Some Sturmian Symbolic Dynamics

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Overview

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- nonperiodic Sturmian 0,1 sequences
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- nonperiodic \textit{Sturmian} $0,1$ sequences

- and also \textit{periodic} “Sturmian” sequences, involving
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- and also periodic “Sturmian" sequences, involving
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- ideals in $C^*$ algebras.
Characterizations of nonperiodic Sturmian sequences

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- **Balanced**: For any two blocks \(u, v\) of the same length, \(|u|_1 - |v|_1| \leq 1\).
- **Codings of irrational rotations**: There are \(x\) and irrational \(\theta\) such that for all \(n\), \(\omega(n) = 1_{[1-\theta, 1]}(x + n\theta)\) or for all \(n\), \(\omega(n) = 1_{(1-\theta, 1]}(x + n\theta)\). (A *Sturmian system* is then the closure of the orbit of \(\omega\) under the shift. It is minimal, uniquely ergodic, and isomorphic to the irrational translation.)
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- Codings of irrational rotations: There are $x$ and irrational $\theta$ such that for all $n$, $\omega(n) = 1_{[1 - \theta, 1)}(x + n\theta)$ or for all $n$, $\omega(n) = 1_{(1 - \theta, 1]}(x + n\theta)$. (A Sturmian system is then the closure of the orbit of $\omega$ under the shift. It is minimal, uniquely ergodic, and isomorphic to the irrational translation.)
- Staircase coding: There are $x$ and irrational $\theta$ such that for all $n$, $\omega(n) = \lfloor x + (n + 1)\theta \rfloor - \lfloor x + n\theta \rfloor$ or for all $n$, $\omega(n) = \lceil x + (n + 1)\theta \rceil - \lceil x + n\theta \rceil$. (Look at jumps between lattice points above or below line through origin of slope $\theta$. Get jump (of floor) when $n\theta$ is in $[1 - \theta, 1)$.)
Upper and lower staircase codings, by jumps

1 0 1 0 1 0 1 0 0

0 0 1 0 1 0 1 0 0

0 0 1 0 1 0 1 0 1
Farey, Stern-Brocot, or C. Haros Diagram
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- I learned about the Farey shift from papers of Jeff Lagarias and about this “Farey diagram with memory" from Oliver Jenkinson and Florin Boca.
Ordinary and intermediate continued fractions

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Ordinary continued fractions for $x = [a_1, a_2, \ldots]$:

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\( x = [2, 3, 2, 4, \ldots] \approx 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{7}{16}, \frac{10}{23}, \frac{17}{39}, \frac{24}{55}, \frac{31}{71}, \ldots \)
Farey Diagram of Blocks
The word at position corresponding to fraction $\frac{p}{p+q}$ has $p$ 1’s and $q$ 0’s (hence length $p+q$).
Balanced periodic sequences

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- Every balanced word of length $p + q$ with exactly $p$ 1’s is a rotation of the word in the Farey diagram that corresponds to $\frac{p}{p+q}$. There are exactly $p + q$ of them.
- Infinite nonperiodic Sturmian sequences are found as “ends” of infinite paths in the Farey diagram.
Farey Diagram of Blocks

0

0 0001 001 001001 00101 0010101 01 0101011 01011 01011011 011 0110111 0111 01111 1

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**Times 2 map**

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The following are equivalent:

- Two integer vectors $(q, p)$ and $(q', p')$ span the integer lattice $\mathbb{Z}^2$.
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1. Two integer vectors \((q, p)\) and \((q', p')\) span the integer lattice \( \mathbb{Z}^2 \).
2. \( pq' - qp' = \pm 1 \).
3. The parallelogram spanned by the vectors \((q, p)\) and \((q', p')\) has no point of the integer lattice \( \mathbb{Z}^2 \) in its interior.
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- $pq' - qp' = \pm 1$.
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Parallelogram containing no interior lattice points

(2,1)  (5,2)
First part of coding of (7,3) follows (5,2)
Last part of coding of (7,3) follows translate of (2,1)
Bratteli Diagrams

Infinite downward directed graphs

Level \((n)\)

\[ k = 0 : k = 1 \]
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Level \((n)\)

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
k = 0 \quad : \quad k = 1
\end{array}
\]
Bratteli Diagrams

- Infinite downward directed graphs
- Vertices, denoted by \((n, k)\), are partitioned into levels, \(V_n\)
- Edges connect vertices in consecutive levels
- **Incidence matrices** describe the number of edges connecting levels \(n\) and \(n + 1\)

\[
\begin{align*}
\text{Level } (n) & \\
0 & \quad A_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \\
1 & \\
2 & A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \\
3 & A_3 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\end{align*}
\]

