Rational Functions

This unit lesson plan consists of all the necessary topics leading to and describing rational functions for the secondary education level. Rational functions are an essential part of our everyday lives. They are used in a variety of different ways that range from computing the gas mileage of a car at different speeds to Ohm’s law for electricity. Despite constant awareness or unawareness, people use rational functions all the time. Thus, it is important for students to learn how to identify, solve, and apply rational functions.

Rational functions are most prominently taught in Algebra 2 after students obtain an understanding of polynomials and different types of single-variable functions. Some prerequisite material for this unit includes, but is not limited to, basic algebraic operations, graphing equations, domain, range, intercepts, vertical and horizontal asymptotes, exponent operations, combining like terms, factoring strategies, and polynomials including degree. This lesson plan is mainly referenced from K. Elayn Martin-Gay’s *Intermediate Algebra: Second Edition*, along with several other sources.

The teaching style that is the focus of my approach is endogenous for the most part although, I hope to incorporate many different teaching styles into my lessons allowing optimal opportunities for all of my students to learn efficiently. Questions will be the basis of my teaching material to attempt to create constructivism, where the students construct the knowledge for themselves. I also believe student interaction is also essential for classroom learning, in the form of both collaborative learning and teacher-student interaction. Intrinsic motivation will be
another important aspect of my teaching style in order to keep students interested and related to the material.

The North Carolina Standard Course of Study standards that this unit covers or partially applies to, according to those stated on the NC DPI website, are the following:

**1.03** Operate with algebraic expressions (polynomial, rational, complex fractions) to solve problems.

**2.01** Use the composition and inverse of functions to model and solve problems; justify results.

**2.05** Use rational equations to model and solve problems; justify results.

a. Solve using tables, graphs, and algebraic properties.

b. Interpret the constants and coefficients in the context of the problem.

c. Identify the asymptotes and intercepts graphically and algebraically.

Some of the NCTM Algebra expectations for grades 9-12, as discussed in *Principles and Standards for School Mathematics*, are the following (296):

- Understand patterns, relations, and functions
  - Analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior
  - Understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions

I believe that the incorporation of technology into the lesson can be extremely beneficial in some cases, but harmful in others. I believe a good balance between the use of technology to enhance learning and a replacement of understanding of the material is essential. First, the student must learn the material, then technology should allow them to better comprehend
concepts and problems. In my classroom, I plan to use graphing calculators, PowerPoint presentations, and the internet to provide additional learning techniques that build off of, and strengthen, the basic understanding of the material. For a majority of my notes I will use PowerPoint presentations in the classroom for definitions and introductory examples, while writing extra examples on the board so they will be more interactive. These PowerPoint presentations will also be available for students to download and I will ask them to take notes only on the material that is not covered in the presentations. I believe that math is much easier to comprehend when one is paying full attention to what is going on in the problem as opposed to trying to learn the material and copy everything down at the same time.

I believe that discourse in the mathematics classroom has the ability to make a concept more understandable and allow students to express their own opinions and beliefs about the math that they are learning. I believe that many of the students in class that have trouble understanding the material never speak up and say something. If they were to write out a small journal entry twice a week and before test reviews, then they would be able to express their own likes, dislikes, and problems with the material. Hopefully, I will be able to find similarities in the journal entries that I may address in the next lesson. This will allow me to create a more effective lesson plan for the future, as well as provide extra explanation of these “trouble spots” to the current students.

My lesson plan will be rich with problem solving opportunities for the students. I believe that allowing the students to use a combination of information they have learned thus far in this class and concepts/methods from previous courses to solve a problem shows a student’s true understanding of material. I intend to have a range of worthwhile problems that span from basic understanding of a concept to higher level learning questions that challenge individuals and
groups of students. The ability to think through and solve problems when a solution method is not always given or immediately apparent will be of assistance not only in other courses of study, but also in the real world as they encounter problems in life.

Special needs must be addressed in the correct fashion in the classroom. I will treat all students with the same respect and attempt to create a culturally and racially unbiased environment so not to inhibit any one student’s ability to learn. I believe in allowing students with culture differences and learning disabilities to mingle with other students, especially in collaborative learning exercises. In group situations, I will personally choose the groups to prevent any possible situations that may hinder the group’s effectiveness. Much of this observation will come from trial and error of different combinations of students for the group settings and seeing how they interact with one another.

Assessment of students’ work will vary throughout each individual lesson, but assessment over the entire unit is necessary as well. I will read all of the journal entries, compiling similarities and major trouble points of the unit and address these issues at the beginning of the next class period and in future lesson planning. Also, I intend to leave time at the beginning of each lesson to go over any issues on the homework that is assigned for each class period. This time will allow me to observe which concepts are difficult for the students to grasp. Then, at the end of the unit, I intend to give a graded quiz on the information that has been introduced during the unit. This will allow me to further compare similar discrepancies and address those issues before major, multi-unit tests, as well as allow me to compile information about a student’s performance over the course of the semester. I will use all of this information to create an evaluation of student performance at two different points in the semester where I will hold a conference with each individual student and a parent/guardian. This will provide an
opportunity for the student and guardian to be fully aware of the student’s performance and create suggestions for how each party, including myself, can improve and enhance the student’s learning.

Section 1: Rational Functions and Simplifying Rational Expressions

(75-100 min.)

Prior Knowledge:

- Definition of a rational number or fraction
  - the quotient: \( \frac{p}{q} \), where \( p \) and \( q \) are integers and \( q \neq 0 \)

- Polynomial operations
  - addition, subtraction, multiplication, exponents, etc.

