Equivariant Isometric Embeddings of Homogeneous Spaces
Into Hilbert Space

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Let $G$ be a Lie group, $K \subset G$ a compact subgroup. Denote the associated Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$. Let $\pi$ be a unitary representation of $G$ on a Hilbert space $H$, and let $u \in H$ be a smooth vector. We denote the associated Lie algebra representation of $\mathfrak{g}$ by $d\pi$. Let us assume that, given $X \in \mathfrak{g}$,

$$d\pi(X)u = 0 \iff X \in \mathfrak{k},$$

that

$$g \in K \implies \pi(g)u = u,$$

and that, given $g \in G$,

$$\pi(g)u = u \implies g \in K.$$

Under these hypotheses, define

$$\varphi : G \to H, \quad \varphi(g) = \pi(g)u.$$ By (2), this gives rise to a map

$$\psi : G/K \to H, \quad \psi([g]) = \varphi(g),$$

and by (3) this map is one-to-one. Note that, given $X \in \mathfrak{g} = T_eG$,

$$D\varphi(e)X = d\pi(X)u,$$

so the hypotheses above imply $\psi$ is an embedding. Let $\tilde{X}$ denote the left invariant vector field on $G$ associated to $X$:

$$\tilde{X}(g) = \frac{d}{dt}g\text{Exp}(tX)\bigg|_{t=0} \in T_gG.$$ Note that

$$D\varphi(g)\tilde{X}(g) = \frac{d}{dt}\varphi(g\text{Exp}(tX))\bigg|_{t=0}$$

$$= \frac{d}{dt}\pi(g\text{Exp}(tX))u\bigg|_{t=0}$$

$$= \pi(g)d\pi(X)u.$$
Hence, if also \( Y \in \mathfrak{g} \), we have a (degenerate) inner product on \( T_g G \) given by
\[
\langle \tilde{X}(g), \tilde{Y}(g) \rangle = \text{Re}(D\varphi(g)\tilde{X}(g), D\varphi(g)\tilde{Y}(g))_H
\]
\[
= \text{Re}(\pi(g)d\pi(X)u, \pi(g)d\pi(Y)u)_H,
\]
which by unitarity of \( \pi \) is equal to
\[
\langle X, Y \rangle = \text{Re}(d\pi(X)u, d\pi(Y)u)_H.
\]
Consequently, the embedding (5) induces a Riemannian metric on \( M = G/K \) that is \( G \)-invariant, and with respect to which (5) is an isometric embedding, onto a manifold in \( H \) that is a \( G \)-orbit.

Now \( M = G/K \) might come with a \( G \)-invariant metric tensor, and the question arises whether it must coincide (up to a constant factor) with the metric produced by (5). Indeed, sometimes the following holds:
\[
M = G/K \text{ has a unique } G\text{-invariant Riemannian metric, up to a constant factor.}
\]
To restate this condition, take \( p = [e] \in M \) and note the natural action of \( K \) on \( T_p M \). The condition (9) is equivalent to
\[
T_p M \text{ has a unique } K\text{-invariant inner product, up to a constant factor.}
\]
This is equivalent to saying that \( \mathfrak{g}/\mathfrak{k} \) has no proper \( \text{ad(}K)\)-invariant linear subspace. Such a condition holds, for example, if \( M \) is a rank-1 symmetric space. See [CR] for the special case of balls with the hyperbolic metric.

On the other hand, (11)–(12) can fail, for example when \( K = \{e\} \) and \( G > 1 \). The set of \( K \)-invariant inner products on \( T_p M \) is a nonempty, open, convex cone \( \Gamma \) in \( S^2_K T_p^* \), the linear space of \( K \)-invariant symmetric bilinear forms on \( T_p \). Even when \( \dim S^2_K T_p^* > 1 \), the following is true.

**Proposition 1.** If \((M, h)\) is a Riemannian manifold on which \( G \) acts transitively, as a group of isometries, with \( K \subset G \) the compact subgroup fixing \( p \in M \), then \( M \) has an equivariant isometric embedding into a Hilbert space.

To begin the proof, we bring in the regular representation of \( G \) on \( L^2(M, h) \):
\[
L(g)u(x) = u(g^{-1}x), \quad u \in L^2(M, h).
\]
Pick \( u \in C_0^\infty(M) \) to be a positive, monotonically decreasing function of \( d(x, p) \). Then (1)–(3) hold, for \( \pi = L \). Thus we have an embedding
\[
\psi : M \rightarrow L^2(M, h), \quad \psi(g \cdot p) = L(g)u,
\]
giving as in (8)–(10) a \( G \)-invariant metric tensor \( \tilde{g} \).

Let us denote by \( Q \) the inner product on \( T_p M \) given by \( h \) at \( p \), and also the associated degenerate inner product on \( \mathfrak{g} \). With \( u \) as above, set
\[
u_\delta(x) = \delta^{1-n/2}u(\delta^{-1}x),
\]
in exponential coordinates centered at \( p \), and set
\[
Q_\delta(X, Y) = (dL(X)u_\delta, dL(Y)u_\delta)_{L^2}.
\]
The following is readily established.
Lemma 2. There exists \( A \in (0, \infty) \) such that, for all \( X, Y \in \mathfrak{g} \),

\[
\lim_{\delta \to 0} Q_\delta(X, Y) = AQ(X, Y).
\]

Replacing \( u \) by \( A^{-1/2}u \), we have

\[
\lim_{\delta \to 0} Q_\delta(X, Y) = Q(X, Y).
\]

If \( \dim S^2_K T^*_p = \ell = 1 \), we are of course done. If \( \ell > 1 \), we proceed as follows. Let \( A_j, 1 \leq j \leq \ell \) be a basis of \( S^2_K T^*_p \), and pick \( \eta > 0 \) so small that

\[
Q_{\pm j} = Q \pm \eta A_j \quad \text{all belong to} \quad \Gamma.
\]

Then, by arguments as above, pick \( u_{\pm j} \) such that if \( u_{\pm j, \delta}(x) = \delta^{1-n/2} u_{\pm j}(\delta^{-1} x) \),

\[
Q_{\delta,\pm j}(X, Y) = (dL(X)u_{\pm j, \delta}, dL(Y)u_{\pm j, \delta})_{L^2},
\]

then

\[
\lim_{\delta \to 0} Q_{\delta,\pm j}(X, Y) = Q_{\pm j}(X, Y).
\]

Picking \( \delta \) sufficiently small, we have

\[
Q \in \text{convex hull of} \quad \{Q_{\delta,\pm j}\}.
\]

Then we can achieve the metric tensor \( h \) by embedding \( M \) into a finite sum of copies of \( L^2(M, h) \), via (13) with \( u \) replaced by \( u_{\pm j, \delta} \).

**Remark.** Sometimes the procedure described above yields finite dimensional equivariant isometric embeddings. For example, take \( M = \mathbb{R}P^2 \), covered by \( S^2 \). If we let \( \pi \) denote the natural representation of \( SO(3) \) on the space of spherical harmonics on \( S^2 \) corresponding to harmonic polynomials on \( \mathbb{R}^3 \) that are homogeneous of degree 2 (a 5-dimensional space), we get an equivariant isometric embedding of \( \mathbb{R}P^2 \) into \( \mathbb{R}^5 \).

**Reference**