Hidden Markov Chains Found Again
(Continuous Images of Measures on Shifts of Finite Type).

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Setting

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$\pi : X \to Y$ 1-block factor map (continuous, shift-commuting)

$\mu = \sigma$-invariant Borel probability measure on $X$

$\nu = \pi \mu$ on $Y$: $\nu(B) = \mu(\pi^{-1}B)$
Sierpinski (or Dean Smith) Carpet
Nonconformal Carpet
Nonconformal Carpet Coded

Figure 0: McMullen-type generalized Sierpinski carpet.
Disallow some transitions 31
More worn carpet
Information Loss

Models information loss, “deterministic noise”: 
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1. Blackwell

\[ a \leftarrow\begin{array}{c}
\uparrow \frac{1}{3} \\
\downarrow \frac{2}{3}
\end{array}\] \quad \begin{array}{c}
\downarrow \frac{2}{3} \\
\uparrow \frac{1}{3}
\end{array}
\rightarrow \quad \begin{array}{c}
\circ \frac{2}{3}
\end{array}\]

\[ b_1 \quad \begin{array}{c}
\downarrow \frac{2}{3}
\end{array}\]

\[ \quad \begin{array}{c}
\downarrow \frac{1}{3}
\end{array}\]

\[ b_2 \quad \begin{array}{c}
\circ \frac{1}{3}
\end{array}\]

\[ a \xrightarrow{\pi} b \]
II. Some Bad Examples

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Image measure is not Markov.
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Image measure is not Markov.

Its entropy is hard to compute.
Markovian

But

\[
\begin{align*}
a & \xleftarrow{1/2} b_1 \xrightarrow{1/2} b_2 \\
\end{align*}
\]
But

So the code is *Markovian*:

some Markov measure maps to a Markov measure.
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Actually *no* Markov lifts to a Markov.

**MPW**: Blackwell-type example of a metrically sofic $\nu$ on $Y$ that is not the finite-to-one image of any Markov measure of any order anywhere.
3. Walters

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2-block recoding:

\[ 00 \rightarrow 01 \]

\[ 11 \rightarrow 10 \]
Finite-to-one map, hence Markovian
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Bernoulli $1/2, 1/2$ measure on $\Sigma_2$ is mapped to itself.
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Bernoulli \( \frac{1}{2}, \frac{1}{2} \) measure on \( \Sigma_2 \) is mapped to itself.
Every Markov \( \nu \) on \( Y \) has a unique relatively maximal lift (in fact unique preimage), which is Markov
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Bernoulli $1/2, 1/2$ measure on $\Sigma_2$ is mapped to itself.

Every Markov $\nu$ on $Y$ has a unique relatively maximal lift (in fact unique preimage), which is Markov

For every ergodic $\nu$ on $Y$, all of $\pi^{-1}\{\nu\}$ consists of relatively maximal measures over $\nu$, all having the same entropy as $\nu$. 
If $p \neq 1/2$, the two measures on $X$ that correspond to $B(p, 1-p)$ and $B(1-p, p)$ both map to $\nu_p$ on $Y$, which is fully supported.
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If \( p \neq 1/2 \), the two measures on \( X \) that correspond to \( B(p, 1 - p) \) and \( B(1 - p, p) \) both map to \( \nu_p \) on \( Y \), which is fully supported.

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Then \( \{ \text{relatively maximal measures over } \nu_p \} = \pi^{-1} \{ \nu_p \} = \text{equilibrium states of } V_p \circ \pi + G \circ \pi = V_p \circ \pi \) (\( G = 0 \)) (Walters)
If $p \neq 1/2$, the two measures on $X$ that correspond to $\mathcal{B}(p, 1-p)$ and $\mathcal{B}(1-p, p)$ both map to $\nu_p$ on $Y$, which is fully supported.

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Then \{relatively maximal measures over $\nu_p$\} = $\pi^{-1}\{\nu_p\}$ = equilibrium states of $V_p \circ \pi + G \circ \pi = V_p \circ \pi$ ($G = 0$) (Walters)

So this potential function $V_p \circ \pi$ has many equilibrium states.
Entropy increase

4. Marcus-P-Williams

For $\Sigma_3 \to \Sigma_2$ as above, there is a 2-step Markov $\mu$ that projects to $\pi \mu = B(1/2, 1/2)$
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while its 1-step Markovization $\mu^1 \rightarrow \pi \mu^1 \neq B(1/2, 1/2)$. 
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while its 1-step Markovization $\mu^1 \rightarrow \pi \mu^1 \neq B(1/2, 1/2)$.

