) are a right (as well as left) \( Q \)-basis of \( Q_w \). But also

(4.5) COROLLARY. Define the matrix \( C = (c_{e,w})_{e,w \in W} \), where \( c_{e,w} \) is defined in (I_{32}). By Proposition (4.3)(c), \( C \) is a “lower triangular” matrix with non-zero diagonal entries and hence, in particular, \( \{ x_w \}_{w \in W} \) is a right (as well as left) \( Q \)-basis of \( Q_w \).

Now, clearly, \( Q \) has the structure of a left \( Q_w \)-module, defined explicitly by

\[
(\delta_w q) q' = w(qq') \quad \text{for} \quad w \in W \quad \text{and} \quad q, q' \in Q. \quad (I_{33})
\]

Our main result centers around the subring \( R \subset Q_w \), defined by

\[
R = \{ x \in Q_w : x \cdot S \subset S \}.
\]

Obviously \( S_w \subset R \). Furthermore, one can easily see that \( x_i \) (and hence \( x_w \), for any \( w \in W \)) belong to \( R \). By applying the involution \( t \), one gets another subring \( R' \) of \( Q_w \). One has the following crucial structure theorem for \( R \).

The proof of (a) below can be simplified in the finite case using the theory of “harmonics.”

(4.6) THEOREM. (a) \( R \) is free as a right (or left) \( S \)-module. In fact the elements \( \{ x_w \}_{w \in W} \) form a right (or left) \( S \)-basis of \( R \). In particular, any \( x \in R \) can be uniquely written as

\[
x = \sum x_w p_w \quad \text{with some} \quad p_w \in S.
\]

(b) Furthermore, one has \( R \cap R' = S_w \).

(4.7) Remark. Note that \( R \) is a finitely generated ring over \( \mathbb{C} \), since it is generated by \( \{ x_i \}_{1 \leq i \leq l} \) and \( S \).

We need a few lemmas to prove the above theorem.

(4.8) LEMMA. Let \( e \neq w \in W \) be such that \( w \) fixes pointwise a hyperplane \( \Pi \) in \( h \). Then \( w = vr_i v^{-1} \), for some \( v \in W \) and for some simple reflection \( r_i \). In particular, \( \Pi \) is the real-root plane \( \text{Ker}(vx_i) \).
Proof. Step I. We first prove that, as an element of \( \text{Aut} \, h \), \( w \) is semi-simple. Since \( w \) fixes a hyperplane in \( h \), the only other possibility is that \( w \in \text{Aut} \, h \) is unipotent and, in fact, \((w-1)^2 = 0\), i.e., \( w^2 + 1 = 2w \). Multiplying by \( w^{-1} \), we get \( w-1 = 1 - w^{-1} \). Considering the (dual) representation of \( W \) in \( h^* \) and evaluating at \( \rho \), we get \( -(\rho - w\rho) = \rho - w^{-1}\rho \). But, by [9, Proposition 2.5], \( \rho - w^{-1}\rho \) and \( \rho - w\rho \) are both sums of \( l(w) \) positive roots. Since, by assumption, \( w \neq e \), we get a contradiction. So \( w \) is semi-simple.

Step II. We want to show that \( \Pi \) is a real-root plane, i.e., \( \Pi = \ker(v\alpha_i) \), for some \( v \in W \) and simple root \( \alpha_i \).

We need some notations. Let \( h_{\R} \) be the real points of \( h \). (In the Definition (1.1), one can take any field \( k \) of characteristic zero in place of \( C \) and define \( g_k, h_k \), etc.) As in [13, Sect. 3.12], define

\[
C = \{ h \in h_{\R} : \alpha_i(h) > 0, \text{for all } 1 \leq i \leq l \}
\]

\[
C^0 = \{ h \in h_{\R} : \alpha_i(h) > 0 \text{ for all } 1 \leq i \leq l \}.
\]

Set

\[
X = \bigcup_{v \in W} v \cdot C \quad \text{and} \quad X^0 = \bigcup_{v \in W} v \cdot C^0.
\]

The Tits cone \( X \) is a convex cone by [13, Proposition 3.12]. Let \( \Pi_{\R} = \Pi \cap h_{\R} \), \( \Pi \) being a hyperplane, \( h_{\R} \setminus \Pi_{\R} \) has exactly two connected components \( \Pi_{\R}^+ \) and \( \Pi_{\R}^- \). Since \( w \) is semi-simple and \( W \) leaves an integral lattice in \( h_{\R} \) stable, the only eigenvalue of \( w \in \text{Aut}(h_{\R}) \), different from 1, is \(-1\). Since \( X^0 \) is \( W \)-stable, it is easy to see that \( \Pi_{\R}^+ \cap X^0 \) and \( \Pi_{\R}^- \cap X^0 \) are both non-empty. Choose a point \( h^+ \) (resp. \( h^- \)) \( \in \Pi_{\R}^+ \cap X^0 \) (resp. \( \Pi_{\R}^- \cap X^0 \)). The line joining \( h^+ \) and \( h^- \) intersects the plane \( \Pi_{\R} \) in a point (say) \( h^0 \). Since \( \Pi_{\R}^+ \cap X^0 \) and \( \Pi_{\R}^- \cap X^0 \) are both open in \( h_{\R} \) (in the Hausdorff topology) and \( X \) is convex, there is an open subset \( N \) (containing \( h^0 \)) of \( \Pi_{\R} \) contained in \( X \). In particular, \( X \) being a cone, \( \Pi_{\R} \) itself is contained in \( X \). Further, since any point of \( X^0 \) has no isotropy with respect to the \( W \)-action [13, Proposition 3.12(a)], we get that

\[
\Pi_{\R} \subset X \setminus X^0 = \bigcup_{v \in W, \, 1 \leq i \leq l} v(\ker(\alpha_i)_{v \mid h_{\R}}).
\]

Since \( \Pi \) is a hyperplane, we get that \( \Pi = \ker(v\alpha_i) \), for some \( v \in W \) and some simple root \( \alpha_i \), as required.

Step III. Finally, we want to show that \( w = vr_iw^{-1} \). The element \( v^{-1}w \) has fixed plane \( v^{-1}\Pi \), which is the same as \( \ker \alpha_i \) by Step II. Choose a point \( h \in \ker(\alpha_i)_{v \mid h_{\R}} \), such that \( \alpha_j(h) > 0 \) for all \( j \neq i \). By [13,
Proposition 3.12(a)], the only non-trivial isotropy element (with respect to the $W$-action) at $h$ is $r_i$. This shows that $v^{-1}wv = r_i$, proving the lemma.

(4.9) **Lemma.** Let $p$ be an irreducible polynomial $\in S$ and let

$$
\left( \sum_{\langle w \rangle \leq k} p_wx_w \right) \cdot S \subseteq pS
$$

(R)

where $p_w \in S$ (for all $w$), any $p_w$ is either 0 or co-prime to $p$ and $p_w \neq 0$ for some $w$ of length $k$. Then $p$ is a real root.

**Proof.** Rewrite $x = \sum_{\langle w \rangle \leq k} p_wx_w = \sum_{\langle w \rangle \leq k} q_w\delta_w$, where $q_w \in Q$. In fact, $q_w$ has in its denominator only products of real roots. By Proposition (4.3)(c), for $\langle w \rangle = k$, we have $q_w = p_w \cdot c_w \cdot \delta_w^1$, where $c_w \cdot \delta_w^1 = \prod_{v \in W \setminus r_i} \delta_v^1$.

Define $V = \bigcup_{r \in W \setminus r_i} \ker(v - 1)$ ($v - 1$ acting on $h$). We first prove that $Z(p) \subseteq V$, where $Z(p)$ denotes the zero set of $p$. If not, pick any $h_0 \in Z(p) \setminus V$. In particular $h_0$ has no $W$-isotropy. Pick $w_0$ of maximal length $k$ such that $p_{w_0} \neq 0$. There exists a polynomial $p_0 \in S$ such that $p_0(h_0^{-1}h_0) = 1$ and $p_0(w^{-1}h_0) = 0$ for all those (finite in number) $w \neq w_0$ satisfying $q_w \neq 0$. Evaluating $x \cdot p_0$ at $h_0$, we get from (R), $p_{w_0}(h_0) \cdot c_{w_0}^{-1} = 0$ for all such $w$. (Since $h_0$ has no $W$-isotropy, $c_{w_0}^{-1}(h_0)$ makes sense and, of course, is non-zero.) Hence $p_{w_0}(h_0) = 0$, i.e., $p$ divides $p_{w_0}$, which is a contradiction to the assumption. So $Z(p) \subseteq V$.

Since $p$ is irreducible, we get that $Z(p) \subseteq \ker(v - 1)$, for some $v \in W$ and moreover $Z(p)$ being a hyper surface in $h$, $\ker(v - 1)$ is a hyperplane. The lemma follows now by Lemma (4.8).

(4.10) **Lemma.** Let $\{p_{w_1}\}_{\langle w \rangle \leq k}$ be polynomials $\in S$ such that $(\sum_{\langle w \rangle \leq k} p_wx_w) \cdot S \subseteq \alpha, S$, for some simple root $\alpha$. Then $\alpha$ divides all the $p_w$'s.

**Proof.** Denote by $x = (1/\alpha) \sum p_wx_w \in Q_w$. Write $x = x^+ + x^-$, where $x^+$ (resp. $x^-$) $= \frac{1}{2}(x + \delta_i, x)$ (resp. $\frac{1}{2}(x - \delta_i, x)$). Since $x^+$ again satisfies $x^+ \cdot S \subseteq S$ and, by Proposition (4.3), $x^+$ is of the form $(1/\alpha) \sum p', x_i$ for some $p' \in S$ (a similar statement holds good for $x^-$), we can assume that either $\delta_i, x = x$ or $\delta_i, x = -x$. Write

$$
\alpha, x = \sum_{\langle w \rangle \leq k} p_wx_w = \sum_{\langle w \rangle \leq k} q_w \delta_w,
$$

(134)

where, as in the proof of the previous lemma, we have

$$
q_w = p_w \cdot c_w \cdot \delta_w^1 \quad \text{if} \quad \langle w \rangle = k.
$$

(135)
Fix $w_0$ of length $k$ such that $p_{w_0} \neq 0$ and rewrite (I$_{3a}$)

$$\alpha_i x = x_0 + q_{w_0} \delta_{w_0} + q_{r_i w_0} \delta_{r_i w_0},$$

where

$$x_0 = \sum_{w \notin \{r_i w_0, w_0\}} q_w \delta_w.$$  \hspace{1cm} (I$_{36}$)

Also fix a point $h_0 \in \text{Ker} \alpha_i \cap C$ ($C$ is defined in the proof of Lemma (4.8)) such that $h_0$ does not lie on any other real-root plane and choose a polynomial $p_0 \in S$ such that $p_0(w_0^{-1} h_0) = 1$ and $p_0$ at $w^{-1} h_0$ has a “deep” zero for any (finite in number) $w$ such that $q_w \neq 0$ and $w \neq w_0, r_i w_0$. (Since $W$-isotropy at $h_0$ is precisely \{1, $r_i$\}, this is possible.)

