DESCENT OF LINE BUNDLES TO GIT QUOTIENTS OF FLAG VARIETIES BY MAXIMAL TORUS

SHRAWAN KUMAR

Department of Mathematics
UNC at Chapel Hill, Chapel Hill
NC 27599-3250, USA
shrawan@email.unc.edu

Dedicated to Bertram Kostant on his eightieth birthday

Abstract. Let $G$ be a connected, simply connected semisimple complex algebraic group with a maximal torus $T$ and let $P$ be a parabolic subgroup containing $T$. Let $L_P(\lambda)$ be a homogeneous ample line bundle on the flag variety $Y = G/P$. We give a necessary and sufficient condition for $L_P(\lambda)$ to descend to a line bundle on the GIT quotient $Y(\lambda)/T$. As a consequence of this result, we get the precise list of $P$-regular weights $\lambda$ for which the line bundle $L_P(\lambda)$ descends to the GIT quotient $Y(\lambda)/T$.

1. Introduction

Let $G$ be a connected, simply connected semisimple complex algebraic group with a maximal torus $T$ and let $P$ be a parabolic subgroup containing $T$. We denote their Lie algebras by the corresponding Gothic characters. The following theorem is one of our main results.

Theorem 3.9. Let $L_P(\lambda)$ be a homogeneous ample line bundle on the flag variety $Y = G/P$. Then, the line bundle $L_P(\lambda)$ descends to a line bundle on the GIT quotient $Y(\lambda)/T$ if and only if, for all the semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ (in particular, $\text{rank } \mathfrak{s} = \text{rank } \mathfrak{g}$),

$$\lambda \in \sum_{\alpha \in \Delta^+(\mathfrak{s})} \mathbb{Z}\alpha,$$

where $\Delta^+(\mathfrak{s})$ is the set of positive roots of $\mathfrak{s}$.

Using the above theorem, we explicitly get exactly for which $\lambda$ the line bundle $L_P(\lambda)$ descends to $Y(\lambda)/T$. This is our second main result.

In the following $Q$ (resp., $\Lambda$) is the root (resp., weight) lattice and we follow the indexing convention as in Bourbaki [B].

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Theorem 3.10. With the notation as in the above theorem, the line bundle $L_P(\lambda)$ descends to a line bundle on the GIT quotient $Y(\lambda)/T$ if and only if $\lambda$ is of the following form depending upon the type of $G$:

(a) $G$ of type $A_\ell$ ($\ell \geq 1$): $\lambda \in Q$.
(b) $G$ of type $B_\ell$ ($\ell \geq 3$): $\lambda \in 2Q$.
(c) $G$ of type $C_\ell$ ($\ell \geq 2$): $\lambda \in \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_{\ell-1} + \mathbb{Z} \alpha_{\ell} = 2\Delta$.
(d) $G$ of type $D_4$: $\lambda \in \{n_1 \alpha_1 + 2n_2 \alpha_2 + n_3 \alpha_3 + n_4 \alpha_4 \mid n_i \in \mathbb{Z} \text{ and } n_1 + n_3 + n_4 \text{ is even}\}$.
(d1) $G$ of type $D_\ell$ ($\ell \geq 5$): $\lambda \in \{2n_1 \alpha_1 + 2n_2 \alpha_2 + \cdots + 2n_{\ell-2} \alpha_{\ell-2} + n_{\ell-1} \alpha_{\ell-1} + n_{\ell} \alpha_{\ell} \mid n_i \in \mathbb{Z} \text{ and } n_{\ell-1} + n_{\ell} \text{ is even}\}$.
(e) $G$ of type $G_2$: $\lambda \in \mathbb{Z} 6 \alpha_1 + \mathbb{Z} 2 \alpha_2$.
(f) $G$ of type $F_4$: $\lambda \in \mathbb{Z} 6 \alpha_1 + \mathbb{Z} 6 \alpha_2 + \mathbb{Z} 12 \alpha_3 + \mathbb{Z} 12 \alpha_4$.
(g) $G$ of type $E_6$: $\lambda \in 6 \Delta$.
(h) $G$ of type $E_7$: $\lambda \in 12 \Delta$.
(i) $G$ of type $E_8$: $\lambda \in 60Q$.

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2. Notation

Let $G$ be a connected, simply connected, semisimple algebraic group over the field $\mathbb{C}$ of complex numbers. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ contained in $B$. Let $B^-$ be the opposite Borel subgroup of $G$ (i.e., the Borel subgroup $B^-$ such that $B^- \cap B = T$). We denote the unipotent radicals of $B, B^-$ by $U, U^-$, respectively. We denote the Lie algebra of any algebraic group by the corresponding Gothic character. In particular, the Lie algebras of $G, B, T$ are denoted by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$, respectively. Let $\Delta = \Delta(\mathfrak{g})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and let $\Delta^+ = \Delta^+(\mathfrak{g})$ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$) with $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of simple roots. For $\alpha \in \Delta$, we denote the corresponding root space in $\mathfrak{g}$ by $\mathfrak{g}_\alpha$. Let $W$ be the Weyl group of $G$ with $\{s_1, \ldots, s_\ell\} \subset W$ the set of simple reflections corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$, respectively. Let $Q := \bigoplus_{i=1}^\ell \mathbb{Z} \alpha_i \subset \mathfrak{t}^*$ be the root lattice and let

$$\Lambda := \{\lambda \in \mathfrak{t}^* \mid \lambda(\alpha_i) \in \mathbb{Z} \text{ for all } i\}$$

be the weight lattice, where $\{\alpha_1, \ldots, \alpha_\ell\}$ are the simple coroots corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$, respectively. Let $\Lambda^+ \subset \Lambda$ be the set of dominant weights, i.e.,

$$\Lambda^+ := \{\lambda \in \mathfrak{t}^* \mid \lambda(\alpha_i) \in \mathbb{Z}^+ \text{ for all } i\},$$

where $\mathbb{Z}^+$ is the set of nonnegative integers. For $\lambda \in \Lambda^+$, we denote by $V(\lambda)$ the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$.

Let $X$ be the group of characters of $T$. Then, $X$ can be identified with $\Lambda$ by taking the derivative of characters. For $\lambda \in \Lambda$, we denote the corresponding character of $T$ by $e^\lambda$. Then, $e^\lambda$ uniquely extends to a character of $B$.