Thus $h(\mu^1) > h(\mu)$, while $h(\pi \mu^1) < h(\pi \mu)$. 
III. Background Concepts

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1. Relative pressure

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$$P(\pi, V)(y) = \limsup_{n \to \infty} \frac{1}{n} \log \left[ \sum_{x \in D_n(y)} \exp \left( \sum_{i=0}^{n-1} V(\sigma^i x) \right) \right].$$
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For all $y \in Y$,

$$P(\pi, 0)(y) = \limsup_{n \to \infty} \frac{1}{n} \log |D_n(y)|$$

(with $V \equiv 0$).
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$$\int P(\pi, V) d\nu = \sup \left\{ h(\mu) + \int V d\mu \right\} - h(\nu).$$

In particular, for a fixed $\nu \in M(Y)$,

$$\sup \{ h_\mu(X|Y) : \pi \mu = \nu \} = \sup \{ h(\mu) - h(\nu) : \pi \mu = \nu \} = \int_Y P(\pi, 0) d\nu.$$
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Relative Pressure—2

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**Theorem.** For each \( n = 1, 2, \cdots \) and \( y \in Y \) let \( E_n(y) \) be a set consisting of exactly one point from each nonempty cylinder \([x_0 \cdots x_{n-1}] \subset \pi^{-1}[y_0 \cdots y_{n-1}]\).

Then for each \( V \in C(Y) \),

\[
P(\pi, V)(y) = \limsup_{n \to \infty} \frac{1}{n} \log \left[ \sum_{x \in E_n(y)} \exp \left( \sum_{i=0}^{n-1} V(\sigma^i x) \right) \right]
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a.e. with respect to every invariant measure on \( Y \).
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Then for each $V \in C(Y)$,

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Thus, we obtain the value of $P(\pi, V)(y)$ a.e. with respect to every invariant measure on $Y$ if we delete from the definition of $D_n(y)$ the requirement that $x \in \pi^{-1}(y)$. 

Ohio State, 4/12/07 – p.20/46
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For $\mu$ relatively maximal over $\nu$,

$$h_\mu(X|Y) = \int_Y \lim_{n \to \infty} \frac{1}{n} \log |\pi^{-1}[y_0 \ldots y_{n-1}]| \, d\nu(y).$$
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Ohio State, 4/12/07 – p.21/46
2. Compensation function (Boyle-Tuncel, Walters)
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A continuous function $F : X \to \mathbb{R}$ such that

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_Idea:_ Because $\pi : \mathcal{M}(X) \to \mathcal{M}(Y)$ is many-to-one, we always have

$$P_Y(V) = \sup \{ h_\nu(\sigma) + \int_Y V \, d\nu : \nu \in \mathcal{M}(Y) \}$$

$$\leq \sup \{ h_\mu(\sigma) + \int_X V \circ \pi \, d\mu : \mu \in \mathcal{M}(X) \}.$$
Compensation Functions

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$$\leq \sup \left\{ h_\mu(\sigma) + \int_X V \circ \pi \, d\mu : \mu \in \mathcal{M}(X) \right\}.$$ 

$F$ takes into account, for all potential functions $V$ on $Y$ at once, the extra freedom, information, or free energy available in $X$ as compared to $Y$ because of the ability to move around in fibers over points of $Y$. 

Properties of Compensation Functions

For SFT’s $X$ and $Y$, there is always a compensation function.
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If $G \in \mathcal{F}(Y)$ (Walters class), then $G \circ \pi$ is a (saturated) compensation function if and only if there is $c > 0$ such that

$$\frac{1}{c} \leq e^{S_n G(y)} |\pi^{-1}[y_0 \ldots y_{n-1}]| \leq c \text{ for all } y, n.$$
3. Example of a Compensation Function

\[ a \rightarrow b_1 \rightarrow b_2 \rightarrow a \]

\[ \pi \rightarrow a \leftarrow b \]
3. Example of a Compensation Function

\[ G(y) = \begin{cases} 
-\log 2 & \text{if } y = .a \ldots \\
0 & \text{if } y = .b \ldots 
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When in \( y \) we see \( ab^{k_1}ab^{k_2}a \ldots ab^{k_r}a \), multiply in: 1 at each \( b \), 2 at each \( a \).
IV. Relatively Maximal Measures

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$(\mathcal{B}(p, 1 - p), \mathcal{B}(1 - p, p))$.