Case I. $\delta_{r_i} x = x$. In this case, by (I$_{36}$), we have $-(r_i q_{w_0}) = q_{r_i w_0}$. In particular, $r_i w_0 < w_0$. Choose a reduced expression $w_0 = r_i r_i^{'} \cdots r_{k-1}^{'}$ (since $r_i w_0 < w_0$, this is possible). By (I$_{35}$), (I$_{36}$), and Proposition (4.3)(c), we have

$$\alpha_i x = x_0 + \frac{p_{w_0}}{\alpha_i(r_i \beta)} \delta_{w_0} + \frac{r_i \cdot p_{w_0}}{\alpha_i(r_i \beta)} \delta_{r_i w_0},$$

where $\beta = \alpha_i \cdots (r_i \cdots r_{k-2} \alpha_i)^{k_i-1}$.

\hspace{1cm} (I$_{37}$)

Evaluating $(\alpha_i^2 \cdot x) \cdot p_0$ at $h_0$ (since $\alpha_i(h_0) = 0$, $p_0(w_0^{-1} h_0) = 1$, and $p_0$ has “deep” zero at points other than $w_0^{-1} h_0$), we get, by (I$_{37}$), $p_{w_0}(h_0) = 0$. Hence $\alpha_i$ divides $p_{w_0}$.

Case II. $\delta_{r_i} x = -x$: In this case, we have $r_i q_{w_0} = q_{r_i w_0}$. In particular, again we have $r_i w_0 < w_0$. Analogous to (I$_{37}$), we get

$$\alpha_i x = x_0 + \frac{p_{w_0}}{\alpha_i(r_i \beta)} \delta_{w_0} - \frac{r_i \cdot p_{w_0}}{\alpha_i(r_i \beta)} \delta_{r_i w_0},$$

where $\beta$ is the same as in (I$_{37}$).

Considering $\alpha_i x(w_0 \cdots \alpha_i \cdot p_0)$ and evaluating at $h_0$, we get again $p_{w_0}(h_0) = 0$, i.e., $\alpha_i$ divides $p_{w_0}$ in this case as well. This proves the lemma.

(4.11) Proof of Theorem (4.6)(a). Let $x \in R$. By Corollary (4.5), we can write $x = (1/p) \sum_{\alpha_i \in \chi} p_{\alpha_i} x_{\alpha_i}$, for some $p_{\alpha_i} \in S$ and some $\alpha_i \in \chi$. We want to prove that $p$ divides $p_{\alpha_i}$ for every $w$. We can assume, of course, that $p$ is irreducible. By Lemma (4.9), if $p$ does not divide some $p_{\alpha_i}$ then $p$ has to be a real root (say) $\alpha_i$, for some $\alpha_i \in \chi$ and simple root $\alpha_i$. Since $\delta_{\alpha_i} R = R$ and $\delta_i(S \alpha_i) = \sum \alpha_i S \alpha_i$ (as is easy to see), we can assume that $p = \alpha_i$. But
then Lemma (4.10) proves that \( x \in \sum_w S \alpha_w \). The rest is clear from Corollary (4.5) together with Proposition (4.3).

(b) Fix \( x \in R \cap R' \). Since \( x \in R \), we can write, by (a), \( x = \sum p_n \alpha_n \), for some \( p_n \in S \). Express \( x = \sum q_n \delta_n \), where \( q_n \in Q \). Upon multiplying by a suitable polynomial, we can assume, without loss of generality, that all the \( q_n \)’s have only one fixed real root (say) \( \nu \), in their denominators. Further since \( \delta_{r-1} R = R \), \( \delta_{r-1} \cdot R' = R' \), and \( \delta_{r-1} \cdot S_w = S_{w'} \), we can further assume that all the \( q_n \)’s have only \( \alpha_i \) in their denominators, i.e.,

\[
x = \frac{1}{\alpha_i} \sum \tilde{p}_w \delta_w \quad \text{for some } \tilde{p}_w \in S.
\]  

(138)

We want to prove that all the \( \tilde{p}_w \)’s are divisible by \( \alpha_i \). Analogous to the proof of Lemma (4.10), considering \( x = \frac{1}{2} (x + \delta_{r} x) + \frac{1}{2} (x - \delta_{r} x) \), we can assume that either \( \delta_{r} x = x \) or \( \delta_{r} x = -x \).

Case I. \( \delta_{r} x = x \). In this case, by (138), we have

\[
\tilde{p}_e = - r_i (\tilde{p}_r).
\]  

(139)

By (138), we get (taking \( t \))

\[
x' = \frac{\tilde{p}_e}{\alpha_i} \delta_e - r_i \frac{\tilde{p}_r}{\alpha_i} \delta_r + \sum_{w \neq r, r} w^{-1} \left( \frac{\tilde{p}_w}{\alpha_i} \right) \delta_{w}.
\]  

(140)

Fix \( h_0 \in \text{Ker} \alpha_i \), such that \( h_0 \) does not lie on any other real-root plane. Choose a function \( \rho_0 \in S \) such that \( \rho_0(h_0) = 1 \) and \( \rho_0(wh_0) = 0 \) if \( w \neq e, r_i \) and \( \tilde{p}_w \neq 0 \). (This is possible because, by the choice, isotropy at \( h_0 \) is precisely \( \{ e, r_i \} \).) Considering \( \alpha_i (x', \rho_0) \) and evaluating at \( h_0 \) we get (by (139) and (140)) \( \tilde{p}_e(h_0) = 0 \), i.e., \( \alpha_i \) divides \( \tilde{p}_e \). To prove that \( \alpha_i \) divides \( \tilde{p}_w \) for general \( w \), we can consider \( x' \delta_w \) and argue as before.

Case II. \( \delta_{r} x = -x \). In this case, by (138), we have

\[
\tilde{p}_e = r_i \tilde{p}_r.
\]  

(141)

Fix \( h_0 \) and \( \rho_0 \) as in the previous case. Considering \( \alpha_i (x, \rho_0) \) and evaluating at \( h_0 \) we get (by (138) and (141)) \( \tilde{p}_e(h_0) = 0 \). So again \( \alpha_i \) divides \( \tilde{p}_e \). This completes the proof of (b) part as well. \( \square \)

(4.12) Definition. The elements \( \{ x_w \} \) have much in common with the standard basis of a Hecke ring. However, \( x_{r_i}^2 = x_i^2 = 0 \). This and a further nilpotence condition, in its action on \( A \) (Sect. 4.19), persuade us to refer to \( R \) as a nil Hecke ring. A departure from usual conditions is that \( S \) is not central in \( R \).
We describe the left action of the Weyl group $W$ in terms of $\tilde{x}_w$ basis ($\tilde{x}_w$ is defined in Proposition (4.2)).

(4.13) \textbf{Proposition.} Fix a simple reflection $r_i \in W$. Then

$$\delta_{r_i} \tilde{x}_w = - \tilde{x}_w$$

if $r_i w < w$ and

$$= \tilde{x}_{r_i}(w^{-1} x_i) - \sum_{r \to r_i w} \tilde{x}_r x_i(v')$$

otherwise.

\textbf{Proof.}

$$\delta_{r_i} \tilde{x}_w = (\delta_{r_i} - \delta_i) \frac{1}{x_i} (\tilde{x}_i \tilde{x}_w) + \tilde{x}_w$$

$$= \tilde{x}_{r_i} \left[ \tilde{x}_w (w^{-1} x_i) - \sum_{r' \to r_i w} \tilde{x}_r x_i(v') \right] + \tilde{x}_w$$

by (I_26).

\textbf{Case I.} $r_i w < w$. In this case only $v'$ with the property that $v' \to w$ and $v' \to r_i v'$ is $r_i w$: Hence, by Proposition (4.3), the above sum reduces to $-2\tilde{x}_w + \tilde{x}_w = - \tilde{x}_w$.

\textbf{Case II.} $r_i w > w$. In this case,

$$\delta_{r_i} \tilde{x}_w = \tilde{x}_{r_i}(w^{-1} x_i) + \tilde{x}_w - \sum_{r' \to r_i w \text{ and } v' \to r_i v'} \tilde{x}_{r_i} x_i(v')$$

(4.12)

Further, as in the proof of Proposition (4.2), it is easy to see that $\{ v' \to v' w \text{ and } v' \to r_i v' \} \to \{ v \to \text{r}_i w \text{ and } v \neq w \}$, under the map $v' \mapsto r_i v'$. Moreover, under this map, $v'$ corresponds with $r_i v'$. (It is clear that $r_i(v')$ is a positive root.) Hence, by (I_42), we get the required result. \qed

(4.14) \textbf{Co-product structure on $Q_w$.} Let $Q_w \otimes_Q Q_w$ be the tensor product, considering both the copies of $Q_w$ as right $Q$-modules. Define the diagonal map $\Delta: Q_w \to Q_w \otimes_Q Q_w$, by

$$\Delta(\delta_w q) = \delta_w q \otimes \delta_w = \delta_w \otimes \delta_w q \quad \text{for } w \in W \text{ and } q \in Q.$$  \hspace{1cm} (I_{43})

$\Delta$ is clearly right $Q$-linear. Moreover, it is easy to see that the co-product $\Delta$ is associative and commutative with a co-unit $\varepsilon: Q_w \to Q$, defined by

$$\varepsilon(\delta_w q) = q.$$

We introduce an \textit{associative product structure}, denoted by $\odot$, in $Q_w \otimes_Q Q_w$, so that $\Delta$ is a ring homomorphism. Define

$$(\delta_v q_v \otimes \delta_w q_w) \odot (\delta_v' q_{v'} \otimes \delta_w' q_{w'}) = \delta_{v' (w')^{-1} w} q_{v' (w')^{-1} q_v} \odot \delta_{w w'} (w'^{-1} q_w) q_{w'}.$$  \hspace{1cm} (I_{44})
In the next proposition, we describe the diagonal map $A$ in terms of \{\tilde{x}_w\} basis.