We also consider (standard) parabolic subgroups $P \supset B$ of $G$. Define

$$\Pi_P = \{\text{simple roots } \alpha_i \mid -\alpha_i \text{ is a root of } \mathfrak{p}\},$$

\[ \Lambda_P = \{ \lambda \in \Lambda \mid \lambda(\alpha_i^\vee) = 0 \text{ for } \alpha_i \in \Pi_P \}, \quad \Lambda_P^+ = \Lambda_P \cap \Lambda^+, \]

and

\[ \Lambda_P^\circ = \{ \lambda \in \Lambda_P \mid \lambda(\alpha_i^\vee) > 0 \text{ for each } \alpha_i \in \Pi \setminus \Pi_P \}. \]

In particular, for \( P = B, \) \( \Lambda_B = \Lambda \) and \( \Lambda_B^+ = \Lambda^+. \) We recall that \( e^\lambda \) extends to a character of \( P \) if and only if \( \lambda \in \Lambda_P. \) For \( \lambda \in \Lambda_P, \) let \( \mathcal{L}_P(\lambda) = G \times_P C_{-\lambda} \) be the homogeneous line bundle on the flag variety \( G/P \) associated to the principal \( P \)-bundle \( G \to G/P \) via the one-dimensional representation \( C_{-\lambda} \) of \( P \) given by the character \( e^{-\lambda}. \) Then, \( \mathcal{L}_P(\lambda) \) is an ample line bundle if and only if \( \lambda \in \Lambda_P^\circ. \) For \( g \in G \) and \( v \in C_{-\lambda}, \) we denote the equivalence class of \((g, v)\) in \( G \times_P C_{-\lambda} \) by \([g, v]_\lambda.\)

3. Proofs

For any set of positive roots \( \beta = \{\beta_1, \ldots, \beta_n\}, \) let \( \mathbb{Z}\beta := \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n \subset Q \) and let \( \mathfrak{g}(\beta) \) be the semisimple subalgebra of \( \mathfrak{g} \) with roots

\[ \Delta(\mathfrak{g}(\beta)) := \Delta(\mathfrak{g}) \cap \mathbb{Z}\beta. \]

We recall the following well-known result.

**Lemma 3.1.** For any \( \lambda \in \Lambda^+ \) and any set of positive roots \( \beta \) such that the index of \( \mathbb{Z}\beta \) in the root lattice \( Q \) is finite, the 0-weight space of the submodule \( U(\mathfrak{g}(\beta)) \cdot v_\lambda^+ \subset V(\lambda) \) is nonzero if and only if \( \lambda \in \mathbb{Z}\beta, \) where \( v_\lambda^+ \) is a nonzero highest weight vector of \( V(\lambda). \) Observe that since \( \mathbb{Z}\beta \) is of finite index in \( Q, \) the full Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{g}(\beta). \)

Let \( P \) be a standard parabolic subgroup. For the action of \( T \) on \( G/P \) via the left multiplication, the isotropy subgroup \( I_{gP} \subset T \) at any \( gP \in G/P \) is clearly given by

\[ I_{gP} = T \cap gPg^{-1}. \]

**Lemma 3.2.** For \( gP \in G/P \) and any \( \mu \in \Lambda_P, \) the isotropy subgroup \( I_{gP} \) acts trivially on the fiber \( \mathcal{L}_P(\mu)|_{gP} \) if and only if

\[ e^{\mu}|_{P \cap g^{-1}Tg} \equiv 1. \]

**Proof.** For \( t \in I_{gP} = T \cap gPg^{-1} \) and any nonzero vector \( v_\mu \in C_{-\mu}, \)

\[ t[g, v_\mu] = [tg, v_\mu] \]

\[ = [gg^{-1}tg, v_\mu] \]

\[ = [g, (g^{-1}tg)v_\mu], \text{ since } g^{-1}tg \in P, \]

\[ = [g, e^{-\mu}(g^{-1}tg)v_\mu]. \]

Thus,

\[ t[g, v_\mu] = [g, v_\mu] \text{ for } t \in T \cap gPg^{-1} \quad \text{if and only if} \quad e^{-\mu}(g^{-1}tg) = 1. \]

This proves the lemma. \( \square \)
Lemma 3.3. For any \( g = \bar{\omega}u, \) for \( \bar{\omega} \in N(T), u \in U^-_P \) and \( p \in P, \) we have
\[
P \cap g^{-1}Tg = p^{-1}(T \cap u^{-1}Tu)p, \tag{1}
\]
where \( N(T) \) is the normalizer of \( T \) in \( G \) and \( U^-_P \) is the unipotent radical of the opposite parabolic \( P^- \) (where the Lie algebra of \( P^- \) is defined by \( \mathfrak{p}^- := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(p)} \mathfrak{g} - \alpha \), \( \Delta(p) \) being the set of roots of \( p \)).

Moreover, in the above, if we just assume that \( u \in U^- \), then
\[
P \cap g^{-1}Tg \supset p^{-1}(T \cap u^{-1}Tu)p. \tag{2}
\]

Observe that \( \bigcup_{\bar{\omega} \in N(T)} \bar{\omega}U^-_P P = G. \) This follows since \( Z := \bigcup_{\bar{\omega} \in N(T)} \bar{\omega}U^-_P P/P \) is a \( T \)-stable open subset of \( G/P \) and the complement \( (G/P) \setminus Z \) does not contain any \( T \)-fixed points of \( G/P \).

**Proof.** We first prove (1). We have
\[
P \cap g^{-1}Tg = P \cap p^{-1}u^{-1}w^{-1}T\bar{\omega}up,
\]
i.e.,
\[
P \cap g^{-1}Tg = p^{-1}(P \cap u^{-1}Tu)p. \tag{3}
\]
Now, for any \( t \in T \), if \( u^{-1}tu \in P \), then \( u^{-1}tu = t \). To prove this, observe that \( u^{-1}tu^{-1}t \in P \cap U^-_P = (1) \). Thus, \( p^{-1}(P \cap u^{-1}Tu)p \subset p^{-1}Tp \) and hence \( P \cap g^{-1}Tg \subset p^{-1}Tp \cap g^{-1}Tg \). The reverse inclusion is, of course, obvious. This proves equality (1).

Inclusion (2) of course follows from equality (3). \( \square \)

For \( Y = G/P \) and an ample line bundle \( \mathcal{L}_P(\lambda) \) on \( Y \) (i.e., \( \lambda \in \Lambda^\circ_P \)), by \( Y^{ss}(\lambda) \) we mean the set of semistable points in \( Y \) with respect to the action of \( T \) via the left multiplication on \( Y \) and \( T \)-linearized ample line bundle \( \mathcal{L}_P(\lambda) \). Then, as is well known, \( Y^{ss}(\lambda) \subset Y \) is a Zariski open subset. Moreover, by Lemma 3.1, for any \( n \geq 1 \), \( V(n\lambda)^T \neq 0 \) if and only if \( n\lambda \in Q \). Since \( Q \) is of finite index in \( \Lambda \), we get from the next Lemma 3.4 that
\[
Y^{ss}(\lambda) \neq \emptyset.
\]

**Lemma 3.4.** Let \( \lambda \in \Lambda^\circ_P \). Then, \( gP \in Y^{ss}(\lambda) \) if and only if \( gv^+_n\lambda \) has a nonzero component in the zero weight space for some \( n \geq 1 \), where \( v^+_n\lambda \) is a highest weight vector of \( V(n\lambda) \).