Objectives:

- Define a rational expression and a rational function.
- Find the domain of a rational function.
- Write a rational expression in lowest terms.

Review Exercise:

Use polynomial addition/subtraction and factor the resulting polynomial:

1. \((x^2 + 10x + 38) + (2x - 3)\)
2. \((5x - 38) - (-x^2 + x + 7)\)

To be completed individually as warm-up. Ask for volunteers to come to the board and write their solutions. Talk about solution methods and corrections or issues.

Answer Key: 1. \((x+5)(x+7)\) 2. \((x-5)(x+9)\)

(7-10 min.)

Open Question:

- What would you do to simplify an example like this one:

\[
\frac{x^2 - 12x + 35}{x^2 + 4x - 45}
\]

Ask for volunteers to talk about what they might do. Address the misconception that the student should use long division for this problem.

(3-5 min.)

Definitions:

- Recall definitions.
  - rational number
  - domain
- Present applicable definitions.
  - rational expression
  - rational function
- Provide relevant visual examples
Find domain of a rational function.

- All real numbers except those values that make the denominator 0.

(12-15 min.)

Fundamental Principle of Rational Expressions:

For any rational expression \( \frac{P}{Q} \) and any polynomial \( R, R \neq 0 \),

\[
\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q}
\]

Ex.

\[
\frac{(x+2)^2}{x^2-4} = \frac{(x+2)(x+2)}{(x+2)(x-2)} = \frac{x+2}{x-2}
\]

Address misconception that there can be cancellation through addition and subtraction.

(5-8 min.)

Return to Open Question:

- Form predetermined groups of 4 based on past group work and equal distribution of learning capabilities.

- Students with special needs, such as learning disabilities, intermixed and helped more often.

- Now what would you do with this rational expression:

\[
\frac{x^2 - 12x + 35}{x^2 + 4x - 45}
\]
(10-12 min.)

Extra Collaborative Work:

- Handout to be completed for homework if not completed during time in groups.
  - Answer key available to teacher
Simplifying Rational Expressions

Simplify each expression.

1) \( \frac{-26x^3}{42x^2} \) 
2) \( \frac{16r^3}{16r^7} \)

3) \( \frac{16p^3}{28p} \) 
4) \( \frac{32n^2}{24n} \)

5) \( \frac{70n^2}{28n} \) 
6) \( \frac{15n}{30n^3} \)

7) \( \frac{2r - 4}{r - 2} \) 
8) \( \frac{45}{10a - 10} \)

9) \( \frac{x - 4}{3x^2 - 12x} \) 
10) \( \frac{15a - 3}{24} \)

11) \( \frac{v - 5}{v^2 - 10v + 25} \) 
12) \( \frac{x + 6}{x^2 + 5x - 6} \)
13) \( \frac{27}{27x + 18} \)
14) \( \frac{v^2 - 7v - 30}{v^2 - 5v - 24} \)

15) \( \frac{x^2 + 8x + 12}{x^2 + 3x - 18} \)
16) \( \frac{x^2 - 11x + 18}{x^2 + 2x - 8} \)

17) \( \frac{b^2 + 3b - 28}{b^2 - 49} \)
18) \( \frac{v^2 - 3v - 40}{v^2 - 11v + 24} \)

19) \( \frac{4r - 4}{6r - 20} \)
20) \( \frac{v^2 - 5v - 14}{v^2 + 4v + 4} \)

21) \( \frac{6v^3 + 42v^2}{2v^2 + 26v + 84} \)
22) \( \frac{x^2 - x^2 - 42x}{2x^2 - 20x + 42} \)

23) \( \frac{2v^2 + 10v - 48}{8v + 64} \)
24) \( \frac{9x^2 + 81x}{x^2 + 8x^2 - 9x} \)

25) \( \frac{x^2 + 2x - 80}{2x^2 - 24x^2 + 64x} \)
26) \( \frac{3r^2 - 39x + 50}{r^2 - 3r - 70} \)
Simplifying Rational Expressions

Simplify each expression.

1) \(-\frac{36x^2}{42x^3}\)

2) \(-\frac{6x}{7}\)

3) \(-\frac{16r^2}{16r^3}\)

4) \(-\frac{1}{r}\)

5) \(-\frac{16r^3}{28p}\)

6) \(-\frac{4p}{7}\)

7) \(-\frac{2r - 4}{r - 2}\)

8) \(-\frac{45}{10a - 10}\)

9) \(-\frac{5a - 2}{28n}\)

10) \(-\frac{3a^2}{24}\)

11) \(-\frac{v - 5}{v^2 - 10v + 25}\)

12) \(-\frac{1}{v - 5}\)
13) \[ \frac{27}{27x + 18} = \frac{3}{2x + 2} \]

14) \[ \frac{y^2 - 7y - 39}{y^2 - 5y - 24} = \frac{y - 10}{y - 8} \]

15) \[ \frac{x^2 + 8x + 12}{x^2 + 3x - 18} = \frac{x + 2}{x - 3} \]

16) \[ \frac{x^2 - 11x + 18}{x^2 + 2x - 8} = \frac{x - 9}{x + 4} \]

17) \[ \frac{h^2 + 36 - 28}{h^2 - 49} = \frac{h - 4}{h - 7} \]

18) \[ \frac{y^2 - 3y - 40}{y^2 - 11y + 24} = \frac{y + 5}{y - 3} \]

19) \[ \frac{4\pi - 4}{6\pi - 20} = \frac{2(\pi - 1)}{3\pi - 10} \]