\textbf{(4.15) Proposition.} For any $w \in W$, we have

$$A(\tilde{x}_w) = \sum_{u,v \leq w} \tilde{x}_u \otimes \tilde{x}_v (p^w_{u,v})$$

for some homogeneous polynomials $p^w_{u,v} \in S$ of degree $l(u) + l(v) - l(w)$. In particular $p^w_{u,v} = 0$ unless $l(u) + l(v) \geq l(w)$.

\textbf{Proof.} We prove the proposition by induction on $l(w)$. For $w = r_i$, we have

$$A(\tilde{x}_{r_i}) = \delta_{r_i} \left( \frac{1}{\alpha_i} \right) \otimes \delta_{r_i} \left( \frac{1}{\alpha_i} \right) \otimes \delta_{r_i} \left( \frac{1}{\alpha_i} \right)$$

$$= (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) \otimes (\delta_{r_i} - \delta_{r_i}) + \delta_{r_i} \otimes (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$+ (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) \otimes \delta_{r_i} \left( \frac{1}{\alpha_i} \right)$$

$$A(\tilde{x}_{r_i}) = \tilde{x}_{r_i} \otimes \tilde{x}_{r_i} (\alpha_i) + \delta_{r_i} \otimes \tilde{x}_{r_i} + \tilde{x}_{r_i} \otimes \delta_{r_i}. \quad (I_{45})$$

Now take arbitrary $w$ and write $w = w' r_i$, for some simple reflection $r_i$ so that $w' < w$. Since $A$ is multiplicative, we have $A(\tilde{x}_w) = A(\tilde{x}_{w'}) \circ A(\tilde{x}_{r_i})$.

$$A(\tilde{x}_w) = \left[ \sum_{u', v' \leq w'} \tilde{x}_{u'} \otimes \tilde{x}_{v'} p^w_{u', v'} \right] \otimes \left[ \delta_{r_i} \otimes \delta_{r_i} \left( \frac{1}{\alpha_i} \right) \otimes \delta_{r_i} \left( \frac{1}{\alpha_i} \right) \right] \quad (I_{46})$$

(by induction hypothesis).

It is easily seen, from (I_{44}), that

$$(x \otimes y) \circ (\delta_{w_0} \otimes \delta_{w_0} q) = x \cdot \delta_{w_0} \otimes y \cdot \delta_{w_0} q$$

for any $x, y \in Q_{w'}$, $w_0 \in W$, and $q \in Q$.

Hence from (I_{46}), we get

$$A(\tilde{x}_w) = \sum_{u', v' \leq w'} \tilde{x}_{u'} \delta_{r_i} \otimes \tilde{x}_{v'} \delta_{r_i} (r_i, p^w_{u', v'}) \left( \frac{1}{\alpha_i} \right) - \sum_{u', v' \leq w'} \tilde{x}_{u'} \otimes \tilde{x}_{v'} \left( \frac{1}{\alpha_i} \right)$$

$$= \sum_{u', v' \leq w'} \tilde{x}_{u'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) \otimes \tilde{x}_{v'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$\otimes \tilde{x}_{r_i} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) (r_i, p^w_{u', v'}) + \sum_{u', v' \leq w'} \tilde{x}_{u'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$\otimes \tilde{x}_{r_i} \left( r_i, p^w_{u', v'} \right) \otimes \tilde{x}_{v'} \left( r_i, p^w_{u', v'} \right) \left( \frac{1}{\alpha_i} \right)$$

$$= \sum_{u', v' \leq w'} \tilde{x}_{u'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) \otimes \tilde{x}_{v'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$\otimes \tilde{x}_{r_i} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) (r_i, p^w_{u', v'}) + \sum_{u', v' \leq w'} \tilde{x}_{u'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$\otimes \tilde{x}_{r_i} \left( r_i, p^w_{u', v'} \right) \otimes \tilde{x}_{v'} \left( r_i, p^w_{u', v'} \right) \left( \frac{1}{\alpha_i} \right)$$

$$= \sum_{u', v' \leq w'} \tilde{x}_{u'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) \otimes \tilde{x}_{v'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$\otimes \tilde{x}_{r_i} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right) (r_i, p^w_{u', v'}) + \sum_{u', v' \leq w'} \tilde{x}_{u'} (\delta_{r_i} - \delta_{r_i}) \left( \frac{1}{\alpha_i} \right)$$

$$\otimes \tilde{x}_{r_i} \left( r_i, p^w_{u', v'} \right) \otimes \tilde{x}_{v'} \left( r_i, p^w_{u', v'} \right) \left( \frac{1}{\alpha_i} \right)$$
\[
\Delta(\tilde{x}_w) = \sum_{u', r' \leq u' \text{ with } u' < u r', \text{ and } r' < r r'} (\tilde{x}_{u'} \otimes \tilde{x}_{r'})(r_i p_{u', r'}^{w'}) \cdot \chi_i \\
+ \sum_{u', r' \leq u' \text{ with } u' < u r', \text{ and } r' < r r'} \tilde{x}_{u'} \otimes \tilde{x}_{r'}(r_i p_{u', r'}^{w'}) \\
- \sum_{u', r' \leq u' \text{ with } u' < u r', \text{ and } r' < r r'} \tilde{x}_{u'} \otimes \tilde{x}_{r'} E_{x_i}(p_{u', r'}^{w'}) \quad \text{(by Proposition (4.3))} \quad (I_{47})
\]

where \(E_{x_i}\) is the classical Bernstein–Gel'fand–Gel'fand operator \([3, \text{ Sect. 1}]\) acting on \(S\) and defined by \(E_{x_i}(p) = (p - r_i p)/\chi_i\), for any \(p \in S\).

The proposition follows now by observing that \(E_{x_i}: S \to S\) decreases the degree by 1 and further using \([6, \text{ Theorem 1.1}].

(4.16) Remark. We will determine \(\{p_{u', r'}^{w'}\}\) more specifically later in (4.31) and (4.32). The fact that Proposition (4.13) (resp. Proposition (4.15)) gives the Weyl group action (resp. the cup product) on the cohomology ring \(H^*(G/B)\) would be clear in the next section.

Now we dualize the concepts (and results) introduced (proved) so far in this section. These dual objects will play an important role in determining various structures on the cohomology of infinite dimensional flag varieties.

(4.17) The algebra \(Q^w\). Regarding \(Q^w\) as a right \(Q\)-module, let \(\Omega = \text{Hom}_Q(Q^w, Q).\) Since any \(\psi \in \Omega\) is determined by its restriction to the (right) \(Q\)-basis \(\{\delta_w\}_{w \in W}\) and conversely, we can (and often will) regard \(\Omega\) as the \(Q\)-module of all the functions: \(W \to Q\) with pointwise addition and scalar multiplication, i.e.,

\[
(q \psi)(w) = q \cdot \psi(w) \quad \text{for } q \in Q, \psi \in \Omega, \text{ and } w \in W. \quad (I_{48})
\]

Furthermore, \(\Omega\) inherits a (commutative) \(Q\)-algebra structure with the product as pointwise multiplication of functions on \(W\). In fact, this multiplication is precisely the one obtained by dualizing the \(Q\)-linear co-multiplication \(A\) in \(Q^w\) (see \((I_{31})\)).

More subtly, \(\Omega\) also has the structure of a left \(Q_w\)-module defined by

\[
(x \cdot \psi)(y) = \psi(x \cdot y) \quad \text{for } x, y \in Q_w \text{ and } \psi \in \Omega. \quad (I_{49})
\]

Moreover the action is \(Q\)-linear. In particular, we have the Weyl group action as well as the Hecke-type operators \(A_w\) \((w \in W)\) on \(\Omega\), defined by

\[
w \psi = \delta_w \cdot \psi \quad (I_{50})
\]

\[
A_w \psi = x_w \cdot \psi \quad \text{for } w \in W \text{ and } \psi \in \Omega. \quad (I_{51})
\]
(Recall that $x_w$ is defined in Proposition (4.2)). Taking $w$ to be one of the simple reflections $r_i$, from (I$_{51}$), we get

$$(A_r, \psi) v = \frac{\psi(r_i v) - \psi(v)}{v^{-1}a_i} \quad \text{for} \quad \psi \in \Omega \text{ and } v \in W. \quad \text{(I$_{52}$)}$$

We would generally abbreviate $A_r$ by $A_i$.

(4.18) Remark. Observe that $\Omega$ is a $Q$-module (under (I$_{48}$)) as well as a left $Q_w$-module (under (I$_{49}$)). Further, $Q$ injects into $Q_w$ by $q \to \delta_v q$ (for $q \in Q$); in particular, $\Omega$ inherits a $Q$-module structure (from the restriction of $Q_w$-module structure). But these two $Q$-structures are different, in general. Whenever, we refer to $\Omega$ as a $Q$-module, we would mean the first $Q$-action.

We define the following important subring $A$ of $\Omega$.

(4.19) Definition. $A = \{ \psi \in \Omega: \psi(R') \subseteq S \text{ and } \psi(x_w) = 0 \text{ for all but a finite number of } w \in W \}$. ($R$ is defined just before Theorem (4.6) and $x_w$ is defined in Proposition (4.2).)

One has the following

(4.20) Proposition. (a) $A$ is a $S$-subalgebra of $\Omega$.

(b) $A$ is a free $S$-module. In fact, $\{ \xi^w \}_{w} \subseteq S$ is an $S$-basis of $A$, where $\xi^w \in \Omega$ is defined (uniquely) by $\xi^w(\tilde{x}_v) = \delta_{v,w}$, for $w, v \in W$.

