A Model of Endogenous Government Formation

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Abstract

Political parties bargain over the allocation of cabinet portfolios when forming coalition governments. Non-cooperative theories of legislative bargaining typically predict that the “formateur” enjoys a disproportionate share of government ministry positions. However, empirical evidence indicates that parties receive shares of portfolios proportional to their share of legislative seats that a government party contributes to the government coalition, in support of Gamson’s Law of portfolio allocation. This paper examines government formation as a process in which both the government coalition and the formateur are determined endogenously. In equilibrium, if parties have similar preferences over cabinet portfolios, the share of seats they are allocated is proportional to the parties’ sizes.
The defining feature of a parliamentary democracy is that the executive branch - which often consists of a prime minister and a Cabinet (“government”) - is not elected directly by the citizens, but it is invested and responsible to the legislature (Laver, 1998; Laver & Shepsle, 1996; Laver & Schofield, 1998; Müller & Strøm, 2000; Schofield & Laver, 1985). Since in most parliamentary democracies it is rare for a single party to control the majority of the seats in parliament, coalition governments are usually undertaken by the parties to achieve the majority vote (or “vote of investiture”) by the Parliament.

The key issue of the bargaining process that parties must resolve when forming a coalition government is how to allocate the resources of government - for example cabinet portfolios - among themselves. If this is the primary concern of the parties, only coalitions that include parties whose support is needed in order to obtain the majority of the votes will form, thus coalitions are predicted to have a minimal winning size (Riker, 1962).

A long and distinguished literature in the study of coalition formation has sprouted from the foundational work of Riker (See Baron, 1993, for a detailed overview). Yet this literature leaves important questions unanswered. First, it has little predictive power, since many different Minimal Winning Coalitions (henceforth MWC) can typically be formed at any given point in time. Second, it is characterized by a striking paradox concerning how resources are distributed inside the coalition. An important stream of literature (from Baron & Ferejohn, 1989 to recent work by Yildirim, 2007) analyzes government formation as a bargaining game between parties seeking to form a government coalition, predicting
that a party with the power to propose an allocation of government resources ("formateur") can exploit its privileged position and receive a disproportionately larger share of cabinet portfolios than its coalition partners.

Set against this, a prominent empirical regularity suggests that the share of cabinet portfolios that each government party receives is almost perfectly proportional to the share of legislative seats it contributes to the government coalition (Browne & Franklin, 1973), with no evidence of a formateur advantage even when portfolio payoffs are weighted by salience (Warwick & Druckman, 2001). Although Ansolabehere et al. (2005) do find a statistically significant formateur bonus, Warwick & Druckman (2006) show that this is not the result of the formateur party’s bargaining strength, but only of its greater size.

These empirical results support Gamson’s conjecture (1961), which claims that parties seeking to form a coalition government demand a share of portfolios proportional to the amount of seats in the assembly that each contributed to the coalition. Figure plots the share of cabinet portfolios against the shares of legislative seats government parties contribute to the coalition for all governments formed between 1945 and 2000. Strong proportionality is displayed by all governments, including Minority and Super-majority governments. As Laver (1998) notes, “Gamson’s Law shows one of the highest non-trivial R-squares in political science.” In this article I break new ground by providing a formal model that makes predictions in line with Gamson’s law.

[Figure 1 about here]

Many implications of coalition theories have been empirically assessed in
comparative politics and an extensive literature has resulted in further understanding of government formation (see Laver & Schofield, 1998 and Müller & Strøm, 2000 for a comprehensive analysis). More recently, Martin & Stevenson (2001) have found that minimal winning and ideologically connected coalitions are more likely to form than others. Bäck & Dumont (2008) showed that party’s size and moderate policy preference increase the likelihood of being formateur. Martin & Stevenson (2010) and Glasgow et al. (2011), found that incumbents can affect the new formateur selection, but that incumbency is an advantage only when the incumbent party has performed well in government.