20) \[ \frac{y^2 - 5y - 14}{y^3 + 4y + 4} = \frac{y - 7}{y + 2} \]

21) \[ \frac{6v^2 + 42v^2}{2v^2 + 26v + 84} = \frac{2v^2}{v + 6} \]

22) \[ \frac{x^2 - x^2 - 42x}{2x^2 - 20x + 42} = \frac{x(x + 6)}{2(x - 3)} \]

23) \[ \frac{2v^2 + 10v - 48}{8v + 64} = \frac{v - 3}{4} \]

24) \[ \frac{9x^2 + 81x}{x^3 + 8x^2 - 9x} = \frac{9}{x - 1} \]

25) \[ \frac{x^2 + 2x - 80}{2x^2 - 24x^2 + 64x} = \frac{x + 10}{2x(x - 4)} \]

26) \[ \frac{3r^2 - 39r + 90}{r^2 - 3r - 70} = \frac{3(r - 3)}{r + 7} \]

(15-25 min.)
Higher Level Learning:

- Provide the following example for groups:
  \[
  \frac{x^2 - px + 4}{x - q} = x - s \quad \text{Solve for } p, q, \text{and } s.
  \]

- Allow time to work on this problem before coming back together as a class.

- Disassemble groups and discuss higher level example as a class.

- Ask for ideas and volunteers to write on the board.

- Provide possible solution method.
  
  o Possible answers for \( p, q, \text{and } s \), respectively:

  - 5, 4, 1
  - 4, 2, 2

(17-20 min.)

Leads To:

- Multiplying rational expressions.

- Later in lesson, evaluating the graphs of rational functions.

Assessment:

- Assign homework checked for effort the next day:

  o Complete handout from group work.
Problems in the book:

- p. 339 (5, 9, 10, 27, 32, 42, 53)

(3-5 min.)

- Graded quiz on the last day of the unit including domain and simplifying rational expression problems.

Section 2: Multiplying Rational Expressions

(75-100 min.)

Prior Knowledge:

- Polynomial Operations
- Definitions of rational expressions and functions
- Simplification rules for rational expressions

Objectives:

- Multiply rational expressions.

Warm-Up:

1. Find the domain of the rational function:

   \[ f(x) = \frac{3x}{7-x} \]

2. Write the rational function in lowest terms.
\[
\frac{2x^2 + 12x + 18}{x^2 - 9}
\]

To be completed individually. Ask for volunteers to come to the board and write their solutions.

Talk about solution methods and corrections or issues.

Answer Key: 1. \( \{x: x \text{ a real number and } x \neq 7\} \) 2. \( \frac{2x+6}{x-3} \)

(7-10 min.)

Check homework for completion and effort while students work on warm-up.

Rubric:

Scores based on a 1-3 scale:

3-Completion of entire homework displaying great effort on every problem.

2-Partial completion of homework assignments with acceptable effort shown on most problems.

1-Little to no completion or effort displayed on assignments.

Homework Questions:

Display the answers to all homework assignments and ask students if they had trouble on particular problems. Talk about solutions to “trouble” problems and work them out on the board step-by-step.

(10-15 min.)

Activity: Hoop Rates and Percents:
In this activity, students play waste paper basketball and keep track of their makes and misses. Each group, consisting of 4 people, does a variety of total number of shots (out of 5, out of 10, out of 20). Each student then calculates the percentage made for each result. Afterwards, the class makes a list of the results sorting by the percentage made. From this list students see equivalent fractions and conjecture how to create other equivalents.

In preparation, you will need to have waste paper baskets, boxes or other containers available for students to shoot into. You may want to have a couple students test out different shooting distances so that it is not too easy or too hard to make a shot.

After students have collected their data, bring the class together to share results. Ask a student for one of their “out of 5” results. Ask if anyone else got the same percent, but out of a different number of shots. Continue to this for each of the other “out of 5” results. Depending on the number of equivalent fractions you get, you may want to move to the “out of 10” results as well.

Once the data has been sorted, tell the students that each of the fractions or rates that give the same percent is called an equivalent fraction. Compare this terminology to that of equivalent rational expressions. Ask students if they see a relationship between the fractions in each category. They should see that the “out of 10” and “out of 20” results are just the “out of 5” result multiplied by 2 and 4 respectively.

For example:

\[
\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} = 0.60 = 60\%
\]

\[
\frac{3}{5} = \frac{3 \cdot 4}{5 \cdot 4} = \frac{12}{20} = 0.60 = 60\%
\]
Ask the class what other fractions would be equal to \( \frac{3}{5} \). Then ask the class, in general, what they can do to a fraction and keep the result equivalent. They should see that as long as the numerator and denominator are multiplied by the same number, the fractions will be equivalent. Symbolically, that would mean:

\[
\frac{a}{b} = \frac{a \cdot c}{b \cdot c}
\]
Student Worksheet

Section 2: Hoop Rates and Percents

Rates and percents are important ideas when working with rational expressions. In problems I, II and III, you will collect some data on the rate that you make waste paper basketball shots.

I. Take 5 shots and record how many you made. Repeat this 5 times.

\[ \frac{5}{10} = \frac{5}{10} = \frac{5}{10} = \frac{5}{10} = \frac{5}{10} \]

Convert each of these rates to a percent by dividing the numerator by the denominator.

II. Take 10 shots and record how many you made. Repeat this 5 times.

\[ \frac{10}{10} = \frac{10}{10} = \frac{10}{10} = \frac{10}{10} = \frac{10}{10} \]

Again, convert each of these rates to a percent.