(c) $A$ is stable under the left action of $R < Q_w$.

Proof. To prove (a), use Proposition (4.15). Since, by Theorem (4.6)(a), $\{ x_w \}_{w}$ is a right $S$-basis of $R'$, we get (b). Again using Theorem (4.6)(a) and Proposition(4.3)(b), (c) follows.

The matrix $D$ (defined below) is very basic to our paper.

(4.21) The matrix $D$. Define the matrix $D = (d_{v,w})_{v,w \in W}$ by $d_{v,w} = \xi^w(v)$. The relevance of $D$ to the cup product and Weyl group action on the cohomology of infinite dimensional flag varieties will be clear in the next section.

(4.22) Remark. In the finite case, the matrix $D$ can be extracted from [3, Theorem 5.9] (see Sect. 6 of this paper).

(4.23) Definition. Let $\mathcal{B} = \mathcal{B}_w$ be the space of all the functions $B: W \times W \to Q$, with the property that there exists a $d_B \geq 0$ such that $B(v, w) = 0$ whenever $l(v) - l(w) > d_B$. $\mathcal{B}$ is an associative algebra over $Q$. 
under pointwise addition and convolution as multiplication (i.e., 
\[ B_1 \cdot B_2(v, w) = \sum_u B_1(v, u) B_2(u, w), \text{ for } B_1, B_2 \in \mathcal{B}. \]

We can think of \( \mathcal{B} \) as an appropriate subspace of all the \( W \times W \) matrices over \( Q \). Under this identification, the multiplication in \( \mathcal{B} \) is nothing but the matrix multiplication.

We collect various properties of \( \{ \xi^w \} \) in

(4.24) **Proposition.** For any \( v, w \in W \), we have

\[
\begin{align*}
(\text{a}) & \quad \xi^v(w) = d_{v,w} = 0, \text{ unless } v \leq w \text{ and } \xi^w(w) = d_{v,w} = \prod_{v \in W} \chi_{v \cap A} v. \\
& \text{In particular, } d_{v,w} \neq 0 \text{ and the matrix } D \text{ (defined in (4.21)) belongs to } \mathcal{B}. \text{ In fact, it is "upper triangular" with non-zero diagonal entries and hence invertible.} \\
(\text{b}) & \quad A_v \xi^w = \xi^w \text{ if } v < w, = 0 \text{ otherwise.} \\
(\text{c}) & \quad \xi^w(w) - \chi_i - w^{-1} \chi^w (\chi_i \text{ is defined in (1.4)).} \\
(\text{d}) & \quad \xi^w(w) \text{ is a homogeneous polynomial of degree } l(v). \\
(\text{e}) & \quad C' = D^{-1}, \text{ where the matrix } C \text{ is defined in Corollary (4.5), } C' \text{ denotes the transposed matrix, and } D^{-1} \text{ denotes the inverse of the element } D \in \mathcal{B}_W. \\
(\text{f}) & \quad \xi^v \xi^w = \sum_{v < w} p_{v,w} \xi^w, \text{ where } p_{v,w} \text{ is defined in Proposition (4.15).} \text{ We recall that } p_{v,w} \text{ is a homogeneous polynomial of degree } l(u) + l(v) - l(w).
\end{align*}
\]

(4.25) **Remark.** We will give a characterization of the matrix \( D \) in Proposition (5.5).

**Proof.** (a) Assume that \( v \leq w \) and assume further, by induction, that for any \( u < w \), we have \( \xi^v(u) = 0. \) By Proposition (4.3)(c), we can write

\[
\delta_v = \left[ \prod_{v \in W} \chi_{v \cup A} \right] x_v + \sum_{u < w} q_u \delta_{u^{-1}} \text{ for some } q_u \in Q. \quad (153)
\]

Taking \( t \), we get

\[
\delta_v = x_v \left[ \prod_{v \in W} \chi_{v \cup A} \right] + \sum u < w \delta_u q_u. \quad (154)
\]

This proves (a).

(b) Follows by dualizing Proposition (4.3)(a).
(c) If \( w = e \), (c) follows from (a). Otherwise write \( w = r_j v \), for some \( r_j \) so that \( v < w \) and assume, by induction, that \( \xi'(v) = \chi_i - v^{-1} \chi_i \). Now

\[
(A_j \xi'(v)) v = \frac{\xi'(r_j v) - \xi'(v)}{v^{-1} \chi_i}
\]

by \((I_{52})\)

\[
= \left[ \xi'(w) - (\chi_i - v^{-1} \chi_i) \right] \frac{1}{v^{-1} \chi_i} \quad \text{(by induction)}.
\]

But \( A_j \xi' = \delta_{i,j} \) by \((b)\). Hence we have

\[
\xi'(w) = \delta_{i,j} (v^{-1} \chi_i) + \chi_i - v^{-1} \chi_i = \chi_i - w^{-1} \chi_i.
\]

This proves (c).

(d) If \( w = e \), there is nothing to prove. So, assume that \( w \neq e \) and write \( w = r_j w' \), for some \( r_j \) so that \( w' < w \). Now, by \((b)\) and \((I_{52})\), we have

\[
(A_j \xi')(w') = \frac{\xi'(w) - \xi'(w')}{(w')^{-1} \chi_i} = 0 \quad \text{if } r_j v > v,
\]

\[
= \xi'(w') \quad \text{if } r_j v < v.
\]

But, by induction, we can assume that, in the case \( r_j v < v \), \( \xi'(w') \) is a homogeneous polynomial of degree \( l(r_j v) \), and also \( \xi'(w') \) is a homogeneous polynomial of degree \( l(v) \). This proves (d).

(e) Since \( D \) is invertible, it suffices to show that \( D \cdot C' = \text{Id} \). Fix \( v, w \in W \). By definition, \( \xi''(\tilde{x}_r) = \delta_{w,v} \). But by \((I_{12})\), \( x_r = \sum c_{r,u} \delta_{u,v} \), i.e.,

\( \tilde{x}_r = \sum \delta_{r,u} c_{r,u} \). Hence \( \xi''(\tilde{x}_r) = \delta_{w,v} = \sum d_{w,u} c_{r,u} \). This proves (e).

(f) Follows easily by dualizing Proposition (4.15).

(g) Observe that, by Proposition (4.13), we have

\[
r_j \xi^w = \sum \xi''(\delta_{r_j} \cdot \tilde{x}_u) \xi^w
\]

\[
= \sum_{r,v < u} -\xi''(x_u) \xi^w
\]

\[
+ \sum_{r,u > u} \left[ \xi''(\tilde{x}_r u) (u^{-1} \chi_i) - \xi''(\tilde{x}_u) + \sum_{v < u} \xi''(\tilde{x}_r v) \chi_i (v) \right] \xi^w.
\]

The rest of the proof is along the lines of the proof of Proposition (4.13).

The following lemma is trivial to verify.
(4.26) **Lemma.** Let $\psi_1, \psi_2 \in \Omega$, then

$$A_i(\psi_1 \cdot \psi_2) = (A_i \psi_1) \psi_2 + (r_i \psi_1)(A_i \psi_2).$$

More generally, by induction on $l(w)$, we get

(4.27) **Lemma.** Let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression. Then for any $\psi_1, \psi_2 \in \Omega$, we have

$$A_w(\psi_1 \cdot \psi_2) = \sum_{0 \leq j \leq n} A_{i_1} \circ \cdots \circ A_{i_{j-1}} \circ \hat{A}_{i_j} \circ \cdots \circ A_{i_n}(\psi_1) \cdot (A_{i_1} \circ \cdots \circ A_{i_n} \psi_2)$$

where the notation $\hat{A}_i$ means that the operator $A_i$ is replaced by the Weyl group action $r_i$.

(4.28) **Corollary.** For any $w \in W$ and $\psi \in \Omega$, we have

$$A_w(\xi \cdot \psi) = \sum_{v \in ^{-1} w \cdot w} \chi_i(v)(A_i \psi) + (w \xi \cdot \psi) \cdot A_w \psi$$

**Proof.** By Proposition (4.24)(b), (g), and Lemma (4.27), we get

$$A_w(\xi \cdot \psi) = (w \xi \cdot \psi) \cdot (A_w \xi) + \sum_{1 \leq j \leq n} (r_{i_1} \cdots A_{i_j} \cdots r_{i_n} \xi) \cdot (A_{i_1} \circ \cdots \circ A_{i_n} \psi)$$

where $w = r_{i_1} \cdots r_{i_n}$ is a reduced expression. Thus

$$A_w(\xi \cdot \psi) = (w \xi \cdot \psi) \cdot (A_w \psi) + \sum_{\text{those } 1 \leq j \leq n \text{ such that } w_j = r_{i_1} \cdots r_{i_{j-1}} r_{i_j} \cdots r_{i_n} \text{ is reduced}} [A_{i_j}(r_{i_{j+1}} \cdots r_{i_n} \xi)] \cdot (A_{w_j} \psi).$$

(1.55)

By Proposition (4.24)(c), it can be easily seen that

$$A_{i_j}(r_{i_{j+1}} \cdots r_{i_n} \xi) = \chi_i(v) \quad \text{where } w_j^{-1} \xrightarrow{v} w^{-1}. \quad (1.56)$$

Substituting (1.56) in (1.55) and using [3, Proposition 2.8], we get the corollary.

As an immediate consequence of Proposition (4.24)(a) and (b), we get
(4.29) Lemma. Let $\psi \in \Omega$. Express $\psi = \sum_w q^w \xi^w$, for some (unique) $q^w \in Q$ (infinitely many of $q^w$'s could be non-zero). Then $q^w = (A_w \cdot \psi) e$.

The following proposition follows easily from Corollary (4.28) and Lemma (4.29).