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The Path Space

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- For $x = x_0 x_1 x_2 \cdots \in X$ denote by $x_i$ the $i$’th edge of $x$, which connects a vertex in level $i$ to a vertex in level $i + 1$. 
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- For $x = x_0 x_1 x_2 \cdots \in X$ denote by $x_i$ the $i$’th edge of $x$, which connects a vertex in level $i$ to a vertex in level $i + 1$.
- $X$ is a compact metric space with metric given by: For $x, y \in X$, $d(x, y) = 2^{-i}$ where $i = \inf\{j | x_j \neq y_j\}$.
Edge ordering yields a partial order on the set of paths
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Define $y > x$ if $y_n > x_n$ the last time they differ.
The adic transformation

\[ T : X \to X, \quad T x = \text{smallest } y > x \text{ if there is one.} \]
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The adic transformation

\( T : X \rightarrow X, \ T x = \text{smallest } y > x \text{ if there is one.} \)
The adic transformation

\[ T : X \rightarrow X, \quad T_x = \text{smallest } y > x \text{ if there is one.} \]

Thanks to Sarah Bailey Frick for this animated introduction.
Ideals in AF algebras

An AF algebra $\mathcal{A}$ is the closure of the increasing union of finite-dimensional algebras $\mathcal{A}_n$, each the direct sum of the matrix algebras at level $n$ of the Bratteli diagram.
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A (two-sided norm-closed) ideal in $\mathcal{A}$ is determined by a subdiagram $\Lambda$ with the following two properties:

**Closed under successors:** If $(n, i) \in \Lambda$ and $(n, i) \prec (n + 1, j)$, then $(n + 1, j) \in \Lambda$;

**Closed under ancestors:** If $(n + 1, j) \in \Lambda$ for all $j$ such that $(n, i) \prec (n + 1, j)$, then $(n, i) \in \Lambda$. 
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\[(n, i)\] → \[(n + 1, j)\]

\[(n + 1, j_1)\] → \[(n, i)\]
\[(n + 1, j_2)\] → \[(n, i)\]
\[\ldots\] → \[(n, i)\]
\[(n + 1, j_r)\] → \[(n, i)\]
\[\ldots\] → \[(n, i)\]
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\[(n + 1, j_1) \quad (n + 1, j_2) \quad \cdots \quad (n + 1, j_r) \quad \cdots \quad (n + 1, j_s)\]
A (two-sided norm-closed) ideal $I \subseteq \mathcal{A}$ is **primitive** if and only if there are not ideals $I_1, I_2$ in $\mathcal{A}$, both different from $I$, such that $I = I_1 \cap I_2$. 
A (two-sided norm-closed) ideal $I \subset \mathcal{A}$ is primitive if and only if there are not ideals $I_1, I_2$ in $\mathcal{A}$, both different from $I$, such that $I = I_1 \cap I_2$.

In terms of the diagram $\Lambda$ determining $I$, this means that if $(n, i), (m, j) \notin \Lambda$, then there are $p \geq n, m$ and $(p, k) \notin \Lambda$ such that $(n, i) \uparrow (p, k)$ and $(m, j) \uparrow (p, k)$.
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In terms of the diagram \( \Lambda \) determining \( I \), this means that if \((n, i), (m, j) \notin \Lambda\), then there are \( p \geq n, m \) and \((p, k) \notin \Lambda\) such that \((n, i) \not\prec (p, k)\) and \((m, j) \not\prec (p, k)\).
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$$\begin{array}{c}
(n, i) \\
\downarrow \\
\searrow \ \\
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\uparrow \\
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\end{array}$$
Primitive ideals in $\mathcal{A}$

A (two-sided norm-closed) ideal $I \subset \mathcal{A}$ is primitive if and only if there are not ideals $I_1, I_2$ in $\mathcal{A}$, both different from $I$, such that $I = I_1 \cap I_2$.

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Ideals and invariant sets

Ideals of an AF algebra correspond to closed invariant sets of the Bratteli-Vershik transformation on the path space of the diagram.
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- Ideals of an AF algebra correspond to closed invariant sets of the Bratteli-Vershik transformation on the path space of the diagram.