III. Take 20 shots and record how many you made. Repeat this 3 times.

\[ \frac{20}{20} = \frac{20}{20} = \frac{20}{20} = \]

Again, convert each of these rates to a percent.

IV. Write the numerators of equivalent fractions: \( \frac{3}{8} = \frac{16}{24} = \frac{40}{60} \)

V. Write the numerators of equivalent fractions: \( \frac{2}{3} = \frac{15}{24} = \frac{60}{90} \)

VI. Write the numerators of equivalent fractions: \( \frac{40}{70} = \frac{7}{35} = \frac{14}{14} \)
Multiplying Rational Expressions:

- Let $P, Q, R, \text{ and } S$ be polynomials. Then

\[ \frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS} \]

as long as $Q \neq 0$ and $S \neq 0$.

- Ex. \[ \frac{2x^3}{9y} \cdot \frac{y^2}{4x^3} = \frac{2x^3y^2}{36x^3y} \]

  - How do I simplify this expression?

- In general, to multiply rational expressions

  - Step 1: Completely factor the numerators and denominators.

  - Step 2: Multiply the numerators and multiply the denominators.
Step 3: Write the product in lowest terms by applying the fundamental principle of rational expressions and dividing both the numerator and the denominator by their greatest common factor.

(10-15 min.)

Leads To:

- Writing a rational expression as an equivalent rational expression with a given denominator.
- Dividing rational expressions.
- Later in lesson, evaluating the graphs of rational functions.

Assessment:

- Assign homework checked for effort the next day:
  - Journal entry on rational expressions so far to be collected the next day. Express thoughts, troubles, etc. about rational expressions.
  - Problems in the book:
    - p. 348 (6, 7, 14, 41, 56, 71)

(3-5 min.)

- Graded quiz on the last day of the unit including multiplying rational expression problems.
Section 3: Dividing Rational Expressions and Algebra of Functions

(75-105 min.)

Prior Knowledge:

- Knowledge of reciprocals
- Definition of a function
- Definition of sum, difference, product, and quotient

Objectives:

- Write a rational expression equivalent to a rational expression with a different denominator.
- Divide by a rational expression.
- Apply the algebra of functions.

Warm-Up:

- Multiply as indicated. Write answers in lowest terms.

1. \( \frac{2x \cdot 5x+10}{5 \cdot 6(x+2)} \)

2. \( \frac{3xy^3}{4x^3y^2} \cdot \frac{-8x^3y^4}{9x^4y^7} \)

To be completed individually. Ask for volunteers to come to the board and write their solutions.

Talk about solution methods and corrections or issues.

Answer Key: 1. \( \frac{x}{3} \) 2. \( \frac{-2}{3x^2y^2} \)
Check homework for completion and effort while students work on warm-up.

Rubric:

Scores based on a 1-3 scale:

3-Completion of entire homework displaying great effort on every problem.

2-Partial completion of homework assignments with acceptable effort shown on most problems.

1-Little to no completion or effort displayed on assignments.

Collect journal assignment to read that night. Return journals after checking for completion and gather issues to be addressed the next day in class about the material.

Homework Questions:

Display the answers to all homework assignments and ask students if they had trouble on particular problems. Talk about solutions to “trouble” problems and work them out on the board step-by-step.

Equivalent Rational Expressions with a Given Denominator

- Recall the fundamental principle of rational expressions.
- Provide the following example:
Ex. Write the rational expression \( \frac{3x}{2y} \) as an equivalent rational expression with denominator \( 10xy^3 \).

- Ask students what \( 2y \) is multiplied by to get a product of \( 10xy^3 \).
  - \( 2y(5y^2) = 10xy^3 \)
- If I multiply on the bottom, I must multiply on the what?
  - Show:
    \[
    \frac{3x}{2y} = \frac{3x(5y^2)}{2y(5y^2)} = \frac{15x^2y^2}{10xy^3}
    \]

(10-15 min.)

Dividing Rational Expressions:

- To divide by a rational expression, we need to multiply by its reciprocal.
  - Recall that two numbers are reciprocals of each other if their product is 1.
    - Thus, if \( \frac{P}{Q} \) is a rational expression, the \( \frac{Q}{P} \) is its reciprocal.

  - What is the reciprocal of \( \frac{2+x^2}{4x-3} \)?

- Let \( P, Q, R, \) and \( S \) be polynomials. Then
  \[
  \frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}
  \]
  as long as \( Q \neq 0 \) and \( S \neq 0 \).
  - Ex. \( \frac{3x}{5y} \div \frac{9y}{x^5} = \frac{3x}{5y} \cdot \frac{x^5}{9y} = \frac{3x^6}{45y^2} \)
Simplify this expression.

(10-15 min.)

**Algebra of Functions:**

- Recall the definition of a function
  - Recall the definition of a rational function
- Open Question:
  - If I told you that
    \[
    f(x) = \frac{3}{x} \text{ and } g(x) = \frac{x + 1}{5}
    \]
    how would you compute their product?
  - Hopefully, students will make the connection that
    \[
    f(x) \cdot g(x) = \frac{3}{x} \cdot \frac{x + 1}{5} = \frac{3(x + 1)}{5x}
    \]
    We can use the notation \((f \cdot g)(x)\) to denote this new function.
- Finding the sum, difference, product, and quotient of functions to generate new functions is called the algebra of functions.
  - Let \(f\) and \(g\) be functions.
    - Their sum, written as \(f + g\), is defined by \((f + g)(x) = f(x) + g(x)\).
    - Their difference, written as \(f - g\), is defined by \((f - g)(x) = f(x) - g(x)\).
    - Their product, written as \(f \cdot g\), is defined by \((f \cdot g)(x) = f(x) \cdot g(x)\).
    - Their sum, written \(\frac{f}{g}\), is defined by \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\).
Higher Level Learning:

- Work independently on the following problem.
  - Business people are concerned with cost functions, revenue functions, and profit functions. Recall that the profit $P(x)$ obtained from $x$ units of a product depends on the revenue $R(x)$ and the cost $C(x)$ of manufacturing the $x$ units. Write an equation expressing this relationship among $P(x), R(x),$ and $C(x)$.
    - Then suppose for an iPod Touch, the revenue $R(x) = 300x$, and the cost $C(x) = x^2 - 350x + 50$. Find the profit $P(x)$ for 400 iPods Touch’s.