(4.30) Proposition. For any $v \in W$ and simple reflection $r_i$, we have

$$\xi^v \cdot \xi^w = \sum_{v \rightarrow w} \chi_i(v^r) \xi^w + \chi^r_i(v) \xi^v.$$ 

More generally, we have

(4.31) Proposition. (generalized cup product formula). For any $v, w \in W$, we have (by Proposition (4.24))(I) $\xi^w \cdot \xi^v = \sum_{r_i, w \in U} p^i_{r_i, w} \xi^w$, where $p^i_{r_i, w}$ is a homogeneous polynomial of degree $l(v) + l(w) - l(u)$. Write $u^{-1} = r_i \cdots r_j$ as a reduced expression, then

$$p^i_{r_i, w} = \sum_{\text{Those } j_1 < \cdots < j_m \text{ such that } r_i \cdots r_{j_m} = w^{-1}} A_{j_1} \circ \cdots \circ A_{j_m} \circ \hat{A}_{i_{j_1}} \circ \cdots \circ \hat{A}_{i_{j_m}} \circ \cdots \circ A_{i_e}(\xi^r)(e)$$

where $m = l(w)$.

Proof. Follows easily by combining Lemmas (4.27) and (4.29) together with Proposition (4.24)(a) and (b). 

Recall the definition of the matrix $D = (d_{r_i, w})_{r_i, w \in W}$ from (4.21).

(4.32) Proposition. (a) Fix $w \in W$. Define two matrices $P_w$ and $D_w \in \mathfrak{B}_W$ by $D_w(u, v) = \delta_{u, v} d_{r_i, v}$ and $P_w(u, v) = p^i_{r_i, w}$, for $u, v \in W$. Of course, by definition, $D_w$ is a diagonal matrix. Then explicitly,

$$P_w = D \cdot D_w \cdot D^{-1}.$$ 

(b) For $u, v \in W$ write

$$u \cdot \xi^v = \sum_{l(v) - l(u) \leq l(w) \leq l(v)} g^u_{v, w} \xi^w \quad (I_{57})$$

where $g^u_{v, w}$ is a homogeneous polynomial of degree $l(v) - l(w)$. (This is possible by Proposition (4.24)(g).) For a fixed $w \in W$, define $G_w$ and $S_w \in \mathfrak{B}_W$ by $G_w(u, v) = g^w_{v, w}$ and $S_w(u, v) = \delta_{w, u, v}$. Then

$$D \cdot S_w \cdot D^{-1} = G_w.$$
Proof. Since $p^v_{w,u} = 0$ unless $u \leq v$, we get $P_w \in \mathcal{B}_W$:

\[
(P_w D)(u,v) = \sum_{w'} P_w(u, w') D(w', v)
\]

\[
= \sum_{w'} p^v_{w,u} \cdot d_{w',v}
\]

\[
= d_{u,v} \cdot d_{u,v} \quad \text{by Proposition (4.24)}
\]

\[
= (D \cdot D_u)(u,v) \quad \text{proving (a)}.
\]

By (I.57) we have $d_{u,w-1} = \sum_w g^w_{u,w} \cdot d_{w-1,v}$, i.e., $(DS_u)(u,v) = (G_u D)(u,v)$. 

To conclude the section, we make the following

(4.33) Definition. Recall the definition of the $S$-algebra $A$ from (4.19). For any subset $X \subset \{1, \ldots, l\}$ and let $W_X \subset W$ be the subgroup of $W$ as defined in (1.1). We define $A^X$ to be the $S$-subalgebra of $A$, consisting of $W_X$-invariant elements in $A$. (Of course the $W$-module structure, in particular a $W_X$-module structure, on $A$ is the one given by (1.50).)

The following lemma describes the structure of $A^X$.

(4.34) Lemma. $A^X = \sum_{w \in W^1_X} S \xi^w$ ($W^1_X$ is defined in (1.1). In particular, $A^X$ is a free module over $S$.

Proof. Recall that $W^1_X$ is characterized as the set of all those $w \in W$ such that $w$ is (the unique) element of minimal length in its coset $W_Xw$. In particular, for any $i \in X$, $r_i w > w$ for $w \in W^1_X$. Hence, by Proposition (4.24)(g), $\xi^w \in A^X$, for any $w \in W^1_X$.

Conversely, take $\xi \in A^X$ and write (Proposition (4.20)) $\xi = \sum_w p^w \cdot \xi^w$, where all but finitely many $p^w$'s are zero. Fix $i \in X$. Since $r_i \xi = \xi$, we have $A_i \xi = 0$. This in particular (by Proposition (4.24)(g)), gives that $p^w = 0$, unless $r_i w > w$. So any $w$, with $p^w \neq 0$, belongs to $W^1_X$. 

(4.35) Remarks. (a) In the next section, we will see that $\mathbb{C}_0 \otimes_S A^X$ is isomorphic with the cohomology algebra $H^*(G/P_X, \mathbb{C})$, where $G$ is the group associated to the Kac–Moody Lie algebra $g$ and $P_X$ is a parabolic (corresponding to the subset $X$) in $G$. Further, when $X = \emptyset$, the isomorphism is $W$-equivariant.

(b) Many concepts and results in this chapter can be extended (with suitable and easy modifications) to an arbitrary finitely generated Coxeter group $W$ with a specified set of Coxeter generators and also equipped with a representation satisfying root-system condition, as given in [7, Sect. 2].
5. MARRIAGE OF THE SECOND AND THE FOURTH CHAPTER

Throughout this chapter, \( g = g(A_r) \) would denote an arbitrary symmetrizable Kac-Moody Lie algebra with Cartan sub-algebra \( h \) and Weyl group \( W \).

(5.1) Recall the definition of \( d, \partial \) harmonic forms \( \{ s^w_0 \}_{w \in W} \subset C(g, h) \) from Section 3 (case \( X = \emptyset \)). \( s^w_0 \) is the unique \( d, \partial \) harmonic form such that \( \int_{[B_{r,v}] B_{r,w}} s^w_0 = \delta_{r,v,w} \), for all \( v \in W \) with \( l(v) = l(w) \). Also recall the definition of the map \( \bar{\eta} : \text{Ker} S \to C\{ W \} \), from Theorem (2.7).

Consider the function \( \bar{D} : W \times W \to \mathbb{C} \), defined by \( \bar{D}(v, w) = \bar{\eta}(s^v_0)(w) \). We want to relate the function \( \bar{D} \) with the function \( D \) introduced in (4.21).

(5.2) PROPOSITION. For any \( v, w \in W \), \( \bar{D} \) satisfies the following:

(a) \( \bar{D}(v, w) = 0 \), if \( l(w) \leq l(v) \) and \( v \neq w \).

(b) \( \bar{D}(w, v) = \prod_{\rho \in \Phi} v(h(\rho)) \), where \( h(\rho) \) is the unique element of \( h \) satisfying \( \chi(h(\rho)) = \sigma(\chi, \rho) \), for all \( \chi \in h^* \).

(c) For any simple reflection \( r_i \),

\[ \bar{D}(r_i, w) = (\chi_i - w^I \chi_i) \cdot h(\rho) \] (\( \chi_i \) is defined in (1.4))

(d) Cup-product formula

\[ \bar{D}^r \bar{D}^s = \sum_{r \to r'} \chi_i \bar{D}^r + \bar{D}(r_i, v) \bar{D}^v \] as elements of \( C\{ W \} \)

where \( \bar{D}^r \in C\{ W \} \) is defined by \( \bar{D}^r(w) = \bar{D}(v, w) \), for any \( v, w \in W \) and \( \bar{D}^r \cdot \bar{D}^s : W \to \mathbb{C} \) denotes the function, obtained by the pointwise multiplication of \( \bar{D}^r \) and \( \bar{D}^s \).

Proof. (a) and (b) follows easily from the definition of \( \bar{D} \), by observing that \( \sum_{i \geq 1} R^i(h^\nu) \in \text{Im} L \) (see (3.5)).

(c) We show that

\[ s^w_0 = \frac{1}{2\pi \sqrt{-1}} d_g(\bar{\chi}_i) \] (1.58)

where \( \bar{\chi}_i \in g^* \) is the element satisfying \( \bar{\chi}_{i|h} = \chi_i \) and \( \chi_{i|g} = 0 \), for any root \( \chi \).

\( d_g \) denotes the differential (of degree \( +1 \)) of the co-chain complex \( C(g, \mathbb{C}) \).

We clearly have \( i(h)(d_g(\bar{\chi}_i)) \) and \( \text{ad}(h)(d_g(\bar{\chi}_i)) \) equal to 0, for all \( h \in h \).
Hence \( d_g(\tilde{\chi}_i) \in C^2(g, h) \). Further \( \partial d_g(\tilde{\chi}_i) \in C^1(g, h) \) and \( C^1(g, h) \) can be easily
seen to be 0. So \( d_g(\tilde{\chi}_i) \) is \( d, \partial \) harmonic. Now

\[
d_g(\tilde{\chi}_i)(e_j f_j) = -\chi_i(h_j) = -\delta_{i,j} \quad \text{for all} \quad 1 \leq i, j \leq l. \tag{I_{59}}
\]

Moreover, by Theorem (3.1) and the expression of \( s^{\alpha} \) as given in (I',h'),
we have

\[
s^{\alpha}_0(e_j, f_j) = \frac{\sigma(x_\alpha, x_\alpha) \sqrt{-1}}{4\pi} e(h_\alpha \otimes a_\alpha)(e_j, f_j), \quad \text{i.e.,}
\]

\[
s^{\alpha}_0(e_j, f_j) = \frac{\sqrt{-1}}{2\pi} \delta_{i,i} \quad \text{since} \quad \|e_j\|^2 = \|f_j\|^2 = \frac{2}{\sigma(x_j, x_j)}. \tag{I_{60}}
\]

Combining (I_{59}) and (I_{60}), (I_{58}) follows. Further, it is easy to see that

\[
d_g(\tilde{\chi}_i) = \sum_{\phi \in \tilde{A}_i} -\sigma(\tilde{\chi}_i, \Phi) e(h_\phi \otimes a_\phi) \tag{I_{61}}
\]

(\( \tilde{A}_i, h_\phi \), and \( a_\phi \) are defined in (3.5).

