Some results and systems related to the super-$K$ property

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Introduction

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Stationary processes

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The time-0 partition of \(\Omega\) is a generator for the m.p. system \((\Omega, \mu, \sigma)\).
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It is the intersection of the algebras generated by all the cylinder sets $\{ T^n x \in a_{i_n}, \ldots, T^{n+j} x \in a_{i_{n+j}} : n, j \geq 0 \}$. 
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When $\alpha$ is a generator, $\mathcal{T}^+(\alpha)$ is the Pinsker algebra of $(X, \mathcal{B}, \mu, T)$. 
The $K$ property

A system $(X, B, \mu, T)$ is $K$ (has the Kolmogorov property) if there is a generator $\alpha$ such that $T^+(\alpha)$ is trivial, i.e. consists only of sets of measure 0 or 1.
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Therefore, for any partition \( \alpha \), \( \mathcal{T}^- (\alpha) \) is trivial if and only if \( \mathcal{T}^+ (\alpha) \) is trivial (because for any \( \beta \leq \alpha \), \( h_\mu (T, \beta) = h_\mu (T^{-1}, \beta) \)).
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it can be recoded to an isomorphic process that is **2-sided deterministic**: if the remote past and remote future can communicate and cooperate, they can determine what is going on near the present.
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The ordinary tail fields are the fields of saturated sets for the Borel equivalence relation under \textit{finite coordinate changes}. Now consider some finer tail fields that allow for saving a limited amount of information as the present recedes into the distance.

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$$\psi^m_n(x) = \psi(T^m x) \cdots \psi(T^n x), \text{ in abelian case } \sum_{k=m}^{n} \psi(T^k x)$$
E.g., if $\psi : \Omega \to \mathbb{Z}^d$ is defined by $\psi(\omega) = e_i \in \mathbb{Z}^d$ if $\omega_0 = a_i$, then $\psi_{n-1}^0(\omega)$ gives in each entry $i$ the number of times that $a_i$ appears in the first $n$ entries in $\omega$. 
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$$F^\pm(\alpha) = \bigcap_{n \geq 0} B\{\psi_j^j : j \geq 0\}$$
Equivalence relations

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\[ \omega \sim \omega' \text{ if and only if } \omega, \omega' \text{ differ in only finitely many coordinates and } \sum_{0 \text{ or } -\infty}^{\infty} [\psi(\sigma^k \omega) - \psi(\sigma^k \omega')] = 0. \]
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When \( \psi \) is the symbol-counting cocycle, these equivalence relations are the orbit relation of the group of \textit{finite coordinate permutations}. 
Relations among fields

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Note that $F^+\psi(\alpha) \supset T^+$ and $F^-\psi(\alpha) \supset T^-$.

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and sometimes $F_{\psi}^\pm(\alpha) \not\supset F_{\psi}^+(\alpha), F_{\psi}^-(\alpha)$. 
We say that a process \((\alpha, T)\) is super-\(K^+\) if \(\mathcal{F}_\psi^+(\alpha)\) is trivial, with \(\psi\) the symbol-counting cocycle.

Super-\(K\)

For example, Bernoulli processes are super-\(K^+\), super-\(K^-\), and super-\(K^\pm\) (Hewitt-Savage, 1988). There are also such results for the 2-sided case by Blackwell-Freedman for Markov processes, Georgii for Gibbs states, Berbee-den Hollander for integer-valued processes, and others.
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We can have $\mathcal{F}^+_{\psi}(\alpha)$ trivial and find a refinement $\beta \geq \alpha$ with $\mathcal{F}^+_{\psi}(\beta)$ nontrivial (in fact equal to $\mathcal{B}$).
Triviality of two-sided fine tails

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Interpretation: History is useless and science is impossible.

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Super-$K^+$ generators

**JPT-KP, 2004:** If an ergodic system $(X, \mathcal{B}, \mu, T)$, with generator $\alpha$, is isomorphic to the direct product of a positive-entropy Bernoulli system $(\mathcal{B}, \sigma)$ and some other system $(Y, S)$, then there is a generator $\beta$ for $(X, \mathcal{B}, \mu, T)$ such that $\mathcal{F}^+(\beta) = \mathcal{T}^+(\beta) = \mathcal{T}^+$. Consequently, every $K$ process with a direct Bernoulli factor has a super-$K^+$ generator (since then $\mathcal{T}^+$, the Pinsker algebra, is trivial). The idea of the proof is to construct a partition $\beta$ with $F^+(\beta) \subset T^+(\beta)$, so that no new information is provided by counting $\beta$-symbols.
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$$|H(\beta^K_{-k}|\mathcal{P}(T)) + H(\beta_n^\infty|\mathcal{P}(T)) - H(\beta^k_{-k} \lor \beta_n^\infty|\mathcal{P}(T))| < \epsilon$$
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This implies that if for all $k, \epsilon$ there is $N$ such that if $n \geq N$ then