- Allow students to work alone for several minutes and write homework on the board so those who finish early will have something to work on.

- Ask students to compare answers with one of their neighbors and discuss any differences they might have in solutions or solution methods.

- Ask for volunteers to talk about their solution and how they got it, providing work on the board if needed.
  - Students should see that the relationship can be written as: $P(x) = R(x) - C(x)$.
  - Next students should plug in the formulas for the revenue and cost of an iPod Touch.
    - Then plug in 400 for $x$ to find the profit $P(x)$ for 400 iPods Touch’s.
  - An alternative method would be to solve the individual equations for revenue and cost before plugging these values into the formula.
    - Either way, the answer should be $P(400) = 99950$. 
(20-25 min.)

Leads To:

- Adding and subtracting rational expressions.
- Later in lesson, evaluating the graphs of rational functions.

Assessment:

- Assign homework checked for effort the next day:
  - Problems in the book:
    - p. 348 (23, 28, 43, 60, 67, 77, 84, 88)

(3-5 min.)

- Graded quiz on the last day of the unit including dividing rational expression and algebra of function problems.

Section 4: Adding and Subtracting Rational Expressions

(75-105 min.)

Prior Knowledge:

- Adding and subtracting of rational numbers
- Basic factoring skills
- Knowledge of the least common denominator (LCD)
• Writing a rational expression as an equivalent rational expression with a given denominator

Objectives:

• Add or subtract rational expressions with common denominators.
• Identify the least common denominator of two or more rational expressions.
• Add and subtract rational expressions with unlike denominators.

Warm-Up:

• Perform the indicated operation. Write all answers in lowest terms.

1. \[
\frac{5a^2-20}{3a^2-12a} \div \frac{a^3+2a^2}{2a^2-8a} \cdot \frac{9a^3+6a^2}{2a^2-4a}
\]

• If \( f(x) = -2x \), \( g(x) = x^2 + 2 \), and \( h(x) = 4x + 3 \), find the following.

2. \( \left( \frac{f}{g} \right)(x) \)

To be completed individually. Ask for volunteers to come to the board and write their solutions.

Talk about solution methods and corrections or issues.

Answer Key: 1. \( \frac{15a+10}{a} \) 2. \( \frac{-2x}{4x+3} \)

(7-10 min.)

Check homework for completion and effort while students work on warm-up.

Rubric:
Scores based on a 1-3 scale:

3-Completion of entire homework displaying great effort on every problem.

2-Partial completion of homework assignments with acceptable effort shown on most problems.

1-Little to no completion or effort displayed on assignments.

Homework Questions:

Display the answers to all homework assignments and ask students if they had trouble on particular problems. Talk about solutions to “trouble” problems and work them out on the board step-by-step.

Address any misconceptions, problems, or interesting thoughts about the students’ journal entries. Allow students to ask questions about any of the material in the unit up until this point, or conjecture about the future lessons or techniques.

(15-25 min.)

Open Question:

- What would you do if I asked you to solve a problem like this:
  
  - Perform the indicated operation.

$$\frac{5k}{k^2 - 4} - \frac{2}{k^2 + k - 2}$$

- Ask students to conjecture about possible solution methods or places to start.

(3-5 min.)
Adding or Subtracting Rational Expressions with Common Denominators:

- If $\frac{P}{Q}$ and $\frac{R}{Q}$ are rational expressions, then $\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$ and $\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$.

  - In other words, to add or subtract rational expressions with common denominators, add or subtract the numerators and write the sum or difference over the common denominator.

  - Ex.

    $\frac{x}{4} + \frac{5x}{4} = \frac{x + 5x}{4} = \frac{6x}{4} = \frac{3x}{2}$

(7-10 min.)

Finding the Least Common Denominator:

- To add or subtract rational expressions with unlike denominators, first write them as equivalent rational expressions with common denominators.

- The least common denominator (LCD) is usually the easiest common denominator to work with. The LCD of a list of rational expressions is a polynomial of least degree whose factors include the denominator factors in the list.

- To find the LCD:
  - **Step 1**: Factor each denominator completely.
  - **Step 2**: The LCD is the product of all unique factors formed in step 1, each raised to a power equal to the greatest number of times that the factor appears in any one factored denominator.

  - Ex. Find the LCD of the rational expressions.
\[
\frac{2}{15x^5y^2}, \frac{3z}{5xy^3}
\]

- Factor each denominator:
  - \(15x^5y^2 = 3 \cdot 5 \cdot x^5 \cdot y^2\)
  - \(5xy^3 = 5 \cdot x \cdot y^3\)

- Unique factors are 3, 5, \(x\), and \(y\).
  - The greatest number of times that 3 appears in one denominator is 1.
  - The greatest number of times that 5 appears in one denominator is 1.
  - The greatest number of times that \(x\) appears in one denominator is 5.
  - The greatest number of times that \(y\) appears in one denominator is 3.