Different hypothesis about the institutions that rule the government formation process have resulted in pioneering theories like the “alternating offers model” of Baron & Ferejohn (1989), the “alternating demands model” of Morelli (1999), and the bargaining model of Diermeier & Merlo (2000). The Baron-Ferejohn model (1989), applied to government formation, assumes a randomly selected formateur makes offers to potential coalition partners and keeps the balance. The Morelli protocol (1999) assumes that each player asks for a share of cabinet positions, leaving to subsequent players the possibility of asking for their respective shares. A coalition emerges between parties making compatible demands. Diermeier & Merlo (2000) analyze a government formation process in which parties bargain over office and policy. The formateur party, in order to obtain support in the legislature, can either make “policy concessions” by enacting a compromising policy, or propose “side payments” to other parties, leading to equilibrium governments that can be minority, minimum winning, or surplus.
Yet, the most important empirical law in government formation studies is that coalition partners share cabinet portfolios in proportion to their relative seat shares, which contradicts the predictions of the entire theoretical literature (see Laver et al. 2011). While computational analyses have shown that the proportionality phenomenon could be explained by zero-intelligence players (Golder et al. 2012), the mechanism that leads to this phenomenon is still unclear. Recently, experimental and empirical works have tested whether Gamson’s Law is caused by fairness, envy, bounded rationality (Diermeier & Morton, 2005; Fréchette et al. 2003; Fréchette et al. 2005), or social norms (Bäck et al., 2009) showing no support for any of these hypothesis. As Warwick and Druckman (2006) conclude, this exceptionally strong relationship is still “in acute need of a firm theoretical foundation”.

Why do bargaining models fail to characterize correctly the allocation of portfolios in coalition governments? This article argues that traditional theoretical models fail because they abstract away essential features of the government formation process - notably the way in which the formateur of the government coalition is selected and designated. While the formal constitutional position, in most parliamentary systems, is that the power of appointing a formateur rests in the hands of the head of state, in practice, heads of states rarely use the discretion allowed by the constitution, and typically follow the directions of the parties. This is because their scope for action is limited by parties that agree and publicly commit to certain coalitions before the formateur appointment (as in Austria and Italy), or because their discretion is bounded by clear parliamentary
majorities supporting different candidates (as it happens in Denmark, France, and Finland), or because it is strong constitutional norm to seek and follow the advice of the parties directly, or through the use of “informateurs”\(^1\) (typically in Norway, Luxembourg, the Netherlands, and Sweden).

Hence, subject to a few exceptions where heads of states have actively intervened (Kang, 2009), in most cases, as Müller & Strøm (2000) suggest, formateurs are chosen on the suggestions of parties that have agreed to coalesce and have reached a previous agreement about the head of the government.

For this reason, instead of assuming an \textit{exogenous} recognition rule, as in Baron-Ferejohn style bargaining models, the theoretical model set out below proposes a non-cooperative theory of government formation according to which the role of proposer, or “formateur,” is determined \textit{endogenously} as a product of bargaining among the parties which agree to coalesce. I posit a “free-style” bargaining model with endogenous formateur selection as parties compete to become the proposer. Parties that have previously agreed to coalesce are willing to offer their partners a share of cabinet portfolios in order to be vested with the role of formateur, and to be in the position to propose the distribution of resources.

My model assumes that government formation proceeds in four stages. In the first stage, parties simultaneously propose to form a coalition with a subset of

\(^1\) “Informateurs” are appointed to gather information about parties’ preferences and to identify the parties that are likely to form a majority coalition.
other parties. The parties with identical matching proposals become partners of a “proto-coalition” and proceed to the next stage. In the second stage, each party in the proto-coalition offers a share of the cabinet portfolios to each proto-coalition partner in exchange for the role of formateur. The eventual formateur in any given proto-coalition is the party offering the highest share of cabinets to the proto-coalition partners. In the third stage, the coalition parties bargain over the allocation of cabinet positions, with the formateur having been selected in the second stage. In the fourth stage, the coalition proposal is voted on by the legislature. This structure reflects the sequence of events during government formation in most parliamentary democracies.

Take, for example, the so-called “First Republic” in Italy. In all government formations between 1948 and 1992, we can identify three distinct phases: a first phase of “talks” among the parties represented in Parliament, which continues from the general election (or from the fall of the incumbent government) to the time the head of state asks the leaders of all of the parties to consult; a second stage, in which the head of state nominates a prime minister-to-be; and a third stage, when the nominated prime minister-to-be bargains with the parties to compile a list of ministers (Mershon, 1994; Pridham, 1984; Wertman, 1987).