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This implies that if for all $k, \epsilon$ there is $N$ such that if $n \geq N$ then

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then $\beta^{-\infty} \perp_{\mathcal{P}(T)} \mathcal{F}^+(\beta)$,
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then \(\beta^{\infty}_{-\infty} \perp_{\mathcal{P}(T)} \mathcal{F}^+(\beta)\),

and hence \(\mathcal{F}^+(\beta) \subset \mathcal{P}(T) = \mathcal{T}^+(\beta)\).
Recoding

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We use a special marker block $W = 1^{tq}2^{tq} \cdots |\rho|^{tq}$. Appearances of $W$ in sequences $\omega \in B$ cut $\mathbb{Z}$ into marked and free intervals.
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On each marked interval, where $W$ appears in $B$, we do not change the $B$ coding, but we change the $Y$ coding so that each $\beta_0$ symbol appears the same number of times.
Coding free intervals

On each free interval, we recode the $\gamma \times \rho$ name by cutting into subintervals and using permutations of a string of all $\beta_0 \times \rho$ symbols (one of each symbol), plus we add one extra symbol, which depends only on the length of the free interval.
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Now the $\beta$-symbol count across a union of free and marked intervals is constant on the marked intervals and a function of $B$, hence asymptotically adds no information to the ordinary tail.

If the count ends inside a marked or free interval, with high probability we have a bounded translate of a count across a complete union of intervals, so it is not too different.
Super-K generators for K systems

JPT, 2008: If $(X, \mathcal{B}, \mu, T)$ is ergodic, finite entropy, and weak Pinsker (for every $\epsilon > 0$, $X \approx B \times Y$ with $B$ Bernoulli and $h(Y) < \epsilon$), then there is a finite generator $\alpha$ with $\mathcal{F}_\psi^\pm(\alpha) = \mathcal{P}(T)$. 
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**Corollary:** If $(X, B, \mu, T)$ is $K$, it has a super-$K^{\pm}$ generator.
Odometers

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Invariant measures

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The measure of maximal entropy on $\Sigma_M$ assigns pretty much the same measure of all cylinder sets of a fixed length.
Graphs for the fine tail fields

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with $x = (x_k) \in A^\mathbb{N}$ giving the *edge labels* of a path in $\mathbb{Z}^d$:

$x_k$ labels the edge from $s_{k-1}(x)$ to $s_k(x)$. 
Adic transformations

The fine tail equivalence relation on $A^\mathbb{N}$ has $x \sim y$ if there is $N$ such that $s_n(x) = s_n(y)$ for all $n \geq N$: the paths are cofinal—eventually coincide.
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The equivalence classes are the orbits of any adic (Bratteli-Vershik) transformation that is defined on most of $A^\mathbb{N}$ once the incoming edges to each vertex are given a total order.
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The invariant sets of each such adic transformation are $\mathcal{F}_\psi^+(\alpha)$.

Thus these systems visually present the future fine tail fields—we can see the corresponding equivalence relations.
The Pascal walk
The Delannoy walk
The Delannoy graph
Xavier Méla’s $X_3$ walk
Systems that present tail fields

Xavier Méla’s $X_3$
Frick’s $2x + 1$ walk
Frick’s $2x + 1$ system
A walk with 4 vectors
An isotropic adic system based on a walk with 4 vectors
Ordering incoming edges to define the transformation
Some questions about the systems

Ergodic measures

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Expansiveness and complexity

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For the Delannoy, $p(n) \sim n^3/24$ (Frick).
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**Varying orders**

These properties depend on the choice of *order* of the incoming edges.

What is the maximum complexity over all possible orders?

What is the expected complexity if the orders at the vertices are chosen independently according to a fixed Bernoulli measure?

It seems that for the Pascal, for every order $p(n)$ is asymptotically no more than $n^{5/3}$. 
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When the simple walks that give rise to isotropic adic systems are allowed to evolve according to reinforcement schemes, even more interesting systems arise, for example the *Eulerian system* studied by Frick-Keane-KP-Salama, Frick-KP, KP-Varchenko, Gnedenin-Olshanski.
Some questions about the systems

The Eulerian adic
Some questions about the systems

The Eulerian adic with path counts
Some questions about the systems

C* algebra connections

Study of such systems leads to interesting combinatorial questions and connections with C* algebras and group representations (Kerov).
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**$C^*$ algebra connections**

Study of such systems leads to interesting combinatorial questions and connections with $C^*$ algebras and group representations (Kerov).

Indeed, the Pascal graph is an example of an AF $C^*$ algebra (the “CCR" algebra) in Bratteli’s 1972 paper.