# Propagation of uncertainty

From Wikipedia, the free encyclopedia

Jump to: navigation, search

In statistics, **propagation of uncertainty** (or **propagation of error**) is the effect of variables' uncertainties (or errors) on the uncertainty of a function based on them. When the variables are the values of experimental measurements they have uncertainties due to measurement limitations (e.g. instrument precision) which propagate to the the combination of variables in the function.

The uncertainty is usually defined by the absolute error. Uncertainties can also be defined by the relative error $\Delta x/x$, which is usually written as a percentage.

Most commonly the error on a quantity, $\Delta x$, is given as the standard deviation, σ, . Standard deviation is the positive square root of variance, $\sigma^2$. The value of a quantity and its error are often expressed as $x \pm \Delta x$. If the statistical probability distribution of the variable is known or can be assumed, it is possible to derive confidence limits to describe the region within which the true value of the variable may be found. For example, the 68% confidence limits for a variable belonging to a normal distribution are ± one standard deviation from the value, that is, there is a 68% probability that the true value lies in the region $x \pm \sigma$.

If the variables are correlated, then covariance must be taken into account.

## Contents


- 1 Linear combinations
- 2 Non-linear combinations
    - 2.1 Example
- 3 Caveats and warnings
- 4 Example formulas
- 5 Partial derivatives
    - 5.1 Example calculation: Inverse tangent function
    - 5.2 Example application: Resistance measurement
- 6 Notes
- 7 External links
- 8 See also


## [edit] Linear combinations

Let $f(x_1,x_2,...,x_n)$ be a linear combination of $n$ variables $x_1,x_2,...,x_n$ with combination coefficients $a_1,a_2,...,a_n$.

$$f = \sum_{i=1-n} a_i x_i = \mathbf{a}^\mathbf{T}\mathbf{x}$$

and let the variance-covariance matrix on x be denoted by **M**.

$$\mathbf{M} = \begin{pmatrix} \sigma_1^2 & COV_{12} & COV_{13} & ... \\ COV_{12} & \sigma_2^2 & COV_{23} & ... \\ COV_{13} & COV_{23} & \sigma_3^2 & ... \\ ... & & & \end{pmatrix}$$

Then, the variance of $f$ is given by

$$\sigma_f^2 = \sum_i \sum_j a_i M_{ij} a_j = \mathbf{a}^\mathbf{T}\mathbf{M}\mathbf{a}$$

Each covarance term, $M_{ij}$ can be expressed in terms of the correlation coefficient $\rho_{ij}$by $M_{ij} = \rho_{ij}\sigma_i\sigma_j$, so that an alternative expression for the variance of is is

$$\sigma_f^2 = \sum_i a_i^2\sigma_i^2 + \sum_i \sum_{j(j\neq i)} a_i a_j \rho_{ij}\sigma_i\sigma_j$$

In the case that the variables x are uncorrelated this simplifies to

$$\sigma_f^2 = \sum_i a_i^2\sigma_i^2$$

# [edit] Non-linear combinations

When $f$ is a non-linear combination of the variables $x$, it must usually be linearlized by approximation to a first-order Maclaurin series expansion, though in some cases, exact formulas can be derived that do not depend on the expansion [1].

$$f \approx f^0 + \sum_i \frac{\partial f}{\partial x_i} x_i + \sum_i \sum_j \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} x_i x_j (i \neq j)$$

where $\frac{\partial f}{\partial x_j}$ denotes the partial derivative of $f$ with respect to the $j$-th variable. Since $f^0$ is a constant it does not contribute to the error on $f$. Therefore, the propagation of error follows the linear case, above, but replacing the linear coefficients, $a_i$ by the partial derivatives, $\frac{\partial f}{\partial x_j}$ in the linearized function.

### [edit] Example

Any non-linear function, *f(a,b)*, of two variables, *a* and *b*, can be expanded as

$$f \approx f^0 + \frac{\partial f}{\partial a}a + \frac{\partial f}{\partial b}b + 2\frac{\partial f}{\partial a}\frac{\partial f}{\partial b}ab$$

Whence

$$\sigma_f^2 = \left(\frac{\partial f}{\partial a}\right)^2\sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2\sigma_b^2 + 2\frac{\partial f}{\partial a}\frac{\partial f}{\partial b}COV_{ab}$$

In the particular case that *f=ab*, $\frac{\partial f}{\partial a} = b, \frac{\partial f}{\partial b} = a$. Then

$$\sigma_f^2 = b^2\sigma_a^2 + a^2\sigma_b^2 + 2abCOV_{ab}$$

or

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + 2\left(\frac{\sigma_a}{a}\right)\left(\frac{\sigma_b}{b}\right)\rho_{ab}$$

## [edit] Caveats and warnings

Error estimates for non-linear functions are biased on account of using a truncated series expansion. The extent of this bias depends on the nature of the function. For example, the bias on the error calculated for log x increases as x increases since the expansion to 1+x is a good approximation only when x is small.

In data-fitting applications it is often possible to assume that measurements errors are uncorrelated. Nevertheless, parameters derived from these measurements, such as least-squares parameters, will be correlated. For example, in linear regression, the errors on slope and intercept will be correlated and this correlation should be taken into account when deriving the error on a calculated value.

$$y = mz + c : \sigma_y^2 = z^2\sigma_m^2 + \sigma_c^2 + 2z\rho\sigma_m\sigma_c$$

In the special case of the inverse 1 / $B$ where $B = N(0,1)$, the distribution is a Cauchy distribution and there is no definable variance. In such cases, there can be defined probabilities for intervals which can be defined either by Monte Carlo simulation, or, in some cases, by using the Geary-Hinkel transformation [2].