In practice, in the first phase, parties agree to coalesce and designate a prime-minister to propose to the head of state; in the second phase, the head of state

\(^2\)Proposals that are exact counterparts of one another.
ratifies the decision of the coalition. Of particular interest is the case of the VIII Legislature (1979), during which parties could not reach an agreement in the first phase of informal talks after the general election, leading to repeated failures of all attempts to form a government (Bianchi 1979 a-f; La Spina 1979). This case illustrates how a head of state alone has next to no discretionary powers in the process of government formation. Although the head of state may assume the role of a facilitator in the case of an impasse in the inter-party negotiations, coalition formation is the result of “free-style bargaining” between parties.

The model set out in this article is an important departure from current legislative bargaining models. This is because allowing parties to compete for the role of formateur by pursuing a buyout completely exhausts any “proposer advantage”. The main prediction of my model is that if parties are only office-motivated and the set of cabinet portfolios is a homogeneous and perfectly divisible good, then the share each party in the government coalition is allocated in equilibrium is perfectly proportional to the number of seats that each controls in the legislature.

Having set out and solved the baseline model, I then extend this to accommodate heterogeneous preferences whereby parties have different preferences among ministerial positions. The fact that political parties may care differently about the same ministries is not a novel idea in the literature, although its effect on government formation has not received much formal attention. In equilibrium, if the coalition parties have similar preferences about ministries, then the share of these they receive is proportional to their legislative seat share. Other-
wise, the allocation is not perfectly proportional, but this imbalance is not driven by a formateur advantage, as in the Baron & Ferejohn model (1989), but by the Pareto efficient solution of the parties’ utility maximization problem.

A case study: Germany 1969

Germany after 1969 is a good illustration of how coalition governments are formed as a result of free-style bargaining among the parties. The Basic Law does not provide any guidance about how electoral results should affect government formation, or about the role of the parties in the latter process.

In the election to the Bundestag on September 28, 1969, the Christian Democrats (CDU/CSU) received 242 seats together (192 and 50 respectively), the Social Democrats (SPD) 224, and the Free Democrats (FDP) 30. The National Democratic Party (NDP) received 4.5% of the votes, less than the 5% hurdle and thus they remained unrepresented.

As soon as the election results were disclosed, early in the evening of election day, the CDU/CSU, having obtained the largest number of seats, started pondering whether to open formal negotiations to form a majority government coalition with the SDP or the FDP. On the other side, the SPD with its 224 seats (up from 202 in the past legislature) immediately invited the FDP to form a coalition. The FDP, with a slim 5.8% of the votes, suddenly became the party in the position to decide the shape and direction of the new government.

On the very night of the election, the SPD announced its determination to
form a government coalition with the FDP, with the tacit acknowledgement of the FDP which meant that the party was ready to start negotiations.

What happened on those late hours of election night set a new course for Germany. Excerpts from the Transcript of the Session of the FDP Federal Executive Committee on September 30, 1969 (reprinted in Daniel Hofmann, 2000) show that the FDP was heavily courted by both parties. The CDU/CSU seemed to be eager to lock the FDP into formal negotiations, promising a generous allocation of cabinet portfolio, but without any specifics. SPD leaders, however, met informally with FDP’s leaders and offered them the Foreign and Interior Office, along with a reduction of the total number of ministries through a reform of the cabinet (making de facto the impact of FDP in the Cabinet larger).