Let \( \Phi_w = A_+ \cap wA_+ = \{ \beta_1, \ldots, \beta_p \} \quad (p = l(w)) \). Recall the definition of \( \eta \)
from Lemma (2.4).

\[
\eta\left( \frac{1}{2\pi} \sqrt{-1} d_g(\tilde{\chi}_i)(e(h_{\beta_1} \wedge \cdots \wedge h_{\beta_p})) \right)
\]

\[
= \frac{(-1)}{2\pi} \sqrt{-1} \eta\left( \sum_{\phi \in \tilde{A}_i} \sigma(\tilde{\chi}_i, \Phi) e(h_\phi \otimes a_\phi)(e(h_{\beta_1} \wedge \cdots \wedge h_{\beta_p})) \right)
\]

(by (I_{61}).

\[
= \sum_{\phi \in \tilde{A}_i} \sigma(\tilde{\chi}_i, \Phi) e(h_\phi \otimes a_\phi) \cdot (\tau(a_\phi) e(h_{\beta_1} \wedge \cdots \wedge h_{\beta_p}))
\]

(by the definition of \( \eta \))

\[
= \sum_{j=1}^{p} \sigma(\tilde{\chi}_i, \beta_j) e(h_{\beta_1} \wedge \cdots \wedge h_{\beta_p})
\]

\[
= \sigma(\chi, \rho - w\rho) e(h_{\beta_1} \wedge \cdots \wedge h_{\beta_p}) \quad \text{since} \quad \sum \beta_j = \rho - w\rho.
\]

Hence \( \tilde{\eta}(s^{\alpha}_0)(w) = (\tilde{\chi}_i - w^{-1} \tilde{\chi}_i) h(\rho) \), by the definition of \( \tilde{\eta} \) as in (2.7). This proves (c).

(d) Since \( \tilde{D}^r \tilde{D}^r \in \tilde{J}(r+1) \) (by Lemma (2.11) and Corollary (2.14)),
we have

\[
\tilde{D}^r \tilde{D}^r = \sum_{l(w) - l(r) + 1} z^w \tilde{D}^w + \sum_{l(w) < l(r)} z^w \tilde{D}^w \tag{I_{62}}
\]
for some constants $z^w$. Let $p_0$ be the minimum integer, such that $z^{w_0} \neq 0$, for some $w_0$ of length $p_0$. Then $p_0 \geq l(v)$. For otherwise, by part (a) of this proposition, we would get, by evaluating $(L_{\delta_2})$ at $w_0$, $z^{w_0}D^{w_0}(w_0) = 0$, which is a contradiction by part (b). Exactly the same argument shows that $z^w = 0$ for all $w \in W$ with $l(w) = l(v)$ and $w \neq v$. To determine $z^w$, evaluate $(L_{\delta_2})$ at $v$ to get $D^v(v) = z^w$. Finally, by Theorem (2.12), $Gr \mathbb{C} \{ W \} \cong H^*(g, h)$ as an algebra. In view of Proposition (3.10) and [23, Theorem 4.5] along with [24, Theorem 1.6], the (d) part follows.

(5.3) **Lemma.** Let $v \neq w \in W$. Then $(v\chi_i - w\chi_i)h(p) \neq 0$, for some $1 \leq i \leq l$.

**Proof.** Define $h_0^* = \{ \chi \in h^* : \chi(h_i) = 0, \text{ for all } 1 \leq i \leq l \}$. Clearly $h_0^* \cap \sum_{i=1}^l \mathbb{C}\chi_i = \{0\}$. Hence, by dimension count, $h_0^* + \sum_{i=1}^l \mathbb{C}\chi_i = h^*$. Assume that the lemma is false, i.e., $\chi_i(v^{-1}h(p) - w^{-1}h(p)) = 0$, for all $i$. Further $\chi_i(v^{-1}h(p) - w^{-1}h(p)) = 0$, for any $\chi \in h_0^*$ (since $v^{-1}h(p) - w^{-1}h(p)$ lies in the span of $\{ h_i \}_{1 \leq i \leq l}$). Hence $v^{-1}h(p) = w^{-1}h(p)$, which gives $v^{-1}p = w^{-1}p$. This is possible only if $v = w$, by [9, Corollary 2.6].

(5.4) **Definition.** Recall the definition of the matrix $D$ from (4.21), which can also be viewed as a function: $W \times W \to S$. Fix any $h \in h$. There is an evaluation map $ev_h : S \to \mathbb{C}$, defined by $ev_h(p) = p(h)$. We define the function $D_{h(p)} : W \times W \to \mathbb{C}$ by $D_h(v, w) = ev_h(D(v, w))$.

We prove the following characterization of $D_{h(p)}$.

(5.5) **Proposition.** Let $E : W \times W \to \mathbb{C}$ be any function satisfying

1. $E(v, w) = 0$ if $l(w) \leq l(v)$ and $w \neq v$.
2. $E(w, w) = D_{h(p)}(w, w)$ for all $w \in W$
3. $E(r_i, w) = D_{h(p)}(r_i, w)$ for all simple reflections $r_i$ and all $w \in W$, and
4. the cup-product formula holds for $E$, i.e.,

$$E^*E^v = \sum_{r \to r^{-1}w} \chi_j(v) E^w + E(r_i, v) E^v$$

for all $r_i$ and $v \in W$.

(The notation $E^*$ is similar to the one in Proposition (5.2)(d).) Then $E = D_{h(p)}$.

**Proof.** We prove the proposition by induction on $l(v, w) = l(w) - l(v)$. For $(v, w)$ with $l(v, w) \leq 0$, by (1) and (2) and Proposition (4.24)(a), we
have $E(v, w) = D_{h(p)}(v, w)$. So assume that $E(v, w) = D_{h(p)}(v, w)$, for $v, w$ with $l(v, w) \leq n$ and let $(v_0, w_0)$ be such that $l(v_0, w_0) = n + 1$. We have, by (4), $[E^r(v_0) - E^r(v_0)] E'^q(w_0) = \sum_{r \to w} \alpha_r(v^r) E^q(w_0)$. By induction, $E^q(w_0) = D_{h(p)}(w, w_0)$ (since $l(w, w_0) = n$) and, by (3), $E^r(v_0) - E^r(v_0) = D_{h(p)}(r_i, w_0) - D_{h(p)}(r_i, v_0)$. By Lemma (5.3) and Proposition (4.24)(c), there is a $1 \leq i \leq l$ such that $D_{h(p)}(r_i, w_0) - D_{h(p)}(r_i, v_0) \neq 0$. So, by Proposition (4.30), $E(v_0, w_0) = D_{h(p)}(v_0, w_0)$.

Recall the definition of $\tilde{D}: W \times W \to \mathbb{C}$ from (5.1). The following result provides a bridge between Sections 2 and 4. Combining Propositions (4.24), (5.2), and (5.5), we get

(5.6) **COROLLARY.** $\tilde{D} = D_{h(p)}$ as functions: $W \times W \to \mathbb{C}$.

(5.7) **Remark.** There is nothing very special about $h(p)$ in Proposition (5.5). Any $h \in h$, such that Lemma (5.3) holds for $h(p)$ replaced by $h$ will do.

We recall the following.

(5.8) **DEFINITION.** Let $K$ be the standard real form of a Kac-Moody group $G$ and let $T$ denote the “maximal torus” of $K$ (see (1.2)). There is an action of the Weyl group $W \simeq N_K(T)/T$ ($N_K(T)$ denotes the normalizer of $T$ in $K$) on $K/T$ defined as

$$n \cdot (k \mod T) = (kn^{-1}) \mod T, \quad \text{for } n \in N_K(T) \text{ and } k \in K.$$ 

In particular, $W$ acts on the cohomology $H^*(K/T, \mathbb{Z})$ as well as on the homology $H_*(K/T, \mathbb{Z})$. Also recall, from (3.9), that $\{e^v\}_r \in W$ denotes the $\mathbb{Z}$-basis of $H^*(K/T, \mathbb{Z})$ dual to the closures of the Schubert cells. The following lemma, in the finite case, is due to [3, Theorem 3.14(iii)]. An easy proof of the lemma (in the general case) can be given by using Proposition (3.10).

(5.9) **LEMMA.**

$$r_i e^v = e^v \quad \text{if } r_i v > v,$$

$$= e^v - \sum_{r \to w} \alpha_r(v^r) e^w \quad \text{if } r_i v < v.$$

(5.10) **Remarks.** (a) This lemma does not require the symmetrizability assumption on $G$.

(b) The formula for $\sigma_z \cdot P_w$ given in [3, Theorem 3.14(iii)], in the
case when $l(w_\sigma_w) = l(w) - 1$, is incorrect. The correct formula (in their notation) is

$$\sigma_w \cdot P_w = P_w + \sum_{w_\sigma_w \rightarrow w'} w_\alpha(H_{w'}) P_{w'}.$$  

(5.11) Recall the definition of $A$ from (4.19). Let $C_0 = S/S^+$ be the 1-dimensional (over $C$) $S$-module ($S = S(h^*)$), where $S^+$ is the augmentation ideal (evaluation at $0 \in h$) in $S$. By Proposition (4.20), $C_0 \otimes_S A$ is an algebra and the action of $R$ on $A$ descends to give an action of $R$ on $C_0 \otimes_S A$. Also, from Proposition (4.20)(b), the elements

$$\sigma_w = 1 \otimes \zeta_w \in C_0 \otimes_S A$$

provide a $C$-basis. Moreover by Proposition (4.24)(f), the algebra $A$ is filtered by $\{A_p\}_{p \in Z_+}$, where $A_p = \sum_{(t) \leq p} S^{t \xi^*}$. Again using Proposition (4.24)(f), it is easy to see that this filtered algebra structure gives rise to an (obvious) graded (commutative) algebra structure on $C_0 \otimes_S A$.

Further, besides having a ring structure and being a module for $W$ (described in (5.8)), $H^*(K/T)$ also admits a ring of operators $\mathcal{A}$ (with $C$-basis $\{A_w\}_{w \in W}$), where $A_w$ ($1 \leq i \leq l$) corresponds topologically to the integration on fiber for the fibration $G/B \rightarrow G/P_i$ ($P_i$ is the minimal parabolic containing $r_i$). The ring of operators $\mathcal{A}$ on $H^*(K/T)$ was introduced by Bernstein, Gel'fand, and Gel'fand [3] in the finite case. The definition in the general case is carried out by Kac and Peterson.

We come to one of the main theorems of this chapter.