- The LCD is the product of \(3^1 \cdot 5^1 \cdot x^5 \cdot y^3\), or \(15x^5y^3\).

(10-15 min.)

**Adding or Subtracting Rational Expressions with Unlike Denominators:**

- To add or subtract rational expressions with unlike denominators, we write each rational expression as an equivalent rational expression so that their denominators are alike.
  - **Step 1:** Find the LCD of the rational expressions.
  - **Step 2:** Write each rational expression as an equivalent rational expression whose denominator is the LCD found in step 1.
Step 3: Add or subtract numerators, and write the sum or difference over the common denominator.

Step 4: Simplify, or write the resulting rational expression in lowest terms.

- Return to open question.

- Now what would you do in the following problem?
  - Allow students to take you through, step-by-step.

- Perform the indicated operation.

\[ \frac{5k}{k^2 - 4} - \frac{2}{k^2 + k - 2} \]

- Factor each denominator to find the LCD.

\[ \frac{5k}{k^2 - 4} - \frac{2}{k^2 + k - 2} = \frac{5k}{(k + 2)(k - 2)} - \frac{2}{(k + 2)(k - 1)} \]

- The LCD is \((k + 2)(k - 2)(k - 1)\). Write equivalent rational expressions with the LCD as denominators.

\[ \frac{5k}{(k + 2)(k - 2)} - \frac{2}{(k + 2)(k - 1)} \]

\[ = \frac{5k(k - 1)}{(k + 2)(k - 2)(k - 1)} - \frac{2(k - 2)}{(k + 2)(k - 2)(k - 1)} \]

\[ = \frac{5k(k - 1) - 2(k - 2)}{(k + 2)(k - 2)(k - 1)} \quad \text{Subtract the numerators.} \]

\[ = \frac{5k^2 - 5k - 2k + 4}{(k + 2)(k - 2)(k - 1)} \]

\[ = \frac{5k^2 - 7k + 4}{(k + 2)(k - 2)(k - 1)} \quad \text{Simplify the numerator.} \]
Since the numerator polynomial is prime, the numerator and denominator have no common factors and this rational expression is in lowest terms.

- Ask students if they need more examples. Be sure that students understand all the steps necessary. Provide more examples from the book if needed.

(20-25 min.)

Leads To:

- Evaluating the graphs of rational functions.

Assessment:

- Assign homework checked for effort the next day and allow students to begin working on assignments at the end of class:
  
  - Journal entry to be collected the next day. In your own words, explain how to add and subtract rational expressions with different denominators. Express thought, troubles, etc. about the material in the unit up to this point.

  - Problems in the book:

    - p. 356 (4, 5, 10, 17, 29, 42, 51, 56, 57, 60, 75)

(13-15 min.)

- Graded quiz on the last day of the unit including adding and subtracting rational expression problems.
**Section 5: Graphing Rational Functions:**

(75-100 min.)

**Prior Knowledge:**

- The definition of a rational function and the relationship between the domain, codomain, and range of the function.
- Graphing a function and finding x- and y-intercepts, vertical and horizontal asymptotes, etc.
- Basic algebraic and exponent operations including combining like terms.
- Basic factoring skills including factoring of polynomials with a leading coefficient other than 1.

**Objectives:**

- Find the asymptotes and holes in a rational function.
- Graph a rational function on graph paper.

**Warm-Up:**

- Perform the indicated operation. Write answers in lowest terms.

1. \( \frac{-2}{x^2-3x} - \frac{1}{x^3-3x^2} \)

- Find the domain of the given rational expression.

2. \( \frac{3+2x}{x^3+x^2-2x} \)
To be completed individually. Ask for volunteers to come to the board and write their solutions.

Talk about solution methods and corrections or issues.

Answer Key: 1. \( \frac{-2x-1}{x^2(x-3)} \) 2. \( \{x: x \text{ is a real number and } x \neq 2, x \neq 0, x \neq 1\} \)

(7-10 min.)

Check homework for completion and effort while students work on warm-up.

Rubric:

Scores based on a 1-3 scale:

3-Completion of entire homework displaying great effort on every problem.

2-Partial completion of homework assignments with acceptable effort shown on most problems.

1-Little to no completion or effort displayed on assignments.

Collect journal assignment to read that night. Return journals after checking for completion and gather issues to be addressed the next day in class about the material.

Homework Questions:

Display the answers to all homework assignments and ask students if they had trouble on particular problems. Talk about solutions to “trouble” problems and work them out on the board step-by-step.
Address any misconceptions, problems, or interesting thoughts about the students’ journal entries. Allow students to ask questions about any of the material in the unit up until this point, or conjecture about the future lessons or techniques.

(15-20 min.)

**Graphing Rational Functions:**

- Recall: In a rational function, the domain is all real values such that the denominator, or \( q(x) \), is not 0.
- The solutions, or zeros, of the rational function are also the x-intercepts and are found where the numerator, or \( p(x) \), is 0.
- Both the domain and zeros of the rational function are most easily found by first factoring the polynomials \( p(x) \) and \( q(x) \), and simplifying the rational expression.

\[
\frac{x^2 + 5x + 6}{x^2 - 4} = \frac{(x+2)(x+3)}{(x+2)(x-2)} = \frac{x+3}{x-2}
\]

- Now the domain is \( \{x \in R | x \neq 2\} \) and the only zero is (-3, 0).
- It is essential to note that when factors like \( (x - k) \) cancel, there is either a hole in the graph or a vertical asymptote at \( x = k \).
  - In the previous example there is a hole at \( x = -2 \). There is not a vertical asymptote since there is no longer a \( (x + 2) \) factor left in the denominator.

