An adic dynamical system related to the Delannoy numbers

Karl Petersen

Department of Mathematics
University of North Carolina at Chapel Hill

Information and Randomness 2010
Outline

Nicomachus and Delannoy Diagrams
  The diagrams
  Formulas for Delannoy numbers
  Adic dynamics
Outline

Nicomachus and Delannoy Diagrams
- The diagrams
- Formulas for Delannoy numbers
- Adic dynamics

Invariant Measures
- Nicomachus adic
- Delannoy adic
Outline

Nicomachus and Delannoy Diagrams
- The diagrams
- Formulas for Delannoy numbers
- Adic dynamics

Invariant Measures
- Nicomachus adic
- Delannoy adic

Total Ergodicity
Outline

Nicomachus and Delannoy Diagrams
  The diagrams
  Formulas for Delannoy numbers
  Adic dynamics

Invariant Measures
  Nicomachus adic
  Delannoy adic

Total Ergodicity

Remarks and Questions
The first part of the Nicomachus diagram
Nicomachus of Gerasa, c. 100
Nicomachus on Music

THE MANUAL OF HARMONICS
OF NICOMACHUS THE PYTHAGOREAN
TRANSLATION AND COMMENTARY
BY FLORA R. LEVIN
Nicole Oresme, c. 1350
Oresme
The Nicomachus diagram with added diagonals
The Delannoy graph
The Delannoy graph made into a Bratteli diagram
Recurrence formula and generating function for Delannoy numbers

\[ D(n, 0) = D(0, n) = 1 \text{ for all } n \geq 0; \]
\[ D(n, k) = 0 \text{ if either } n \text{ or } k < 0; \]
\[ D(n, k) = D(n, k - 1) + D(n - 1, k - 1) + D(n - 1, k) \text{ for all } n, k. \]
Recurrence formula and generating function for Delannoy numbers

\[ D(n, 0) = D(0, n) = 1 \text{ for all } n \geq 0; \]
\[ D(n, k) = 0 \text{ if either } n \text{ or } k < 0; \]
\[ D(n, k) = D(n, k - 1) + D(n - 1, k - 1) + D(n - 1, k) \text{ for all } n, k. \]

\[ \sum_{n,k \geq 0} D(n, k)x^n y^k = \frac{1}{1 - (x + y + xy)} \]
Various formulas for Delannoy numbers

Assuming $n \geq k$,

\[ D(n, k) = \sum_{d=0}^{k} \binom{k}{d} \binom{n + k - d}{k} = \sum_{d=0}^{k} 2^d \binom{n}{d} \binom{k}{d} \]
Various formulas for Delannoy numbers

Assuming $n \geq k$,

\[
D(n, k) = \sum_{d=0}^{k} \binom{k}{d} \binom{n+k-d}{k} = \sum_{d=0}^{k} 2^d \binom{n}{d} \binom{k}{d}
\]

\[
= \sum_{d=0}^{k} \binom{k}{d} \binom{n+d}{k} = \sum_{d=0}^{k} \binom{k}{k-d} \binom{n+d}{k}
\]
Various formulas for Delannoy numbers

Assuming $n \geq k$,

$$D(n, k) = \sum_{d=0}^{k} \binom{k}{d} \binom{n+k-d}{k} = \sum_{d=0}^{k} 2^d \binom{n}{d} \binom{k}{d}$$

$$= \sum_{d=0}^{k} \binom{k}{d} \binom{n+d}{k} = \sum_{d=0}^{k} \binom{k}{k-d} \binom{n+d}{k}$$

$$= \sum_{d=0}^{k} \binom{n+k-d}{k-d} \binom{n}{d} = \sum_{d=0}^{k} \binom{n+d}{d} \binom{n}{k-d}.$$
Asymptotics of Delannoy numbers on the diagonal

\[
D(n, n) \sim (3 + 2\sqrt{2})^n (0.57 \sqrt{n} - 0.067 n^{-3/2} + 0.006 n^{-5/2} + \ldots).
\]
Adic systems

- $X =$ compact metric space of infinite paths $x = (e_i(x))$ that begin at the root
Adic systems

- $X =$ compact metric space of infinite paths $x = (e_i(x))$ that begin at the root
- Incoming edges are ordered
Adic systems

- $X =$ compact metric space of infinite paths $x = (e_i(x))$ that begin at the root
- Incoming edges are ordered
- Two paths are *tail equivalent* if they coincide from some level $N$ on
Adic systems

- $X$ = compact metric space of infinite paths $x = (e_i(x))$ that begin at the root
- Incoming edges are ordered
- Two paths are *tail equivalent* if they coincide from some level $N$ on
- Then $x < y$ if $e_N(x) < e_N(y)$.
Adic systems

- $X$=compact metric space of infinite paths $x = (e_i(x))$ that begin at the root
- Incoming edges are ordered
- Two paths are *tail equivalent* if they coincide from some level $N$ on
- Then $x < y$ if $e_N(x) < e_N(y)$.
- $Tx = \text{smallest } y > x$ (if there is one).
Invariant measures for the Nicomachus adic

Theorem

The only ergodic (invariant probability) measures for the Nicomachus adic dynamical system are the two unique measures supported on the two boundary odometers.
Invariant measures for the Nicomachus adic