**Proof.** By definition, the point \( gP \in Y \) is semistable if and only if there exists a section \( \sigma \in H^0(Y, \mathcal{L}_P(n\lambda))^T \) (for some \( n \geq 1 \)), such that \( \sigma(gP) \neq 0 \). Consider the \( G \)-equivariant isomorphism
\[
\chi : V(n\lambda)^* \sim H^0(Y, \mathcal{L}_P(n\lambda)),
\]
where \( \chi(f)(gP) = [g, (g^{-1}f)|_{v^+_n\lambda}] \). Thus,
\[
\chi(f)(gP) \neq 0 \iff f(gv^+_n\lambda) \neq 0,
\]
and hence \( gP \) is semistable if and only if \( gv^+_n\lambda \) has a nonzero component in the zero weight space. \( \square \)
Lemma 3.5. For $gP \in Y^{ss}(\lambda)$,
\[ e^{n\lambda}|_{P \cap g^{-1}Tg} \equiv 1 \quad \text{for some} \quad n \geq 1. \]  
(4)

In particular,
\[ e^\lambda|_{(P \cap g^{-1}Tg)^0} \equiv 1, \]  
where $(P \cap g^{-1}Tg)^0$ is the identity component of $P \cap g^{-1}Tg$.

**Proof.** By Lemma 3.4, $gP$ is semistable if and only if for some $n \geq 1$, $[gv^+_{n\lambda}]_0 \neq 0$, where $[gv^+_{n\lambda}]_0$ denotes its component in the zero weight space. For $t \in T$,
\[ [tgv^+_{n\lambda}]_0 = [gv^+_{n\lambda}]_0. \]  
(6)

But, for $t \in T \cap gPg^{-1}$,
\[ [tgv^+_{n\lambda}]_0 = [gg^{-1}tgv^+_{n\lambda}]_0 = e^{n\lambda}(g^{-1}tg)[gv^+_{n\lambda}]_0. \]  
(7)

Combining (6) and (7), we get $e^{n\lambda}(g^{-1}tg) = 1$. This proves (4).

The identity (5) follows immediately from (4) since a connected torus is a divisible group. □

**Lemma 3.6.** For any $x \in u^-$, let $\beta_x$ be the subset of $\Delta^+$ such that $x = \sum_{\alpha \in \beta_x} x_{-\alpha}$, where $x_{-\alpha} \in g_{-\alpha}$ is nonzero. Then, for $u = \text{Exp } x$,
\[ T \cap u^{-1}Tu = \bigcap_{\alpha \in \beta_x} \{ t \in T \mid e^\alpha(t) = 1 \}. \]  
(8)

In particular, for a weight $\mu \in \Lambda$, we have $e^\mu|_{T \cap u^{-1}Tu} \equiv 1$ if and only if $\mu \in \mathbb{Z}\beta_x$.

**Proof.** Take $t \in T$ such that $utu^{-1} \in T$. Then,
\[ utu^{-1}t^{-1} \in T \cap U^- = \{1\}. \]

Thus, $utu^{-1} = t$, i.e., $ut^{-1} = u$. But,
\[ tut^{-1} = \text{Exp}(\text{Ad } t \cdot x) = \text{Exp}\left( \sum_{\alpha \in \beta_x} e^{-\alpha}(t)x_{-\alpha} \right) = \text{Exp}(x). \]

Since $\text{Exp}|_{u^-}$ is a bijection,
\[ \sum_{\alpha \in \beta_x} e^{-\alpha}(t)x_{-\alpha} = \sum_{\alpha \in \beta_x} x_{-\alpha}. \]

Thus, $e^{-\alpha}(t) = 1$ for any $\alpha \in \beta_x$. This proves the inclusion
\[ T \cap u^{-1}Tu \subset \bigcap_{\alpha \in \beta_x} \{ t \in T \mid e^\alpha(t) = 1 \}. \]
The reverse inclusion follows by reversing the above calculation.

To prove the ‘In particular’ statement, consider the isomorphism $\xi : T \simeq \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{C}^*)$, $\xi(t)\mu = e^\mu(t)$ for $t \in T$ and $\mu \in \Lambda$. From this identification and (8) it is easy to see that $\xi(T \cap u^{-1}Tu) = \text{Hom}_\mathbb{Z}(\Lambda/\mathbb{Z}\beta_x, \mathbb{C}^*)$. Now, if $\mu \in \Lambda \setminus (\mathbb{Z}\beta_x)$, there exists a homomorphism $f : \Lambda/\mathbb{Z}\beta_x \to \mathbb{C}^*$ such that $f(\mu) \neq 1$. From this we conclude the ‘In particular’ statement. \(\square\)

Since $Y^{ss}(\lambda)$ is nonempty and Zariski open in $Y$, we can find a semistable point of the form $\text{Exp}(x)P$, where $x \in u^-$ and $\beta_x = \Delta^+$. More generally, take any subset $\beta \subset \Delta^+$ such that $\mathbb{Z}\beta$ is of finite index in $Q$. Then, for some $n \geq 1$, $n\lambda \in \mathbb{Z}\beta$ and hence, by Lemma 3.1,
\[(U(g(\beta)) \cdot v^+_n)^T \neq (0).\]
In particular,
\[(G(\beta)/P(\beta)) \cap Y^{ss}(\lambda) \neq \emptyset,\]
where $G(\beta)$ is the connected (semisimple) subgroup of $G$ with Lie algebra $g(\beta)$ and $P(\beta) := P \cap G(\beta)$ is a parabolic subgroup of $G(\beta)$. Again, by Zariski density, we can find an element of the form $\text{Exp}(x)P \in Y^{ss}(\lambda)$ such that $x \in u^-$ and $\beta_x$ is the set of all the positive roots of $G(\beta)$ (i.e., all the roots of $B \cap G(\beta)$).

**Lemma 3.7.** For any subset $S \subset \Delta^+$, the quotient group
\[T_S/T_S^0 \simeq \text{Tor}(\Lambda/\mathbb{Z}S),\]
where $T_S := \bigcap_{\alpha \in S} \{ t \in T \mid e^\alpha(t) = 1 \}$, $T_S^0$ denotes its identity component, and $\text{Tor}$ denotes the torsion subgroup.