**Theorem** (Shin). *Suppose that $\nu \in \mathcal{E}(Y)$ and $\pi \mu = \nu$. Then $\mu$ is relatively maximal over $\nu$ if and only if there is $V \in \mathcal{C}(Y)$ such that $\mu$ is an equilibrium state of $V \circ \pi$.**
Lifting Markov Measures

If there is a locally constant saturated compensation function $G \circ \pi$, then every Markov measure on $Y$ has a unique relatively maximal lift, which is Markov, because then the relatively maximal measures over an equilibrium state of $V \in \mathcal{C}(Y)$ are the equilibrium states of $V \circ \pi + G \circ \pi$ (Walters).
Lifting Markov Measures

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- Further, $\mu_X$ is the unique equilibrium state of the potential function 0 on $X$; and the relatively maximal measures over $\mu_Y$ are the equilibrium states of $G \circ \pi$. 

Ohio State, 4/12/07 – p.26/46
2. An answer
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**Theorem (P-Quas-Shin).** For each ergodic $\nu$ on $Y$, there are only a finite number of relatively maximal measures over $\nu$.

In fact, the number of ergodic invariant measures of maximal entropy in the fiber $\pi^{-1}\{\nu\}$ is at most

$$N_\nu(\pi) = \min\{|\pi^{-1}\{b\}| : b \in \mathcal{A}(Y), \nu[b] > 0\}.$$
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**Theorem** (P-Quas-Shin). For each ergodic $\nu$ on $Y$, any two distinct ergodic measures on $X$ of maximal entropy in the fiber $\pi^{-1}\{\nu\}$ are relatively orthogonal.
Relatively Independent Joining

For $\mu_1, \ldots, \mu_n \in \mathcal{M}(X)$ with $\pi \mu_i = \nu$ for all $i$, their relatively independent joining $\hat{\mu}$ over $\nu$ is defined by:
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if $A_1, \ldots, A_n$ are measurable subsets of $X$ and $\mathcal{F}$ is the $\sigma$-algebra of $Y$, then

$$\hat{\mu}(A_1 \times \ldots \times A_n) = \int_Y \prod_{i=1}^{n} \mathbb{E}_{\mu_i}(1_{A_i} \mid \pi^{-1} \mathcal{F}) \circ \pi^{-1} d\nu.$$
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There is zero probability of coincidence.
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Proof of First Theorem

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Suppose that we have \( n > N_{\nu}(\pi) \) ergodic measures \( \mu_1, \ldots, \mu_n \) on \( X \), each projecting to \( \nu \) and each of maximal entropy in the fiber \( \pi^{-1}\{\nu\} \).
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Form the relatively independent joining \( \hat{\mu} \) on \( X^n \) of the measures \( \mu_i \) as above.
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Form the relatively independent joining \( \hat{\mu} \) on \( X^n \) of the measures \( \mu_i \) as above.

Let \( b \) be a symbol in the alphabet of \( Y \) such that \( b \) has \( N_\nu(\pi) \) preimages \( a_1, \ldots, a_{N_\nu(\pi)} \) under the block map \( \pi \).
Since $n > N_\nu(\pi)$, for every $\hat{x} \in \phi^{-1}[b]$ there are $i \neq j$ with $(p_i\hat{x})_0 = (p_j\hat{x})_0$. 
Pigeonholing

Since $n > N_{\nu}(\pi)$, for every $\hat{x} \in \phi^{-1}[b]$ there are $i \neq j$ with $(p_i \hat{x})_0 = (p_j \hat{x})_0$.

At least one of the sets $S_{i,j} = \{ \hat{x} \in X^n : (p_i \hat{x})_0 = (p_j \hat{x})_0 \}$ must have positive $\hat{\mu}$-measure,
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and then also

$$(\mu_i \otimes_\nu \mu_j)\{(u, v) \in X \times X : \pi u = \pi v, u_0 = v_0\} > 0,$$

contradicting relative orthogonality.
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and then also

$$\left( \mu_i \otimes_\nu \mu_j \right) \{(u, v) \in X \times X : \pi u = \pi v, u_0 = v_0 \} > 0,$$

contradicting relative orthogonality.