The official FDP committee met on September 30th and voted to open formal negotiations with the SPD. In the following two weeks SPD and FDP negotiating teams met to hammer out a government policy program and to decide on the Cabinet composition. The teams ratified the allocation of portfolios that the two parties initially agreed upon, allocating a total of 3 ministerial posts to the FDP (Interior, Foreign, and Food, Agriculture and Forestry) and 13 ministries to the SPD.\(^3\)

What was revealed on election night is that the emergence of a government coalition

\(^3\)The share of cabinet portfolios for FDP (18.75%) is slightly more than proportional to the share of legislative seats that the party contributes to the government coalition (11.8%). This is consistent with the analysis of Warwick and Druckman (2006), who find that smaller coalition parties tend to receive more than proportional portfolios shares.
coalition may be led by a few key proposals that are described in the model as “proto-proposals”. The CDU/CSU, the largest party, lacked the willingness and the promptness to offer the “right” compensation in exchange for the role of formateur. Instead, the SPD nailed down an offer that satisfied the FDP and ultimately become formateur. Government policies may be refined and polished, cabinet portfolio allocations can be reshuffled, but key proposals are most likely what lead to the start of formal negotiations and what ultimately shape final agreements.

**A bargaining model with endogenous formateur**

Consider a legislature with \( N \equiv \{1, 2, \ldots, n\} \) parties. Each party \( i \) has a legislative weight \( w_i \in [0, 1] \). We can think of these weights as being the proportion of legislative seats held in the legislature. No party has a majority of votes necessary to pass a government proposal \( (w_i < 1/2, \forall i \in N) \).

The relationship between parties and legislators is one of mutual loyalty. Individual legislators \( l \in \{1, 2, \ldots, L\} \) are assumed to be loyal to the party to which they belong, meaning that they do not switch from one party to another and act as a unitary block following directions from party leaders. Parties maintain members’ discipline through the use of patronage, meaning that loyalty is rewarded with office positions that, correspondingly, are denied to renegade members. Yet, members can have different weights inside the party and, consequently, may
enjoy different shares of the benefits.

A.1 The utility function of party \( i \) is represented by the aggregate utility of each of its members: \( U_i = \sum_{l \in i} u_l \).

The main difference between this model and most of the models discussed thus far is that the formateur (the proposer in the bargaining game) is not exogenously designated but is determined in equilibrium by the characteristics of the parties. Parties are rational players and, looking forward, foresee that the formateur might enjoy an advantage in the bargaining game, thus they are willing to make an offer to the coalition partners in order to achieve this proposer role.

The bargaining game proceeds as follows. Parties behave non-cooperatively and decide how to allocate a perfectly divisible homogeneous bundle of ministerial posts (normalized to sum to 1) among themselves. Let \( N \equiv \{1, 2, ..., n\} \) denote the set of parties, and \( S \equiv \{(s_1, s_2, ..., s_n) | s_i \geq 0, \sum_i s_i \leq 1, \forall i \in N\} \) denote the set of feasible allocations, where \( s_i \) is the share party \( i \) receives. Assume that parties are risk neutral and that they discount the future by a discount factor \( \delta \in (0, 1] \). Parties care only about their own consumption and have preferences that can be represented by a continuous and strongly monotonic – but not necessarily quasi-concave – utility function \( U_i = f(s_i) \). Since the group of agents decides how to allocate a perfectly divisible surplus of unit size among themselves, we can assume without further restrictions that the utility of each party is the share of the surplus that it obtains: \( U_i = s_i \).

Each party is assumed to know the preferences of all other parties, and all
actions are assumed to be observable (perfect information).\textsuperscript{4} The interaction between parties is modeled as a sequential game, where the proposing party needs the consent of a quota equal to the simple majority (L/2+1) for its proposal to be agreed upon.

In each period $t \in T$, government formation proceeds in four stages. Let $H_t \equiv \{h_1, h_2, h_3, h_4\}^T_{t=1}$ be the history of the game that contains the actions taken at each of the four stages in period $t$.

In the first stage $h_1$, parties simultaneously propose to form a coalition with a subset of parties. Let $J \equiv \{1, 2, ..., m\}$ with $m \leq n$ denote a subset of parties. Then party $i \in N$ takes action $a_i(h_1) = c^J_i$, that is, party $i$ proposes to form a coalition $c^J_i \in C_i$ where $C_i$ denotes the set of coalition proposals of party $i$ and $J \subseteq N$.\textsuperscript{5} The parties with identical matching proposals ($c^J_j = c^J, \forall j \in J$) become partners of the proto-coalition $c^J$ and proceed to the next stage.\textsuperscript{6}

\textsuperscript{4}This is not a completely natural assumption in all situations but provides a necessary preliminary to a more general analysis and may be reasonable in some circumstances.