- Primitive ideals of an AF algebra correspond to topologically transitive closed invariant sets of the Bratteli-Vershik transformation on the path space of the diagram.
Half of Farey diagram
Subadics of the Farey diagram

Regard the Farey diagram as a Bratteli-Vershik diagram, with the adic transformation on the metric space of infinite paths.
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For rational rotation number $\theta$ (the frequency of 1’s), there are 3 topologically transitive subadics, each containing a unique minimal set, isomorphic to a translation on a finite cyclic group.
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For irrational rotation number $\theta$, there is a single minimal subadic, isomorphic to the Sturmian system with that number.
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These closed invariant subsets correspond to primitive ideals of the approximately finite $C^*$ algebra determined by the Farey Bratteli diagram.
Farey diagram again
The orbit of $\frac{1}{3} \sim 001001001001 \cdots = \frac{1}{7}$
Mapping $1/3 \sim 001001001001 \cdots = 1/7$
Mapping \(1/3 \sim 001001001001 \cdots = 1/7\)
Mapping $1/3 \sim 001001001001 \cdots = 1/7$
An orbit forward asymptotic to that of $\frac{1}{3} \sim \frac{1}{7}$
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The diagram (non-red) of one ideal for $\frac{1}{3} \sim 001 \sim \frac{1}{7}$
The diagram (non-red) of another ideal for $1/3 \sim 001 \sim 1/7$
Ideal and orbit closure for $\theta = [2, 3, 2, 4, \ldots]$
$\beta$-shifts

Fix $\beta > 1$, let $d = \lceil \beta \rceil$, and $D = \{0, 1, \ldots, d - 1\}$. 
\section*{\textbf{$\beta$-shifts}}

- Fix $\beta > 1$, let $d = \lceil \beta \rceil$, and $D = \{0, 1, \ldots, d - 1\}$.
- Let $\Sigma^+_{\beta} \subset D^N$ denote the closure of the set of all greedy expansions base $\beta$ of all $x \in [0, 1]$,

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots$$
\( \beta \)-shifts

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- \( (\Sigma_\beta^+, \sigma) \) is a symbolic coding (lift) of the \( \beta \)-transformation \( T_\beta : [0, 1] \to [0, 1] \) defined by \( T_\beta x = \beta x \mod 1 \).
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- \((\Sigma^+_\beta, \sigma)\) is a symbolic coding (lift) of the \(\beta\)-transformation \(T_\beta : [0, 1] \rightarrow [0, 1]\) defined by \(T_\beta x = \beta x \mod 1\).
- If the expansion \(a_1a_2\ldots\) of 1 base \(\beta\) is nonterminating, we put \(e_\beta(1) = a_1a_2\ldots\).
\(\beta\)-shifts

- **Fix** \(\beta > 1\), let \(d = \lceil \beta \rceil\), and \(D = \{0, 1, \ldots, d - 1\}\).
- Let \(\Sigma^+_{\beta} \subset D^\mathbb{N}\) denote the closure of the set of all greedy expansions base \(\beta\) of all \(x \in [0, 1]\),

\[
x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots
\]

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- If the expansion \(a_1a_2\ldots\) of 1 base \(\beta\) is nonterminating, we put \(e_{\beta}(1) = a_1a_2\ldots\).
- Otherwise there is a first \(i\) for which \(T_{\beta}^i1 = n \in \mathbb{N}\), and then we put \(e_{\beta}(1) = [a_1 \ldots a_{i-1}(n - 1)]^\infty\).
A sequence $a = a_1 a_2 \cdots \in D^\mathbb{N}$ is in $\Sigma^+_{\beta}$ if and only if $\sigma^k x \leq e_{\beta}(1)$ for all $k \geq 0$. 
\textbf{\(\beta\)-shifts and lexicographic order}

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- A sequence \(a = a_1a_2 \cdots \in D^\mathbb{N}\) is \(e_\beta(1)\) for some \(\beta\) if and only if it dominates all its shifts: \(a \geq \sigma^k a\) for all \(k \geq 0\) (Parry, 1960).
A doubly lexicographic map of the interval

Consider now a Sturmian symbolic dynamical system with rotation number $\theta$. It also has a lexicographically *maximal* element.
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Since $M(\theta)$ is lexicographically maximal in a subshift, it dominates all its shifts and hence is the expansion $e_\beta(1)$ of 1 base $\beta$ for some $\beta = \beta(\theta) \in (1, 2)$. 
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We define $L : (0, 1] \rightarrow (0, 1]$ by $L(\theta) = \beta(\theta) - 1$. 
The map $L$

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Queen Mary, U. of London, June 22, 2009 – p.36/37
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- For $\theta = 2/3$, the minimal element is $011011011\ldots$, the maximal element is $M(\theta) = 110110110\ldots = (1_{[0,2/3]}(n \times 1/3))$, and $\beta(\theta)$ is the reciprocal of the solution of $1 = (x + x^2)(1 + x^3 + \ldots)$, i.e. $1 = x + x^2 + x^3$. 
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- I recently found out that in recent papers and preprints, DoYong Kwon has defined and studied essentially the same function.