**Proof.** Recall the identification $\xi : T \simeq \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{C}^*)$ from the proof of Lemma 3.6. Under this identification
\[\xi(T_S) = \text{Hom}_\mathbb{Z}(\Lambda/\mathbb{Z}S, \mathbb{C}^*).\]
Decompose
\[\Lambda/\mathbb{Z}S \simeq \text{Tor}(\Lambda/\mathbb{Z}S) \oplus F,\]
where $F$ is a free $\mathbb{Z}$-module. Thus,
\[\text{Hom}_\mathbb{Z}(\Lambda/\mathbb{Z}S, \mathbb{C}^*) \simeq \text{Hom}_\mathbb{Z}(\text{Tor}(\Lambda/\mathbb{Z}S), \mathbb{C}^*) \times \text{Hom}_\mathbb{Z}(F, \mathbb{C}^*).\]
But $\text{Hom}_\mathbb{Z}(F, \mathbb{C}^*) \simeq (\mathbb{C}^*)^m$, where $m := \text{rk} F$. Thus, $T_S^0 \simeq \text{Hom}_\mathbb{Z}(F, \mathbb{C}^*)$ and hence
\[T_S/T_S^0 \simeq \text{Tor}(\Lambda/\mathbb{Z}S)^\vee,\]
where $\text{Tor}(\Lambda/\mathbb{Z}S)^\vee := \text{Hom}_\mathbb{Z}(\text{Tor}(\Lambda/\mathbb{Z}S), \mathbb{C}^*). \square$

Let $H$ be a reductive group, $X$ a projective variety, and $L$ an ample $H$-equivariant line bundle on $X$. Then, recall that the GIT quotient $X//H$ is by definition the uniform categorical quotient of the (open) set of semistable points $X^{ss}$ by $H$ (see [MFK, Theorem 1.10]).

We recall the following ‘descent’ lemma of Kempf (see [DN, Theorem 2.3]).
Lemma 3.8. Let $X, H,$ and $\mathcal{L}$ be as above. Then, an $H$-equivariant vector bundle $S$ on $X$ descends to a vector bundle on $X//H$ (i.e., there exists a vector bundle $S'$ on $X//H$ such that its pull-back to $X^{ss}$ under the canonical $H$-equivariant structure is $H$-equivariantly isomorphic to the restriction of $S$ to $X^{ss}$) if and only if for any $x \in X^{ss}$, the isotropy subgroup $I_x$ acts trivially on the fiber $S_x$. In fact (though we do not need this), for the ‘if’ part, it suffices to assume that $I_x$ acts trivially for only those $x \in X^{ss}$ such that the orbit $H \cdot x$ is closed in $X^{ss}$.

Let $P$ be a standard parabolic subgroup and let $L_P(\lambda)$ be a homogeneous ample line bundle (i.e., $\lambda \in \Lambda_P^+$) on $Y = G/P$. Denote the GIT quotient of $Y$ by $T$ with respect to $L_P(\lambda)$ by $Y(\lambda)/T$. The following is one of our main results.

Theorem 3.9. With the notation as above, the line bundle $L_P(\lambda)$ descends to a line bundle on $Y(\lambda)/T$ if and only if for all the semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ (in particular, rank $\mathfrak{s} = \text{rank } \mathfrak{g}$),

$$\lambda \in \mathbb{Z}\Delta^+(\mathfrak{s}),$$

where $\Delta^+(\mathfrak{s}) := \Delta^+ \cap \Delta(\mathfrak{s})$ is the set of positive roots of $\mathfrak{s}$.

Proof. By Lemma 3.8, $L_P(\lambda)$ descends to $Y(\lambda)/T$ if and only if for all the semistable points $gP = \bar{w}uP \in G/P$ (for $\bar{w} \in N(T)$ and $u \in U_P$), the isotropy subgroup $I_{gP}$ acts trivially on the fiber $L_P(\lambda)_{gP}$. By Lemmas 3.2 and 3.3, this is equivalent to the requirement that $e^\lambda\vert_{T \cap u^{-1}Tu} \equiv 1$. Further, by Lemma 3.6, this is equivalent to the requirement that $\lambda \in \mathbb{Z}\beta_x$, where $x \in u^-$ is the element with $\text{Exp} x = u$.

By the discussion above Lemma 3.7, for any semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, there exists an element $x \in u^-$ such that $\text{Exp}(x)P \in G/P$ is semistable and, moreover, $\beta_x = \Delta^+(\mathfrak{s})$. Thus, we get by Lemmas 3.2, 3.3, and 3.6 that $\lambda \in \mathbb{Z}\Delta^+(\mathfrak{s})$ for any such $\mathfrak{s}$ if the line bundle $L_P(\lambda)$ descends to $Y(\lambda)/T$.

Conversely, take any semistable point $\bar{w}\text{Exp}(x)P \in G/P$ for $\bar{w} \in N(T)$ and $x \in u_P$. If $\mathbb{Q}\beta_x \subset Q$ is not of finite index in $Q$, choose simple roots $\alpha_x = \{\alpha_{i_1}, \ldots, \alpha_{i_j}\}$ such that $\mathbb{Q}\beta_x \cap \mathbb{Q}\alpha_x = (0)$ and $\mathbb{Q}\beta_x + \mathbb{Q}\alpha_x = \bigoplus_{i=1}^j \mathbb{Q}\alpha_i$, where $\mathbb{Q}\beta_x := \sum_{\beta \in \mathbb{Q}\beta_x} \mathbb{Q}\beta \subset \mathfrak{t}^*$ and $\mathbb{Q}\alpha_x := \bigoplus_{i=1}^j \mathbb{Q}\alpha_i \subset \mathfrak{t}^*$. With this choice of $\alpha_x$, the torsion submodule

$$\text{Tor}(\Lambda//\mathbb{Z}\beta_x) \hookrightarrow \text{Tor}(\Lambda//(\mathbb{Z}\beta_x + \mathbb{Z}\alpha_x)). \quad (9)$$

To see this observe that we have the following short exact sequence:

$$0 \rightarrow \mathbb{Z}\alpha_x \rightarrow \Lambda//\mathbb{Z}\beta_x \rightarrow \Lambda//(\mathbb{Z}\beta_x + \mathbb{Z}\alpha_x) \rightarrow 0.$$ 

From this the assertion (9) follows easily.

Let $\mathfrak{s}$ be a semisimple subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ such that

$$\mathbb{Z}\Delta^+(\mathfrak{s}) = \mathbb{Z}\beta_x + \mathbb{Z}\alpha_x.$$ 

Choose an element $y = y_\mathfrak{s} \in u^-$ such that $\beta_y = \Delta^+(\mathfrak{s})$ and $\text{Exp}(y)P \in Y^{ss}(\lambda)$. With this choice of $y$, by Lemma 3.6, we see that $T \cap v^{-1}Tv \subset T \cap u^{-1}Tu$, where
$u := \text{Exp } x$ and $v := \text{Exp } y$. We further show that the inclusion $T \cap v^{-1}Tv \subset T \cap u^{-1}Tu$ induces a surjective map

$$(T \cap v^{-1}Tv)/(T \cap v^{-1}Tv)^\circ \to (T \cap u^{-1}Tu)/(T \cap u^{-1}Tu)^\circ. \quad (10)$$

By Lemmas 3.6 and 3.7,

$$(T \cap v^{-1}Tv)/(T \cap v^{-1}Tv)^\circ \simeq \text{Tor}(\Lambda/\mathbb{Z}_b)^\vee$$

and

$$(T \cap u^{-1}Tu)/(T \cap u^{-1}Tu)^\circ \simeq \text{Tor}(\Lambda/\mathbb{Z}_b)^\vee.$$ Combining the above identifications with (9) and the injectivity of $\mathbb{C}^*$, we get (10).