(If you have more measures than preimage symbols, two of those measures have to coincide on one of the symbols: with respect to each measure, that symbol a.s. appears infinitely many times in the same place.)
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We concatenate words from the two processes, using the fact that the two measures are supported on sequences that agree infinitely often. Since $X$ is a 1-step SFT, we can switch over whenever a coincidence occurs.
Weaving In More Entropy

Let \( w \in B(1/2, 1/2) \), symbols 1 and 2.
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Suppose \( u_s = v_s, u_t = v_t, u_r = v_r, \ldots \).

\[ u = \ldots u_s \ldots u_{t-1} u_t \ldots u_{r-1} u_r \ldots \]

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\begin{align*}
    u &= \ldots u_s u_s+1 \ldots u_{t-1} u_t \ldots u_r-1 u_r \ldots \\
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\]

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\]

\[
\pi_3(u, v, w) = \ldots (u_s u_s+1 \ldots u_{t-1})(v_t v_t+1 \ldots v_r-1)(u_r u_r+1 \ldots) \ldots
\]
Why Does It Go Up?

The switching increases entropy.
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The argument uses

- strict concavity of $-t \log t$
- lots of calculations with conditional expectations.
V. Recognizing the hidden Markov measures

1. Identify images of Markov measures (metrically sofic, hidden Markov).
   Heller, Robertson, Furstenberg, Binkowska-Kaminski.
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\[ \phi(\epsilon) = 1, \phi(y_1 \ldots y_n) = \nu[y_1 \ldots y_n] \] extends to linear functional on \( \mathcal{A} \).
Metrically Sofic vs. Finitary

\[ \mathcal{N} = \text{largest left ideal in kernel}(\phi) = \{ a \in A : \phi(wa) = 0 \text{ for all } w \in A^{*} \} \]
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Furstenberg: Characterization of metrically sofic in terms of finite-dimensionality of a related algebra by a different left ideal.
2. Formal languages characterization

Kleene, Schützenberger, Hansel-Perrin, etc.
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Shift-invariant $\mu$ on $A^N$ is a function $A^* \rightarrow \mathbb{R}_+$ or a formal series

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$$(s_1 s_2)(w) = \sum_{u, v \in A^*, uv = w} s_1(u) s_2(v).$$
Module structure

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A submodule \( M \subset \mathcal{F}(A) \) is \textit{stable} if \( w^{-1} M \subset M \) for all \( w \in A^* \).
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3. $F$ is *rational*—can be obtained by starting with a finite set of polynomials (finitely-supported series) and applying finitely many rational operations: sum, product, multiplication by $\mathbb{R}_+$, and $f \to f^* = \sum_{n=0}^\infty f^n$ for $f(\epsilon) = 0$. 
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4. $\mu$ is the image under a 1-block map of a 1-step Markov measure.
VI. Measures of Maximal Hausdorff Dimension

Find measures of maximal Hausdorff dimension for expanding (not necessarily conformal) maps on manifolds restricted to compact invariant sets.
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**Theorem** (Shin). *If there is a saturated compensation function $G \circ \pi$ with $G \in C(Y)$, then the measures which maximize the weighted entropy functional

$$\phi_\alpha(\mu) = \frac{1}{\alpha + 1} \left[ h(\mu) + \alpha h(\pi\mu) \right]$$

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So in some cases they are unique, Bernoulli, etc.
Ledrappier-Young:

\[
\text{HD}(\mu) = \frac{h_\mu(f)}{\lambda_1^\mu(f)} + \left[ \frac{1}{\lambda_2^\mu(f)} - \frac{1}{\lambda_1^\mu(f)} \right] h_{\pi,\mu}(f_*)
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Nonconformal Carpet
Nonconformal Carpet Coded
Disallow some transitions 31
More worn carpet
A candidate for nonuniqueness

\[\pi(1) = 1, \pi(2) = \pi(3) = 2, \pi(4) = \pi(5) = 3.\]
VII. Some Questions
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2. Construction of relatively maximal measures. Our proof uses relative \( g \)-functions and shows that the measures are relatively Markov:

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\alpha \perp_{\sigma^{-1}\alpha \vee \pi^{-1}B(Y)} \alpha_2^\infty, \quad H_\mu(\alpha|\alpha_1^\infty \vee \pi^{-1}B_Y) = H_\mu(\alpha|\sigma^{-1}\alpha \vee \pi^{-1}B_Y).
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Construct them as weak* limits of well-distributed measures on periodic orbits?