(5.12) Theorem. Let $K$ be the standard real form of the group $G$ associated to a symmetrizable Kac–Moody Lie algebra $g$ and let $T$ denote the “maximal torus” of $K$ (see (1.2)). Then the map

$$\theta: H^*(K/T, C) \rightarrow C_0 \otimes_S A$$

defined by $\theta(\varepsilon^w) = \sigma_w$, for any $w \in W$ is a graded algebra isomorphism. Moreover, the action of $w \in W$ and $A_w$ on $H^*(K/T)$ corresponds (under $\theta$) respectively to that of $\delta_w, x_w \in R$ on $C_0 \otimes_S A$ (see (1.49)).

Proof. We give a “geometrical” proof of this theorem. It also admits a more “algebraic” proof (see Remark (5.17)(a)).

Consider the $C$-linear map $f: C_0 \otimes_S A \rightarrow \text{Gr} C\{W\}$, defined by $f(\sigma_w) = \widehat{D}^w \mod f \big|_{(u) \leq 1} \in \text{Gr}^{A_w} C\{W\}$, where $\widehat{D}^w$, $f_p$, and $\text{Gr} C\{W\}$ are defined respectively in Proposition (5.2), (2.8), and Lemma (2.11). By Corollary (2.14), $f$ is a $C$-vector space isomorphism. Further, we recall from Proposition (4.24)(f) that $\xi^w\xi^r = \sum_{u,v \in w} p_{u,v} w^u, \xi^w$, where $p_{u,v}$ is a
homogeneous polynomial of degree $l(u) + l(v) - l(w)$. In particular, $\sigma^v \sigma^r = \sum_{u, v \leq w \text{ and } l(u) + l(v) = l(w)} p_{u,v}^w \sigma^w$. Now the fact, that $f$ is an algebra homomorphism, follows from Corollary (5.6). By [24, Theorem 1.6], there is a natural graded algebra isomorphism $[\gamma]: H^*(g, h) \to H^*(K/T, \mathbb{C})$, given by an “integration” map. We claim that $f \circ \theta \circ [\gamma] = \mathrm{Gr}(\eta) \circ \psi_{d, S}^{-1}$ as maps: $H^*(g, h) \to \mathrm{Gr}(W)$ (see Theorem (2.12)). The claim is easy to establish if we keep track of the definitions of various maps ($f, \theta, \mathrm{Gr}(\eta)$, and $\psi_{d, S}$) involved and observe further (see the proof of [24, Theorem 1.6]) that $[\gamma](s^w) = \varepsilon^w$, for all $w \in W$, where $s^w$ is as defined in (5.1). Now since $\mathrm{Gr}(\eta) \circ \psi_{d, S}^{-1}$ is a graded algebra isomorphism, by Theorem (2.12), we get that $\theta$ is a graded algebra isomorphism as well.

The assertion, that $\theta$ commutes with Weyl group action, follows by combining Proposition (4.24)(g) with Lemma (5.9). Finally the claim, that the action of $A_\nu$ on $H^*(K/T)$ corresponds (under $\theta$) to that of $x_\nu \in R$, follows from Propositions (4.3)(a), (4.24)(b), and the analog of [3, Theorem 3.14(i)] proved by Kac and Peterson.

(5.13) COROLLARIES. (a) We can use either of Propositions (4.31) or (4.32)(a) to determine the cup product $\varepsilon^w \varepsilon^v$ (for arbitrary $u$ and $v \in W$) in terms of the Schubert basis $\{e^w\}$ of $H^*(K/T, \mathbb{C})$. Similarly, we can also write an expression for $u \cdot e^v$ (in terms of the Schubert basis) as given in (4.32)(b). Observe that the Proposition (4.32)(a)–(b) gives cup product as well as the Weyl group action purely in terms of the matrix $D$.

(b) A result of Chevalley (that, in the finite case, $H^*(K/T, \mathbb{C})$ is $W$-isomorphic with the left regular representation of $W$) can be trivially deduced from the above theorem.

(c) Let $X \subset \{1, \ldots, l\}$ be arbitrary. There is a standard parabolic subgroup $P_X$ (as defined in (1.2) of the group $G$ and let $K_X = K \cap P_X$. Also recall the definition of $A_X$, from (4.33).

There is a (unique) graded algebra isomorphism $\theta^X: H^*(K/K_X, \mathbb{C}) \to \mathbb{C}_0 \otimes_S A^X$, making the following diagram commutative:

$$
\begin{array}{ccc}
H^*(K/K_X, \mathbb{C}) & \xrightarrow{\mu^X} & \mathbb{C}_0 \otimes_S A^X \\
\downarrow p_X^\ast & & \downarrow \mathrm{id} \otimes i^X \\
H^*(K/T, \mathbb{C}) & \xrightarrow{\theta} & \mathbb{C}_0 \otimes_S A
\end{array}
$$

where $p_X^\ast$ is induced from the projection $p_X: K/T \to K/K_X$ and $i^X: A^X \subset A$ is the canonical inclusion.

Proof. It is well known (and easy to prove) that the map $p_X^\ast$ is injective and the image of $p_X^\ast$ is precisely the $\mathbb{C}$-span of $\{e^w\}_{w \in W_X^1}$. Further, by
Lemma (4.34), the map $\text{Id} \otimes i^Y$ is injective with image precisely equal to the span of $\{\alpha^w\}_{w \in W^+_L}$. This proves the corollary. 

The following is actually a corollary of the (c) part above. It (the following corollary) has earlier been obtained by Kac and Peterson.

(d) With the notations as in (c), $p^*_X(H^*(K/K_X)) = H^*(K/T)^{W_X}$, where $W_X$ is as defined in (1.1) and $H^*(K/T)^{W_X}$ denotes the $W_X$-invariants in $H^*(K/T)$. In particular $H^*(K/T)^W = H^0(K/T)$.

Proof. Since the map $\theta$ (of Theorem (5.12)) commutes with $W$-actions, it suffices to show that, for $\xi \in \Lambda$, if $r_i(1 \otimes \xi) = 1 \otimes \xi$, for all $i \in X$ then there exists a $\xi' \in \Lambda^X$ such that $1 \otimes \xi = 1 \otimes \xi'$. Write $\xi = \sum p^w \xi^w$, for some $p^w \in S$. Fix $i \in X$. By Proposition (4.24)(g),

$$r_i \xi = \sum_{w < r_i w} p^w \xi^w + \sum_{r_i w < w} p^w \left[ \xi^w - \sum_{r_i w \rightarrow r_i' w} \alpha_i(v^w) \xi^{r_i'} - (w^{-1} \alpha_i) \xi^{r_i w} \right].$$  \hspace{1cm} (I.64)

Since $r_i(1 \otimes \xi) = 1 \otimes r_i \xi = 1 \otimes \xi$ (by assumption), we get by (I.64),

$$1 \otimes \sum_{r_i w < w} p^w \sum_{r_i w \rightarrow r_i' w} \alpha_i(v^w) \xi^{r_i'} = 0. \hspace{1cm} (I.65)$$

We can rewrite (I.65) as

$$1 \otimes \sum_{r_i w < w} 2p^w \xi^w + 1 \otimes \sum_{r_i w < w} p^w \sum_{r_i w \rightarrow r_i' w} \alpha_i(v^w) \xi^{r_i'} = 0. \hspace{1cm} (I.66)$$

For a $w$ with $r_i w < w$ and any $v \in W$ such that $r_i w \rightarrow v$ and $v \neq w$, we have $r_i v > v$. In particular, from (I.66), we get that $p^w \in S^+$, if there exists $i \in X$ such that $r_i w < w$, i.e., $1 \otimes \xi = 1 \otimes \sum_{w \in W^+_L} p^w \xi^w$. 

We further generalize Corollary (5.13)(c) to Schubert varieties in $G/P_X$. We need some preliminaries.

(5.14) Let $\mathcal{H} = \mathcal{H}_X$ be a subset of $W$ with the following properties:

(P₁) $\mathcal{H}$ is left $W_X$-stable.

(P₂) Whenever $w \in \mathcal{H}$ and $w' \leq w$ then $w' \in \mathcal{H}$.

To any such $\mathcal{H}$, we can associate a (left) $B$-stable variety $V_{\mathcal{H}} \subset G/P_X$, defined by

$$V_{\mathcal{H}} = \bigcup_{w \in \mathcal{H}} Bw^{-1}P_X/P_X. \hspace{1cm} (I.67)$$
By property (P₂) of (1), \( V \) is closed in \( G/P \) (see [29]). Conversely, any (left) \( B \)-stable closed subset of \( G/P \) is \( V \), for some appropriate choice of \( \Phi = \Phi_X \).

Let \( \Omega \) denote the \( Q \)-algebra of all the maps: \( \Phi \to Q \). There is a restriction map \( r_\Phi : \Omega \to \Omega \). Define \( A^X_\Phi = r_\Phi (A^X) \). From Proposition (4.24)(a) and Lemma (4.34), we easily get

\[ (5.15) \text{Lemma.} \quad A^X_\Phi \text{ is a free } S \text{-module with a basis} \]

\[ \{ \xi^w_\Phi \mid w \in W^+_X \cap \Phi \}. \]

We have the following generalization of Corollary (5.13)(c).

\[ (5.16) \text{Theorem.} \quad \text{Let } G \text{ be the group associated to a Kac–Moody Lie algebra } g = g(A) \text{ and let } X \text{ be any subset (including } X = \emptyset \text{) of } \{ 1, \ldots, l \}. \text{ Fix} \]

\[ \Phi = \Phi_X \subset W \text{ satisfying } (P_1) \text{ and } (P_2) \text{ as above and let } V_\Phi \subset G/P_X \text{ be the subvariety, defined by } (I_\Phi). \text{ Then there is a unique graded algebra isomorphism } \theta^X_\Phi : H^*(V_\Phi, \mathbb{C}) \to C_0 \otimes_S A^X_\Phi, \text{ making the following diagram commutative:} \]

\[ \begin{array}{ccc}
H^*(K/K_X, \mathbb{C}) & \xrightarrow{\alpha^X} & C_0 \otimes_S A^X \\
\downarrow i^*_\Phi & & \downarrow \rho \\
H^*(V_\Phi, \mathbb{C}) & \xrightarrow{\theta^X_\Phi} & C_0 \otimes_S A^X_\Phi
\end{array} \]

where \( i^*_\Phi \) is induced by the inclusion \( i_\Phi : V_\Phi \to G/P_X \) and \( \rho = \tilde{r}_\Phi \) is induced by the restriction map \( r_\Phi \). (The grading on \( C_0 \otimes_S A^X_\Phi \) comes from the grading of \( C_0 \otimes_S A^X \) via \( \tilde{r} \).)