Since $\widetilde{wuP}$ is a semistable point, by Lemmas 3.3 and 3.5, $e^\lambda_{|T \cap u^{-1}Tu} \equiv 1$. Moreover, by the assumption and Lemma 3.6, $e^\lambda_{|T \cap u^{-1}Tv} \equiv 1$. Thus, $e^\lambda_{|T \cap u^{-1}Tu} \equiv 1$ by (10). This proves the theorem. \qed

Using the above theorem, we explicitly get exactly for which $\lambda$ the bundle $L_P(\lambda)$ descends to the GIT quotient $Y(\lambda)\!//T$.

In the following, we follow the indexing convention as in Bourbaki [B, Planche I–IX].

**Theorem 3.10.** Let $G$ be a connected, simply connected simple algebraic group, $P \subset G$ a standard parabolic subgroup and let $L_P(\lambda)$ be a homogeneous ample line bundle on the flag variety $Y = G/P$ (i.e., $\lambda \in \Lambda_P^\circ$). Then, the line bundle $L_P(\lambda)$ descends to a line bundle on the GIT quotient $Y(\lambda)\!//T$ if and only if $\lambda$ is of the following form depending upon the type of $G$ (in addition to $\lambda \in \Lambda_P^\circ$):

(a) $G$ of type $A_\ell$ ($\ell \geq 1$): $\lambda \in Q$.
(b) $G$ of type $B_\ell$ ($\ell \geq 3$): $\lambda \in 2Q$.
(c) $G$ of type $C_\ell$ ($\ell \geq 2$): $\lambda \in \mathbb{Z}^2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell = 2\Lambda$.
(d1) $G$ of type $D_4$: $\lambda \in \{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 \mid n_i \in \mathbb{Z}$ and $n_1 + n_3 + n_4$ is even\}.
(d2) $G$ of type $D_\ell$ ($\ell \geq 5$): $\lambda \in \{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell \mid n_i \in \mathbb{Z}$ and $n_{\ell-1} + n_\ell$ is even\}.
(e) $G$ of type $G_2$: $\lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$.
(f) $G$ of type $F_4$: $\lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$.
(g) $G$ of type $E_6$: $\lambda \in 6\Lambda$.
(h) $G$ of type $E_7$: $\lambda \in 12\Lambda$.
(i) $G$ of type $E_8$: $\lambda \in 60Q$.

**Proof.** The theorem follows by using Theorem 3.9 and the classification of the semisimple subalgebras of $g$ of maximal rank due to Borel–Siebenthal (see [W, Theorem 8.10.8]). In the following chart, we only list all the proper maximal semisimple subalgebras $s$ of $g$ containing the (fixed) Cartan subalgebra $t$ up to a conjugation under the Weyl group. In the following, $\theta$ denotes the highest root and $\xi_i$ denotes the semisimple subalgebra of $g$ containing $t$ with simple roots $\{\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_\ell, -\theta\}$:

(a) $A_\ell$ ($\ell \geq 1$): none.
(b) $B_\ell (\ell \geq 3)$: $s_i, 2 \leq i \leq \ell$.
(c) $C_\ell (\ell \geq 2)$: $s_i, 1 \leq i \leq \ell - 1$.
(d) $D_\ell (\ell \geq 4)$: $s_i, 2 \leq i \leq \ell - 2$.
(e) $G_2$: $s_i, i = 1, 2$.
(f) $F_4$: $s_i, i = 1, 2, 4$.
(g) $E_6$: $s_i, i = 2, 3, 4, 5$.
(h) $E_7$: $s_i, i = 1, 2, 3, 5, 6$.
(i) $E_8$: $s_i, i = 1, 2, 5, 7, 8$.

We denote by $L(\g; \alpha_1, \ldots, \alpha_\ell)$ the intersection of $\Z \triangle^+(\s)$ (inside the root lattice $Q$) as $\s$ varies over all possible semisimple subalgebras $\s$ of $\g$ containing the (fixed) Cartan subalgebra $\mathfrak{t}$.

In the following, we determine $L(\g; \alpha_1, \ldots, \alpha_\ell)$ for each simple $\g$.

(a) $A_\ell (\ell \geq 1)$: In this case $L(A_\ell; \alpha_1, \ldots, \alpha_\ell) = Q$.
(b) $C_\ell (\ell \geq 2)$: By the above chart,

$$L(C_\ell; \alpha_1, \ldots, \alpha_\ell) = [(\Z \theta + L(C_{\ell-1}; \alpha_2, \ldots, \alpha_\ell))$$

$$\cap (L(C_2; \alpha_1, -\theta) + L(C_{\ell-2}; \alpha_3, \ldots, \alpha_\ell)) \cap \cdots$$

$$\cap (L(C_{\ell-2}; \alpha_{\ell-3}, \ldots, \alpha_1, -\theta) + L(C_2; \alpha_{\ell-1}, \alpha_\ell))$$

$$\cap (L(C_{\ell-1}; \alpha_{\ell-2}, \ldots, \alpha_1, -\theta) + \Z \alpha_\ell)]_W,$$

where $[M]_W$ denotes $\bigcap_{w \in W} w M$. By induction, for $j < \ell$,

$$L(C_j; \alpha_{\ell-j+1}, \ldots, \alpha_\ell) = \Z 2\alpha_{\ell-j+1} + \cdots + \Z 2\alpha_{\ell-1} + \Z \alpha_\ell$$

and

$$L(C_j; \alpha_{j-1}, \ldots, \alpha_1, -\theta) = \Z 2\alpha_{j-1} + \cdots + \Z 2\alpha_1 + \Z \theta$$

$$= \Z 2\alpha_{j-1} + \cdots + \Z 2\alpha_1 + \Z (2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell),$$

since

$$\theta(C_\ell; \alpha_1, \ldots, \alpha_\ell) = 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

Thus,

$$L(C_\ell; \alpha_1, \ldots, \alpha_\ell) = [\Z 2\alpha_1 + \cdots + \Z 2\alpha_{\ell-1} + \Z \alpha_\ell]_W.$$

But, $\Z 2\alpha_1 + \cdots + \Z 2\alpha_{\ell-1} + \Z \alpha_\ell$ is $W$-stable and hence

$$L(C_\ell; \alpha_1, \ldots, \alpha_\ell) = \Z 2\alpha_1 + \cdots + \Z 2\alpha_{\ell-1} + \Z \alpha_\ell.$$

This proves part (c) of the theorem.