(15-20 min.)

**Asymptotes:**
• Vertical Asymptotes: The vertical asymptotes of a rational function are found by finding the roots of \( q(x) \).

• Horizontal Asymptotes: The horizontal asymptote of a rational function is determined by comparing the degrees of the numerator (n) and the denominator (m) as follows:
  
  - If \( n < m \), then the x-axis, or \( y = 0 \), is the horizontal asymptote.
  - If \( n = m \), then \( y = \frac{a_n}{b_m} \) is the horizontal asymptote where \( a_n \) and \( b_m \) are the leading coefficients of n and m, respectively.
  - If \( n > m \), then there is no horizontal asymptote.

• Slant Asymptote: If the degree of the numerator is exactly one more than the degree of the denominator (so that the polynomial fraction is "improper"), then the graph of the rational function will be, roughly, a slanty straight line with some fiddly bits in the middle. Because the graph will be nearly equal to this slanted straight-line equivalent, the asymptote for this sort of rational function is called a "slant" (or "oblique") asymptote. The equation for the slant asymptote is the polynomial part of the rational that you get after doing the long division.
Find the slant asymptote of the following function:

\[ y = \frac{x^2 + 3x + 2}{x - 2} \]

To find the slant asymptote, I'll do the long division:

\[
\begin{array}{c|ccc}
\multicolumn{1}{r}{x - 2} & x^2 & +3x & +2 \\
\hline
x - 2) & x^2 & +5x & \\
\hline
\multicolumn{1}{r}{x^2 - 2x} \\
\hline
\multicolumn{1}{r}{5x + 2} \\
\hline
\multicolumn{1}{r}{5x - 10} \\
\hline
\multicolumn{1}{r}{12}
\end{array}
\]

The slant asymptote is the polynomial part of the answer, not the remainder.

slant asymptote: \[ y = x + 5 \]

(15-20 min.)

Graphing Rational Functions:

- Graphing rational functions consists of factoring out the polynomials to see if there are any holes, finding and plotting the vertical and horizontal asymptotes, and then finding several points on each side of the asymptotes. Using the fact that the graph must “go off” to infinity or negative infinity at the asymptotes, one should be able to graph a relatively accurate depiction of the rational function. A graphing calculator or math computer program can always be used to check the accuracy of the graph.

Ex. Graph the following:

\[ y = \frac{2x + 5}{x - 1} \]

First find the vertical asymptotes, if any, for this rational function. Since I can't graph where the function doesn't exist, and since the function won't exist where there would
be a zero in the denominator, I'll set the denominator equal to zero to find any forbidden points:

- \( x - 1 = 0 \)
  
  \( x = 1 \)

- So I can't have \( x = 1 \), and therefore I have a vertical asymptote there.

- I'll dash this in on my graph:

- Next I'll find the horizontal or slant asymptote. Since the numerator and denominator have the same degree (they're both linear), the asymptote will be horizontal, not slant, and the horizontal asymptote will be the result of dividing the leading coefficients:
  
  \( y = \frac{2}{1} = 2 \)

- I'll dash this in, too:

- Next, I'll find any \( x \)- or \( y \)-intercepts.
- $x = 0$: $y = \frac{0 + 5}{0 - 1} = \frac{5}{-1} = -5$

$y = 0$: $0 = \frac{2x + 5}{x - 1}$

$0 = 2x + 5$

$-5 = 2x$

$-2.5 = x$

- Then the intercepts are at $(0, -5)$ and $(-2.5, 0)$. I'll sketch these in:

Now I'll pick a few more $x$-values, compute the corresponding $y$-values, and plot a few more points.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = \frac{2x + 5}{x - 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-6$</td>
<td>$(2(-6) + 5)/((-6) - 1) = (\frac{-12 + 5}{-7}) = \frac{-7}{-7} = 1$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$(2(-1) + 5)/((-1) - 1) = (\frac{-2 + 5}{-2}) = (\frac{3}{-2}) = -1.5$</td>
</tr>
<tr>
<td>$2$</td>
<td>$(2(2) + 5)/((2) - 1) = (\frac{4 + 5}{1}) = (\frac{9}{1}) = 9$</td>
</tr>
</tbody>
</table>
3 \(\frac{(2(3) + 5)/((3) - 1) = (6 + 5)/(2) = (11)/(2) = 5.5}{(2(6) + 5)/((6) - 1) = (12 + 5)/(5) = (17)/(5) = 3.4}{(2(8) + 5)/((8) - 1) = (16 + 5)/(7) = (21)/(7) = 3}{(2(15) + 5)/((15) - 1) = (30 + 5)/(14) = (35)/(14) = 2.5}

I mostly picked \(x\)-values near the middle of the graph: because of the horizontal asymptote, I already have a good idea of what the graph does off to the sides. (It can be a good idea to do a point or two near the ends anyway, as a check on your work.) Also, since I had no intercepts on the right-hand side of the vertical asymptote to give me hints as to what was happening with the graph, I needed more points there to show me what was going on.

- Now I'll plot these points:
And now I can connect the dots:

- When you draw your graph, make sure you show the graph continuing off to the sides.

- Don't just stop at a point you've drawn, because this will make it look as though the graph actually stops at that point.

(20-25 min.)

Leads To:

- Observing the applications and graphs of rational functions.

Assessment:
• Assign homework checked for effort the next day:

(3-5 min.)

• Study for unit quiz on the next day including graphing rational functions.

**Section 6: Quiz and Project with Graphing Calculators:**

(75-100 min.)