Theorem
The only ergodic (invariant probability) measures for the Nicomachus adic dynamical system are the two unique measures supported on the two boundary odometers.
Invariant measures for the Delannoy adic

**Theorem**

The non-atomic ergodic (invariant probability) measures for the Delannoy adic dynamical system are a one-parameter family \( \{ \mu_\alpha : \alpha \in [0, 1] \} \) given by choosing nonnegative \( \alpha, \beta, \gamma \) with \( \alpha + \beta + \gamma = 1 \) and \( \beta \gamma = \alpha \) and then putting weight \( \beta \) on each horizontal edge, weight \( \gamma \) on each vertical edge, and weight \( \alpha \) on each diagonal edge. (The measure of any cylinder set is then determined by multiplying the weights on the edges that define it.)
The Delannoy adic

- Invariant Measures
- Delannoy adic

![Delannoy adic Diagram](image)
Ingredients of the proofs

- Pemantle-Wilson asymptotics for the Delannoy numbers:

\[
D(n, k) \sim \left( \frac{\sqrt{n^2 + k^2} - k}{n} \right)^{-n} \left( \frac{\sqrt{n^2 + k^2} - n}{k} \right)^{-k} \times
\sqrt{\frac{1}{2\pi}} \sqrt{\frac{nk}{(n + k - \sqrt{n^2 + k^2})^2 \sqrt{n^2 + k^2}}},
\]

uniformly if \( n/k \) and \( k/n \) are bounded.
Ingredients of the proofs

- Pemantle-Wilson asymptotics for the Delannoy numbers:

\[
D(n, k) \sim \left(\frac{\sqrt{n^2 + k^2} - k}{n}\right)^{-n} \left(\frac{\sqrt{n^2 + k^2} - n}{k}\right)^{-k} \times \\
\sqrt{\frac{1}{2\pi}} \sqrt{\frac{nk}{(n + k - \sqrt{n^2 + k^2})^2 \sqrt{n^2 + k^2}}},
\]

uniformly if \( n/k \) and \( k/n \) are bounded.

- Collision argument based on recurrence of symmetric random walk in \( \mathbb{Z}^2 \)
Ingredients of the proofs

- **Pemantle-Wilson asymptotics for the Delannoy numbers:**

\[
D(n, k) \sim \left( \frac{\sqrt{n^2 + k^2} - k}{n} \right)^{-n} \left( \frac{\sqrt{n^2 + k^2} - n}{k} \right)^{-k} \times \sqrt{\frac{1}{2\pi}} \sqrt{\frac{nk}{(n + k - \sqrt{n^2 + k^2})^2 \sqrt{n^2 + k^2}}}.
\]

uniformly if \( n/k \) and \( k/n \) are bounded.

- **Collision argument based on recurrence of symmetric random walk in \( \mathbb{Z}^2 \)**

- **X. Méla’s isotropy argument**
Total ergodicity of the Delannoy adics

Theorem

*With respect to each of its ergodic (invariant probability) measures, the Delannoy adic dynamical system is totally ergodic (i.e., has among its eigenvalues no roots of unity besides 1).*
Total ergodicity of the Delannoy adics

**Theorem**

*With respect to each of its ergodic (invariant probability) measures, the Delannoy adic dynamical system is totally ergodic (i.e., has among its eigenvalues no roots of unity besides 1).*

**Theorem**

*For $p$ prime, $r \geq 0$, and $n = 0, 1, 2, \ldots$,

$$D(n, p^r - 1) \equiv_p (-1)^{n \ mod \ p^r}.$$*
The Delannoy graph with a “blocking set”
Remarks and Questions

- For each of its ergodic measures, the Delannoy system is isomorphic to a subshift on \( \{h, d, v\} \), given by concatenating blocks at the vertices, with a shift-invariant measure.
Remarks and Questions

▶ For each of its ergodic measures, the Delannoy system is isomorphic to a subshift on \(\{h, d, v\}\), given by concatenating blocks at the vertices, with a shift-invariant measure.

▶ The subshift is topologically weakly mixing.
Remarks and Questions

- For each of its ergodic measures, the Delannoy system is isomorphic to a subshift on \( \{ h, d, v \} \), given by concatenating blocks at the vertices, with a shift-invariant measure.
- The subshift is topologically weakly mixing.
- With each ergodic measure, the Delannoy adic is loosely Bernoulli.
Remarks and Questions

- For each of its ergodic measures, the Delannoy system is isomorphic to a subshift on \( \{h, d, v\} \), given by concatenating blocks at the vertices, with a shift-invariant measure.
- The subshift is topologically weakly mixing.
- With each ergodic measure, the Delannoy adic is loosely Bernoulli.
- We do not know about limit laws for return times, weak mixing, multiplicity of the spectrum, or joinings.
Remarks and Questions

- For each of its ergodic measures, the Delannoy system is isomorphic to a subshift on \( \{ h, d, v \} \), given by concatenating blocks at the vertices, with a shift-invariant measure.
- The subshift is topologically weakly mixing.
- With each ergodic measure, the Delannoy adic is loosely Bernoulli.
- We do not know about limit laws for return times, weak mixing, multiplicity of the spectrum, or joinings.
- But there is some progress on the complexity and on generalizing these considerations to a class of systems.