(d) $D_\ell (\ell \geq 4)$: We first consider the case of $D_4$. In this case,

$$L(D_4; \alpha_1, \ldots, \alpha_4) = [L(A_1; \alpha_1) + L(A_1; -\theta) + L(A_1; \alpha_3) + L(A_1; \alpha_4)]_W$$

$$= [\Z \alpha_1 + \Z \theta + \Z \alpha_3 + \Z \alpha_4]_W$$

$$= [\Z \alpha_1 + \Z 2\alpha_2 + \Z \alpha_3 + \Z \alpha_4]_W.$$
Now, the sublattice $L' := \{n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 : n_i \in \mathbb{Z}, n_2 \text{ is even and } \sum_{i=1}^{4} n_i \text{ is even} \} \subset \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ is $W$-stable (as is easy to see). Moreover, clearly the index of $L'$ in $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ is 2 and $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ is not $W$-stable (as can be seen by applying the second simple reflection $s_2$). Thus $[\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4]_W = L'$, proving the theorem in this case.

We now come to the case of general $D_\ell$. By the above chart and the $A_\ell$ case,

$$L(D_\ell; \alpha_1, \ldots, \alpha_\ell) = [(L(D_2; \alpha_1, -\theta) + L(D_\ell-2; \alpha_3, \ldots, \alpha_\ell)) \cap \cdots \cap (L(D_\ell-2; \alpha_{\ell-3}, \ldots, \alpha_1, -\theta) + L(D_2; \alpha_{\ell-1}, \alpha_\ell))]_W,$$

where $D_k$ for $k = 2, 3$ is interpreted as $A_k$. Set

$$L'(\alpha_{\ell-3}, \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell) = \left\{ \sum_{i=\ell-3}^{\ell} n_i \alpha_i : n_i \in \mathbb{Z}, \sum n_i \text{ is even and } n_{\ell-2} \text{ is even} \right\},$$

and if $\ell - k > 4$, set

$$L'(\alpha_{k+1}, \ldots, \alpha_\ell) = \left\{ \sum_{i=k+1}^{\ell} n_i \alpha_i : n_i \in \mathbb{Z}, \sum n_i \text{ is even and } n_{k+1}, \ldots, n_{\ell-2} \text{ are even} \right\},$$

and if $\ell - k < 4$, set

$$L'(\alpha_{k+1}, \ldots, \alpha_\ell) = \mathbb{Z}\alpha_{k+1} + \cdots + \mathbb{Z}\alpha_\ell.$$

By induction and the $A_i$ ($i = 2, 3$) case, for $2 \leq \ell - k < \ell$,

$$L(D_{\ell-k}; \alpha_{k+1}, \ldots, \alpha_\ell) = L'(\alpha_{k+1}, \ldots, \alpha_\ell).$$

Thus,

$$L(D_\ell; \alpha_1, \ldots, \alpha_\ell) = [(\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + L'(\alpha_3, \ldots, \alpha_\ell)) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta + L'(\alpha_4, \ldots, \alpha_\ell)) \cap (L'(\alpha_3, \alpha_2, \alpha_1, -\theta) + L'(\alpha_5, \ldots, \alpha_\ell)) \cap \cdots \cap (L'(\alpha_{\ell-5}, \ldots, \alpha_1, -\theta) + L'(\alpha_{\ell-3}, \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell)) \cap (L'(\alpha_{\ell-4}, \ldots, \alpha_1, -\theta) + \mathbb{Z}\alpha_{\ell-2} + \mathbb{Z}\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell) \cap (L'(\alpha_{\ell-3}, \ldots, \alpha_1, -\theta) + \mathbb{Z}\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell)]_W$$

$$= [L'(\alpha_1, \ldots, \alpha_\ell)]_W.$$

It is easy to see that $L'(\alpha_1, \ldots, \alpha_\ell)$ is $W$-invariant and hence

$$[L'(\alpha_1, \ldots, \alpha_\ell)]_W = L'(\alpha_1, \ldots, \alpha_\ell).$$

This proves the theorem in the case of $D_\ell$. 
(b) $B_\ell (\ell \geq 3)$: By the above chart, we get that

$$L(B_\ell; \alpha_1, \ldots, \alpha_\ell) = [(\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + L(B_{\ell-2}; \alpha_3, \ldots, \alpha_\ell))$$

$$\cap (L(A_3; \alpha_1, \alpha_2, -\theta) + L(B_{\ell-4}; \alpha_4, \ldots, \alpha_\ell))$$

$$\cap (L(D_4; \alpha_3, \alpha_2, \alpha_1, -\theta) + L(B_{\ell-6}; \alpha_5, \ldots, \alpha_\ell)) \cap \cdots$$

$$\cap (L(D_{\ell-3}; \alpha_{\ell-4}, \ldots, \alpha_1, -\theta) + L(B_3; \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell))$$

$$\cap (L(D_{\ell-2}; \alpha_{\ell-3}, \ldots, \alpha_1, -\theta) + L(C_2; \alpha_\ell, \alpha_{\ell-1}))$$

$$\cap (L(D_{\ell-1}; \alpha_{\ell-2}, \ldots, \alpha_1, -\theta) + \mathbb{Z}\alpha_\ell)$$

$$\cap L(D_{\ell}; \alpha_{\ell-1}, \ldots, \alpha_1, -\theta)]_W.$$

By using the result for $D_p$ and $A_3$ and also, by induction, for $B_j (j < \ell)$, we get that (with $L'$ as in the proof of the $D_\ell$ case)

$$L(B_\ell; \alpha_1, \ldots, \alpha_\ell) = [(\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + \mathbb{Z}2\alpha_3 + \cdots + \mathbb{Z}2\alpha_\ell)$$

$$\cap (Z\alpha_1 + Z\alpha_2 + \mathbb{Z}\theta + \mathbb{Z}2\alpha_4 + \cdots + \mathbb{Z}2\alpha_\ell)$$

$$\cap (L'(\alpha_3, \alpha_2, \alpha_1, -\theta) + \mathbb{Z}2\alpha_5 + \cdots + \mathbb{Z}2\alpha_\ell) \cap \cdots$$

$$\cap (L'(\alpha_{\ell-4}, \ldots, \alpha_1, -\theta) + \mathbb{Z}2\alpha_{\ell-2} + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}2\alpha_\ell)$$

$$\cap (L'(\alpha_{\ell-3}, \ldots, \alpha_1 - \theta) + \mathbb{Z}2\alpha_\ell + \mathbb{Z}\alpha_{\ell-1})$$

$$\cap (L'(\alpha_{\ell-2}, \ldots, \alpha_1, -\theta) + \mathbb{Z}\alpha_\ell) \cap L'(\alpha_{\ell-1}, \ldots, \alpha_1, -\theta)]_W$$

$$= [\mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_\ell]_W$$

$$= 2[Q]_W$$

$$= 2Q, \text{ since } Q \text{ is } W\text{-stable.}$$

This proves the theorem for the case of $B_\ell$.