**Proof.** Of course the map \( \tilde{r} \) is, by definition, surjective. It is easy to see that the kernel of \( i^*_\Phi \) is precisely the \( \mathbb{C} \)-span of \( \{ e^w \mid w \in W^+_X \Phi \} \). Moreover, for any \( w \in W^+_X \Phi \setminus \Phi \), \( \xi^w_\Phi = 0 \) by Proposition (4.24)(a). Hence the \( \mathbb{C} \)-span of \( \{ \sigma^w \mid w \in W^+_X \Phi \} \) is in the kernel of \( \tilde{r} \). But then due to Lemma (5.15), by dimensional considerations, it is precisely the kernel of \( \tilde{r} \). This proves the theorem.

\[ (5.17) \text{Remarks.} \quad (a) \text{ Theorem (5.12) admits a more "algebraic" proof using an unpublished result of Kac and Peterson (which Peterson kindly told to the second author), asserting that } A_x e^w = e^{r_x w} \text{ if } r_x w < w \text{ and } =0 \text{ otherwise, together with the "twisted derivation property." For this to be valid, they do not need the symmetrizability assumption on } g. \text{ In particular, Theorem (5.12), the Corollaries (5.13)(a), (c), (d), and Theorem (5.16) are true in the general (not necessarily symmetrizable) situation.} \]
In the finite case, Theorem (2.12) was proved by the first author (unpublished) and also independently by Carrell and Lieberman [5]. Recently (and quite independently) Akyildiz, Carrell, and Lieberman [1] have proved an analog of Theorem (5.16) in the special case when \( G \) is finite dimensional and \( X = \emptyset \). Their methods are very different from ours and it is not clear if their proofs can be extended to the infinite case.

6. Connection between the D-Matrix and Harmonic Polynomials

(6.1) Let \( G \) be the group associated to an arbitrary Kac–Moody Lie algebra \( g = g(A) \) and let \( K \) be the standard real form of \( G \) with "maximal torus" \( T \) (see (1.2)). Denote by \( \beta : S(h^1) \rightarrow H^*(K/T, \mathbb{C}) \) the characteristic homomorphism, given by the classifying map: \( K/T \rightarrow B(T) \) corresponding to the principal \( T \)-bundle \( K \rightarrow K/T \), where \( h^1 = h \cap [g, g] \) is the \( \mathbb{C} \)-span of \( \{h_i\}_{1 \leq i \leq l} \).

Recall the definition of the algebra \( A \) from (4.19). Let \( \tilde{\beta} : S = S(h^*) \rightarrow A \) be the \( \mathbb{C} \)-algebra homomorphism defined by

\[
\tilde{\beta}(p)(w) = (-1)^{\deg w} w^{-1} p \quad \text{for} \quad p \in S \text{ and } w \in W. \tag{1_{68}}
\]

The fact that \( \tilde{\beta}(p) \in A \), follows from Proposition (4.24)(c). It can be easily seen that the composite map \( S \rightarrow \tilde{\beta} A \rightarrow \text{ev}_0 \mathbb{C}_0 \otimes_S A \) (where "evaluation at 0" \( \text{ev}_0 : A \rightarrow \mathbb{C}_0 \otimes_S A \) is given by \( \xi \mapsto 1 \otimes \xi \)) factors through \( S(h^1) \). (See the proof of Lemma (5.3).) We denote the map, thus obtained, by \( \tilde{\beta} : S(h^1) \rightarrow \mathbb{C}_0 \otimes_S A \). We have

(6.2) Lemma. With the notations as above,

(a) The following diagram is commutative:

\[
\begin{array}{ccc}
S(h^1) & \xrightarrow{\tilde{\beta}} & H^*(K/T) \\
\downarrow{\beta} & & \downarrow{\theta} \\
\mathbb{C}_0 \otimes_S A & & \\
\end{array}
\]

where \( \theta \) is defined in the Theorem (5.12).

(b) For any \( p \in S(h^1) \), \( \tilde{\beta}(E_x, p) = x_r \cdot \tilde{\beta}(p) \), where \( E_x \) is the classical Bernstein–Gel'fand–Gel'fand operator defined in the proof of Proposition (4.15) and the element \( x_r \) is defined by (1_{24}).

Proof. (a) Since all the maps in the triangle are algebra homomorphisms, if suffices to show that \( \theta \tilde{\beta}(\chi) = \tilde{\beta}(\chi) \), for all \( \chi \in h^1 \). Of
course, the span of \( \{ \chi_{i,h^1} \}_{1 \leq i \leq l} \) equals \( h^1 \), where \( \chi_i \) is as in (1.4). Now 
\[ \beta(\chi_{i,h^1}) = \varepsilon_i \] (as is known) and \( \theta(\varepsilon_i) = \sigma_i \) (by the definition of \( \theta \)) and hence
(a) follows by Proposition (4.24)(c).

(b) The map \( \beta \) clearly commutes with \( W \)-actions. So by
Lemma (4.26) and the analogous property for \( E_{x_i} \) acting on \( S(h^{1*}) \) (which
can be proved similarly), we again only need to prove that 
\[ \beta(E_{x_i}(\chi)) = x_i \cdot \beta(\chi), \] for \( \chi \in h^{1*} \). Further, for \( 1 \leq i, j \leq l \), \( E_{x_i}(\chi_{j,h^1}) = \delta_{i,j} \) and \( x_i \cdot \beta(\chi_{j,h^1}) = x_i \cdot \sigma_j = \delta_{i,j} \) by Proposition (4.24)(b).

In the next theorem we show, in the finite case, how the \( D \)-matrix can be
obtained from the harmonic polynomials. Recall from (3.9) that \( \{ \varepsilon_w \}_{w \in W} \)
is a \( \mathbb{Z} \)-basis of \( H^*(K/T, \mathbb{Z}) \).

(6.3) Theorem. Let \( g \) be finite dimensional. For any \( w \in W \), choose 
\( f^w \in S(h^{1*}) \) such that \( \beta(f^w) = \varepsilon^w \) and consider the
matriks 
\[ F = (f_{r,w})_{r,w \in W}, \]
where 
\[ f_{r,w} = (-1)^{|r| - |w|} \cdot (f^w). \]
Then the matrix \( F \) (over \( S = S(h^{1*}) \)) can be
decomposed as 
\[ F = E \cdot D \]
where \( D \) is as defined in (4.21) and \( E = (e_{r,w})_{r,w \in W} \) is a lower triangular
matrix over \( S \) with diagonal entries 1 and, in fact, \( e_{r,w} \) is a homogeneous
polynomial of degree \( l(v) - l(w) \).

(6.4) Remarks. (a) Choice of \( \{ f^w \} \), as in Theorem (6.3), is always
possible in the finite case since \( \beta \) is surjective (in this case). In fact we can
choose for \( \{ f^w \} \), \( G \)-harmonic polynomials on \( g \) [18].

(b) Of course such a decomposition, as in the above theorem, is
unique. We call \( D \) (resp. \( E \)) the upper (resp. lower) triangular part of \( F \). In
particular, the upper triangular part of \( F \) does not depend upon the choice
of \( \{ f^w \} \).

(c) A less precise (but illuminating) way to describe the theorem is
that \( "G\)-harmonic polynomials on \( g \) determine the \( D \)-matrix."

Proof of Theorem (6.3). Recall the definition of the map \( \bar{\beta} : S(h^{1*}) \rightarrow A \)
from (6.1). By Proposition (4.20)(b), we can write
\[ \bar{\beta}(f^r) - \sum e_{r,w} \xi^w, \quad (I_{69}) \]
for some unique \( e_{r,w} \in S \). Since, for any \( w \in W \), \( \bar{\beta}(f^r)(w') \) is a homogeneous
polynomial of degree \( l(v) \) and \( \xi^w(w') \) is a homogeneous polynomial of
degree \( l(w) \) by Proposition (4.24)(d), by the uniqueness of decomposition
in \( (I_{69}) \), we get that \( e_{r,w} \) is a homogeneous polynomial of degree \( l(v) - l(w) \).
Now, from Lemma (6.2), it is easy to see that for any \( w \) with \( l(w) = l(v) \), we have \( e_{r,w} = \delta_{r,w} \), i.e., (I_{69}) reduces to

\[
\beta(f^v) = \sum_{l(w) < l(v)} e_{r,w} \xi^w + \xi^v. \tag{I_{70}}
\]

Evaluating (I_{70}) at \( w' \), by (I_{68}), we get \((-1)^{l(v)}(w'^{-1}f^v) = \sum_{l(w) < l(v)} e_{r,w} \xi^w(w') + \xi^v(w') \).

(6.6) Remark. Since, in the finite case, the characteristic homomorphism \( \beta: S(h^*) \rightarrow H^*(K/T) \) is surjective, by complete reducibility of \( W \)-modules, we can choose a \( W \)-equivariant splitting \( s \) of \( \beta \) (e.g., \( G \)-harmonic polynomials on \( g \) provide one such splitting). By composing \( s \) with the \( W \)-equivariant map \( \beta: S \rightarrow A \) (defined in (6.1), we get a \( W \)-equivariant map \( \beta \circ s: H^*(K/T) \rightarrow A \) which splits the surjective map \( \theta^{-1} \circ ev_0: A \rightarrow H^*(K/T) \), where \( ev_0 \) is defined in (6.1) and \( \theta \) is defined in Theorem (5.12). Now, the \( W \)-equivariant map \( \theta^{-1} \circ ev_0 \) is always surjective (i.e., even in the infinite dim case), and, we just have seen that, it admits a \( W \)-equivariant splitting in the finite case. But, in general, it can be seen that it does not admit any \( W \)-equivariant splitting. The counter example exists, e.g., in any affine case.

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