**Prior Knowledge:**

• All the information learned in this unit.

**Objectives:**

• Test the knowledge gained about rational expressions and functions.

• Use a graphing calculator to work with a rational function.

**Warm-Up:**

• Quick review of how to graph a rational function.
  
  o Go over the main points

(7-10 min.)

**Quiz:**
Quiz: Rational Expressions and Functions

1. Find the domain of the rational function:

\[ f(x) = \frac{3x}{9-x} \]

2. Write the rational function in lowest terms.

\[ \frac{2x^2 + 12x + 18}{x^2 - 9} \]

3. Perform the indicated operation. Write all answers in lowest terms.

\[ \frac{5a^2 - 20}{3a^2 - 12a} \div \frac{a^3 + 2a^2}{2a^2 - 8a} \cdot \frac{9a^3 + 6a^2}{2a^2 - 4a} \]

4. If \( f(x) = -5x, \) \( g(x) = x^2 + 3, \) and \( h(x) = 7x + 1, \) find the following:

\[ \left( \frac{f}{g} \right)(x) \]

5. Perform the indicated operation. Write answers in lowest terms.

\[ \frac{-2}{x^2 - 3x} - \frac{1}{x^3 - 3x^2} \]

6. Graph the following rational function and identify any asymptotes, holes, and intercepts:

\[ y = \frac{x + 2}{x^2 + 1} \]
Answer Key:

1. \{x: x a real number and \( x \neq 9 \}\}

2. \( \frac{2x+6}{x-3} \)

3. \( \frac{15a+10}{a} \)

4. \( \frac{-5x}{7x+1} \)

5. \( \frac{-2x-1}{x^2(x-3)} \)

6. No vertical asymptotes, horizontal asymptote at \( y = 0 \) or the x-axis, and intercepts at \((0, 2)\) and \((-2, 0)\).
Rubric for Quiz:

- Holistic Rubric for each question

4 The student answers the question correctly offering a correct strategy.
3 The student provides either a correct answer and incomplete strategy or a complete, correct strategy and an incorrect answer.
2 The strategy displayed is partially correct, showing partial knowledge of the material.
1 The student’s answer is irrelevant, off task, or incorrect, but the response may contain some correct computation.
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Graphing Calculator Activity:

- We can use rational expressions to represent many things in our daily life. For example, you can create a rational expression to represent the average cost of a Karate lesson. If you pay $55 initial fee and $4.50 at the beginning of each lesson, what would be the rational expression for the average cost of a lesson?

\[
\frac{55 + 4.5n}{n}
\]

ACTIVITY : GRAPHING A RATIONAL FUNCTION

TECHNOLOGY DIRECTIONS

To use the graphing calculator on the Activity for this lesson,

1.) press "y="
2.) clear existing equations

3.) enter "55/ (x + 4.5)" in the space for y1

4.) Press "2nd, window"

5.) Make the choice in "Table Setup"

INDPNT: ASK

DEPEND: AUTO

6.) Press "2nd, graph" and you will see an empty table.

7.) Enter the value of n in column x pressing enter after each.

8.) To view the graph, press "window" and make these settings.

Xmin = - 47 Ymin = - 31

Xmax = 47 Ymax = 31

Xscl = 5 Yscl = 5

9. ) Then press "graph"

ACTIVITY: GRAPHING A RATIONAL FUNCTION

1.  Use your calculator to fill in the missing values.
<table>
<thead>
<tr>
<th>Number of lessons, $n$</th>
<th>Average cost of a lesson</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>59.50</td>
</tr>
<tr>
<td>2</td>
<td></td>
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<tr>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Does the function seem to be defined for all positive values of $n$? What happens when $n=0$? Explain.
3. Negative and non-integer values of \( n \) would have no meaning for this example, but is the function defined for negative and non-integer values of \( n \)? Complete the table for two negative and two non-integer values of \( n \).

4. Describe the shape of your graph near the vertical line \( n=0 \). As \( n \) gets larger, what happens to \( A \)?

- In general, the domain of a rational function consists of all real numbers that give a non-zero denominator. At each value where the denominator is zero, this function is undefined. This is where you will find your asymptote.
  - Give examples:

    What are the domains for:

    a) \( y = \frac{1}{x} \)  
    b) \( \frac{x^2 - 4}{x-2} \)  
    c) \( n = \frac{m + 2}{m^2 - 5m + 6} \)

- The rational expressions that I gave you to try had polynomials in the numerator and denominator that were pretty obvious to find the domain for. What if I gave you something like:

\[
\frac{15x^4 - 120x^3 + 260x^2 - 120x + 245}{3x^5 - 24x^4 + 52x^3 - 24x^2 + 49x}
\]

to find the domain for. Do you agree that this would be a pain?
• What if I told you that this expression is equal to $5/x$. Now would it be easier to find the domain? Have I convinced you of the purpose of simplifying expressions?

(33-45 min.)

**Leads To:**

• Comparison between different types of functions and their graphs.
• More complex rational functions including ones that include multiple variables and trigonometry.

**Assessment:**

• Homework:
  
    o Start to read ahead in the next unit.

Rational expressions and functions are very important aspects of algebra. The good news is that, despite the large amount of material that is covered in the section, much of the operations rely on similar operations from past units and courses. The key is allowing the students to connect the dots between what they know and what they want to find. My job is to bridge this gap and enhance the students’ ability to see and make these connections, thus strengthening their understanding of the material in the unit and in successive units.
Works Cited


*Incorporating the Graphing Calculator into a Lesson on Rational Expressions and Functions.*