(e) $G_2$: By the chart and the theorem for the case of $A_2$,

$$L(G_2; \alpha_1, \alpha_2) = [(L(A_2; \alpha_2, -\theta)) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\theta)]_W$$

$$= [(\mathbb{Z}\alpha_2 + \mathbb{Z}\theta) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\theta)]_W$$

$$= [(\mathbb{Z}\alpha_2 + \mathbb{Z}3\alpha_1) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2)]_W$$

$$= [\mathbb{Z}3\alpha_1 + \mathbb{Z}2\alpha_2]_W$$

$$= \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2,$$

since $\mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$ is $W$-stable, whereas $\mathbb{Z}3\alpha_1 + \mathbb{Z}2\alpha_2$ is not $W$-stable and $\mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$ is of index 2 inside $\mathbb{Z}3\alpha_1 + \mathbb{Z}2\alpha_2$.

(f) $F_4$: By the chart,

$$L(F_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = [(\mathbb{Z}\theta + L(C_3; \alpha_4, \alpha_3, \alpha_2))$$

$$\cap (L(A_2; -\theta, \alpha_1) + L(A_2; \alpha_3, \alpha_4))$$

$$\cap L(B_4; -\theta, \alpha_1, \alpha_2, \alpha_3)]_W.$$
By the theorem for $A_2$, $C_3$, and $B_4$, we get

$$L(F_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = [(Z\theta + Z2\alpha_4 + Z2\alpha_3 + Z\alpha_2)$$
$$\cap (Z\theta + Z\alpha_1 + Z\alpha_3 + Z\alpha_4)$$
$$\cap (Z2\theta + Z2\alpha_1 + Z2\alpha + Z2\alpha_3)]_W$$
$$= [(Z2\alpha_1 + Z2\alpha_4 + Z2\alpha_3 + Z\alpha_2)$$
$$\cap (Z3\alpha_2 + Z\alpha_1 + Z\alpha_3 + Z\alpha_4)$$
$$\cap (Z4\alpha_4 + Z2\alpha_1 + Z2\alpha_2 + Z2\alpha_3)]_W$$
$$= [M]_W, \text{ where } M := Z2\alpha_1 + Z6\alpha_2 + Z2\alpha_3 + Z4\alpha_4.$$

Now,

$$Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z12\alpha_4 \subset [M]_W,$$

since $Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z12\alpha_4$ is $W$-stable (as can be easily seen). Conversely, take $\mu \in [M]_W$. Then, $w\mu \in M$ for any $w \in W$. Since

$$s_iw\mu = w\mu - (w\mu)(\alpha_i^\vee)\alpha_i;$$

for any $\mu \in [M]_W$ and any $w \in W$,

$$w\mu(\alpha_2^\vee) \in 6Z \text{ and } w\mu(\alpha_4^\vee) \in 4Z.$$

Since $\alpha_1^\vee \in W.\alpha_2^\vee$ (see [H, §10.4, Lemma C]), we get

$$\mu(\alpha_1^\vee) \in 6Z, \mu(\alpha_2^\vee) \in 6Z, \text{ and } \mu(\alpha_4^\vee) \in 4Z.$$

Take $\mu = n_12\alpha_1 + n_26\alpha_2 + n_32\alpha_3 + n_44\alpha_4 \in [M]_W$. Then, from the above, we get

$$4n_1 - 6n_2 \in 6Z, \ -2n_1 + 12n_2 - 2n_3 \in 6Z, \ \text{ and } \ -2n_3 + 8n_4 \in 4Z.$$

This gives that $n_1 \in 3Z$ and $n_3 \in 6Z$. Thus, $\mu \in Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z4\alpha_4$. Considering $s_3(Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z4\alpha_4)$, we see that

$$[Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z4\alpha_4]_W \subset Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z12\alpha_4.$$

Thus, we conclude that

$$[M]_W = Z6\alpha_1 + Z6\alpha_2 + Z12\alpha_3 + Z12\alpha_4.$$

This completes the proof of the theorem in the case of $F_4$. 
(g) $E_6$: By the chart and the case of $A_\ell$:

$$L(E_6; \alpha_1, \ldots, \alpha_6) = [(Z\theta + L(A_5; \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6)) \\
\cap (Z\alpha_1 + L(A_5; -\theta, \alpha_2, \alpha_4, \alpha_5, \alpha_6)) \\
\cap (L(A_2; \alpha_1, \alpha_3) + L(A_2; -\theta, \alpha_2) + L(A_2; \alpha_5, \alpha_6)) \\
\cap (L(A_5; \alpha_1, \alpha_3, \alpha_4, -\theta) + Z\alpha_6)]_W \\
= [(Z\theta + Z\alpha_1 + Z\alpha_3 + Z\alpha_4 + Z\alpha_5 + Z\alpha_6) \\
\cap (Z\alpha_1 + Z\theta + Z\alpha_2 + Z\alpha_4 + Z\alpha_5 + Z\alpha_6) \\
\cap (Z\alpha_1 + Z\alpha_3 + Z\theta + Z\alpha_2 + Z\alpha_5 + Z\alpha_6) \\
\cap (Z\alpha_1 + Z\alpha_3 + Z\alpha_4 + Z\alpha_2 + Z\theta + Z\alpha_6)]_W \\
= [(Z\alpha_1 + Z2\alpha_2 + Z\alpha_3 + Z\alpha_4 + Z\alpha_5 + Z\alpha_6) \\
\cap (Z\alpha_1 + Z\alpha_2 + Z2\alpha_3 + Z\alpha_4 + Z\alpha_5 + Z\alpha_6) \\
\cap (Z\alpha_1 + Z\alpha_2 + Z3\alpha_4 + Z\alpha_5 + Z\alpha_6) \\
\cap (Z\alpha_1 + Z\alpha_2 + Z\alpha_3 + Z\alpha_4 + Z2\alpha_5 + Z\alpha_6)]_W \\
= [M_6]_W,$$

where $M_6 := Z\alpha_1 + Z2\alpha_2 + Z2\alpha_3 + Z3\alpha_4 + Z2\alpha_5 + Z\alpha_6$.

Clearly, $6\Lambda \subset M_6$ and since $\Lambda$ is $W$-stable,

$$6\Lambda \subset [M_6]_W. \tag{11}$$

Conversely, take $\mu \in [M_6]_W$. Then, for any $w \in W$, $w\mu \in M_6$. Since $s_i(w\mu) = w\mu - (w\mu)(\alpha_i^\vee)\alpha_i$; for any $\mu \in [M_6]_W$ and any $w \in W$, $w\mu(\alpha_i^\vee) \in 2\mathbb{Z}$, and $w\mu(\alpha_i^\vee) \in 3\mathbb{Z}$.

Since $E_6$ is simplylaced, the Weyl group $W$ acts transitively on the coroots [H, §10.4, Lemma C]. Thus, $\mu(\alpha_i^\vee) \in 2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ for all the simple coroots $\alpha_i^\vee$. This proves that $\mu \in 6\Lambda$, i.e.,

$$[M_6]_W \subset 6\Lambda. \tag{12}$$

Comparing (11) and (12) we get $[M_6]_W = 6\Lambda$. This proves the theorem in the case of $E_6$.