## [edit] Example formulas

This table shows the variances of simple functions of the real variables $A, B$ with standard deviations $\sigma_A, \sigma_B$, and precisely-known real-valued constants $a, b$.

| Function | |
|---|---|
| $f = aA$ | $\sigma_f^2 = a^2\sigma_A^2$ |
| $f = aA \pm bB$ | $\sigma_f^2 = a^2\sigma_A^2 + b^2\sigma_B^2 \pm 2abCOV_{AB}$ |
| $f = aAB$ | $\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 + 2\frac{\sigma_a}{A}\frac{\sigma_b}{B}\rho_{AB}$ |
| $f = a\frac{A}{B}$ | $\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 - 2\frac{\sigma_a}{A}\frac{\sigma_b}{B}\rho_{AB}$ |
| $f = aA^{\pm b}$ | $\frac{\sigma_f}{f} = b\frac{\sigma_A}{A}$ |
| $f = a\ln(\pm bA)$ | $\sigma_f = a\frac{\sigma_A}{A}$ |
| $f = ae^{\pm bA}$ | $\frac{\sigma_f}{f} = b\sigma_A$ |

| | |
|---|---|
| $f = a^{\pm bA}$ | $\frac{\sigma_f}{f} = b \ln a \sigma_A$ |

For uncorrelated variables the covariance terms are zero. Expressions for more complicated functions can be derived by combining simpler functions. For example, repeated multiplication, assuming no correlation gives,

$$f = AB(C) : \left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_C}{C}\right)^2$$

## [edit] Partial derivatives

Given $X = f(A, B, C, \cdots)$

| Absolute Error | Varianc |
|---|---|
| $\Delta X = \left\|\frac{\partial f}{\partial A}\right\| \cdot \Delta A + \left\|\frac{\partial f}{\partial B}\right\| \cdot \Delta B + \left\|\frac{\partial f}{\partial C}\right\| \cdot \Delta C + \cdots$ | $\sigma_X^2 = \left(\frac{\partial f}{\partial A}\sigma_A\right)^2 + \left(\frac{\partial f}{\partial B}\sigma_B\right.$ [3] |

### [edit] Example calculation: Inverse tangent function

We can calculate the uncertainty propagation for the inverse tangent function as an example of using partial derivatives to propagate error.

Define

$f(\theta) = \arctan\theta$,

where $\sigma_\theta$ is the absolute uncertainty on our measurement of θ.

The partial derivative of $f(\theta)$ with respect to θ is

$$\frac{\partial f}{\partial \theta} = \frac{1}{1+\theta^2}.$$

Therefore, our propagated uncertainty is

$$\sigma_f = \frac{\sigma_\theta}{1+\theta^2},$$

where $\sigma_f$ is the absolute propagated uncertainty.

### [edit] Example application: Resistance measurement

A practical application is an experiment in which one measures current, $I$, and voltage, $V$, on a resistor in order to determine the resistance, $R$, using Ohm's law, $R = V / I$.

Given the measured variables with uncertainties, $I \pm \Delta I$ and $V \pm \Delta V$, the uncertainty in the computed quantity, $\Delta R$ is

$$\Delta R = \left( \left(\frac{\Delta V}{I}\right)^2 + \left(\frac{V}{I^2}\Delta I\right)^2 \right)^{1/2} = R\sqrt{\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2}.$$

Thus, in this simple case, the relative error $\Delta R/R$ is simply the square root of the sum of the squares of the two relative errors of the measured variables.

## [edit] Notes


1. ^ Leo Goodman (1960). "On the Exact Variance of Products". *Journal of the American Statistical Association* **55** (292): 708-713.
2. ^ Jack Hayya, Donald Armstrong and Nicolas Gressis (July 1975). "A Note on the Ratio of Two Normally Distributed Variables". *Management Science* **21** (11): 1338-1341.
3. ^ [Lindberg] (2000-07-01). Uncertainties and Error Propagation (eng). *Uncertainties, Graphing, and the Vernier Caliper* 1. Rochester Institute of Technology. Archived from the original on 2004-11-12. Retrieved on 2007-04-20. “The guiding principle in all cases is to consider the most pessimistic situation.”


## [edit] External links

- Uncertainties and Error Propagation, Appendix V from the Mechanics Lab Manual, Case Western Reserve University.
- A detailed discussion of measurements and the propagation of uncertainty explaining the benefits of using error propagation formulas and monte carlo simulations instead of simple significance arithmetic.
- Uncertainty calculator can be used to calculate propagated error

## [edit] See also

- Errors and residuals in statistics
- Accuracy and precision
- Delta method
- Significance arithmetic



Categories: Numerical analysis | Statistics

**Views**

- Article
- Discussion
- **Edit this page**
- History

**Personal tools**

- **Sign in / create account**

**Navigation**

- Main Page
- Contents
- Featured content
- Current events
- Random article

**interaction**

- About Wikipedia
- Community portal
- Recent changes
- Contact Wikipedia
- Donate to Wikipedia
- Help

**Search**

Go Search

**Toolbox**

- What links here