(h) $E_7$: By the chart and the result for the cases of $A_\ell, D_\ell$, we get

$$L(E_7; \alpha_1, \ldots, \alpha_7) = [(Z\theta + L(D_6; \alpha_7, \alpha_6, \ldots, \alpha_2)) \\
\cap L(A_7; -\theta, \alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_7) \\
\cap (L(A_2; -\theta, \alpha_1) + L(A_5; \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7)) \\
\cap (L(A_5; -\theta, \alpha_1, \alpha_3, \alpha_4, \alpha_2) + L(A_2; \alpha_6, \alpha_7)) \\
\cap (L(D_6; -\theta, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_2) + Z\alpha_7)]_W \\
= [(Z\theta + L'(\alpha_7, \alpha_6, \ldots, \alpha_2)) \\
\cap (Z\theta + Z\alpha_1 + Z\alpha_3 + Z\alpha_4 + \ldots + Z\alpha_7) \\
\cap (Z\theta + Z\alpha_1 + Z\alpha_2 + Z\alpha_4 + \ldots + Z\alpha_7) \\
\cap (Z\theta + Z\alpha_1 + \ldots + Z\alpha_4 + Z\alpha_6 + Z\alpha_7) \\
\cap (L'(-\theta, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_2) + Z\alpha_7)]_W \\
= [M_7]_W,$$
where $M_7 := \mathbb{Z}4\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}6\alpha_3 + \mathbb{Z}2\alpha_4 + \mathbb{Z}6\alpha_5 + \mathbb{Z}4\alpha_6 + \mathbb{Z}2\alpha_7$.

Clearly, $12\Lambda \subset M_7$ and since $\Lambda$ is $W$-stable, $12\Lambda \subset [M_7]_W$.

For any $\mu \in [M_7]_W$, by considering $(w\mu)(\alpha'_1)$ and $(w\mu)(\alpha'_3)$ as in the proof of the theorem for the case of $E_6$, we get that $\mu(\alpha'_1) \in 4\mathbb{Z} \cap 6\mathbb{Z} = 12\mathbb{Z}$ for all the simple coroots $\alpha'_1$. Thus,

$$[M_7]_W \subset 12\Lambda$$ and hence $[M_7]_W = 12\Lambda$.

This takes care of the case of $E_7$.

Finally, we come to the following:

(i) $E_8$: By the chart and the theorem for $E_6$, $E_7$, $D_8$, and $A_\ell$, we get (denoting the weight lattice of $E_i$ by $\Lambda(E_i)$).

$$L(E_8; \alpha_1, \ldots, \alpha_8) = [L(D_8; -\theta, \alpha_8, \alpha_7, \ldots, \alpha_2)$$
$$\cap L(A_8; \alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_8, -\theta)$$
$$\cap (L(A_4; \alpha_1, \alpha_3, \alpha_4, \alpha_2) + L(A_4; \alpha_6, \alpha_7, \alpha_8, -\theta))$$
$$\cap (L(E_6; \alpha_1, \ldots, \alpha_6) + L(A_2; \alpha_8, -\theta))$$
$$\cap (L(E_7; \alpha_1, \ldots, \alpha_7) + \mathbb{Z}\theta)]_W$$
$$= [L'(\theta, \alpha_8, \alpha_7, \ldots, \alpha_2)$$
$$\cap (Z\alpha_1 + Z\alpha_3 + Z\alpha_4 + \cdots + Z\alpha_8 + Z\theta)$$
$$\cap (Z\alpha_1 + \cdots + Z\alpha_4 + Z\alpha_6 + Z\alpha_7 + Z\alpha_8 + Z\theta)$$
$$\cap (6\Lambda(E_6) + Z\alpha_8 + Z\theta) \cap (12\Lambda(E_7) + Z\theta)]_W$$

$$= [M'_8 \cap (6\Lambda(E_6) + Z\alpha_8 + Z\theta) \cap (12\Lambda(E_7) + Z\theta)]_W,$$

where

$$M'_8 := \{Z4\alpha_1 + n3\alpha_2 + m\alpha_3 + Z2\alpha_4 + Z10\alpha_5 + Z2\alpha_6 + Z2\alpha_7 + Z2\alpha_8$$
$$\quad | n, m \in \mathbb{Z} \text{ and } n + m \text{ is even}\}.$$ 

But the coefficient of $\alpha_3$ in any element of $6\Lambda(E_6) + Z\alpha_8 + Z\theta$ is even. Thus,

$$[M'_8 \cap (6\Lambda(E_6) + Z\alpha_8 + Z\theta) \cap (12\Lambda(E_7) + Z\theta)]_W$$
$$= [M_8 \cap (6\Lambda(E_6) + Z\alpha_8 + Z\theta) \cap (12\Lambda(E_7) + Z\theta)]_W,$$

where

$$M_8 := Z4\alpha_1 + Z6\alpha_2 + Z2\alpha_3 + Z2\alpha_4 + Z10\alpha_5 + Z2\alpha_6 + Z2\alpha_7 + Z2\alpha_8.$$ 

For any $\mu \in [M_8]_W$, by considering $(w\mu)(\alpha'_1)$, $(w\mu)(\alpha'_2)$, and $(w\mu)(\alpha'_5)$, as in the proof of the theorem for the case of $E_6$, we get

$$[M_8]_W \subset 60\Lambda(E_8). \quad (13)$$

Conversely, since $\Lambda(E_8) = Q(E_8)$, we get that

$$60\Lambda(E_8) \subset M_8 \cap (6\Lambda(E_6) + Z\alpha_8 + Z\theta) \cap (12\Lambda(E_7) + Z\theta), \quad (14)$$
and hence

\[ 60\Lambda(E_8) \subset [M_8 \cap (6\Lambda(E_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(E_7) + \mathbb{Z}\theta)]_W. \]

To prove (14), it suffices to show that

\[ 60\alpha_7 \in 6\Lambda(E_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta \]  
(15)

and

\[ 60\alpha_8 \in 12\Lambda(E_7) + \mathbb{Z}\theta. \]  
(16)

To prove (15), observe that

\[ 60\alpha_7 = 20\theta - 40\alpha_8 - 20(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6) \]
\[ = 20\theta - 40\alpha_8 - 60\omega_6(E_6), \]

where \( \omega_6(E_6) \) is the sixth fundamental weight of \( E_6 \). Similarly, to prove (16), observe that

\[ 60\alpha_8 = 30\theta - 30(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7) \]
\[ = 30\theta - 60\omega_7(E_7). \]

This proves (15) and (16) and thus (14). Combining (13) and (14), we get that

\[ [M_8 \cap (6\Lambda(E_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(E_7) + \mathbb{Z}\theta)]_W = 60\Lambda(E_8) = 60Q. \]

This proves the theorem for \( E_8 \) and hence the theorem is completely proved. \( \square \)

**Remark 3.11.** Theorem 3.10(a) was obtained earlier by Howard [Ho].

### References