\textsuperscript{5}For example, if a legislature is composed by party 1, 2, 3, then $C_1 = \\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$.

\textsuperscript{6}If the party coalition proposal is degenerate (it proposes a single-party government by itself) it trivially becomes a “proto-coalition” and proceed to the next stage.
In the second stage $h_2$, each party $i \in c^J$ offers a share of cabinet portfolios to each partner in the coalition ($\Lambda^i_j \in [0, 1]$, $\forall j \in c^J, j \neq i$) in order to be formateur.\(^7\)

Thus, only the parties in a proto-coalition formed in the first stage $i \in c^I$ take action ($a_i(h_2) = \Lambda^i_j$, $\forall i \in c^I$; and $a_i(h_2) = \emptyset$, $\forall i \notin c^I$). The formateur, say $m$, is the party that offers the highest share of cabinet posts to the partners. The formateur and the coalition partners sign a government “proto-proposal” $\tilde{S}^m$, which assigns $\Lambda^m_j$ to the coalition partners $j \in c^J$, the remaining share of cabinet positions ($1 - \sum_{j \in c^J} \Lambda^m_j$) to the formateur $m$, and 0 to the parties outside the proto-coalition $j \notin c^J$.

Intuitively, the proto-proposal is the outcome of the free-style negotiations that precede the official designation of the formateur and the official bargaining stage. The aim of the intra-party talks and negotiations is twofold. First, parties need to explore their possibilities at forming government coalitions with different partners (proto-coalitions). Second, parties within the possible proto-coalitions try to reach a basic agreement about key policy issues and allocations of executive positions. This basic agreement (here called proto-proposal) represents a sketchy arrangement that can be interpreted as the “least common multiple”, often in terms of the numbers of ministries to be allotted, that the proto-coalition parties demand in order to form the government coalition. As Mershon (1994) says: [...Politicians first reach a working agreement on the party composition of

\(^7\)Notice that the formateur is the proposer of the allocation of cabinet portfolios and not necessarily the prime minister designated.
the government... This decision depends on expectations about portfolios and policies, and permits bargaining over portfolios and policy to proceed.

In the third stage $h_3$, the parties belonging to a proto-coalition bargain over cabinet positions in accordance with standard alternating bargaining models. The only difference is that, in case the receiving parties (the parties in the proto-coalition which do not have proposal power) do not accept the proposer’s proposal, the receiving partner can either fall back to the proto-proposal vector $\tilde{S}^m$ that the coalition parties signed in stage $h_2$, or they can terminate the coalition negotiations and so back to stage $h_1$. The bargaining proceeds with the formateur proposing an allocation $S^m = (s_1, s_2, ..., s_n)$ to the coalition partners, and the partners accepting the proposal, rejecting the proposal and falling back to the proto-proposal, or terminating the coalition negotiations. Thus, only the parties in a proto-coalition formed in the first stage $i \in c^l$ take action ($a_i(h_3) = S^m$, if $i \in c^l, i = m$; $a_i(h_3) = \{\text{accept, reject, terminate}\}$, if $i \in c^l, i \neq m$; and $a_i(h_3) = \emptyset$, $\forall i \notin c^l$). If the formateur proposal $S^m$ is accepted, it becomes the “proto-coalition” proposal. If the formateur proposal $S^m$ is rejected, the status quo (the proto-proposal vector $\tilde{S}^m$) becomes the “proto-coalition” proposal. If the formateur proposal $S^m$ is terminated, the game repeats itself at the first stage up to $T$ periods.

If the coalition parties agree to the formateur’s proposal, the proposal is voted upon by the legislative bodies. Elsewhere, coalition parties may vote to move the proto-proposal to the floor (whenever detailed enough) or to terminate the bargaining stage within the proto-coalition.
\[
S^m = \begin{cases}
  s_i & \text{if } i \in c^J, \ i \neq m; \\
  1 - \sum_{i=1}^{m-1} s_i & \text{if } i \in c^J, \ i = m \\
  0 & \text{if } i \notin c^J.
\end{cases}
\]

\[
\tilde{S}^m = \begin{cases}
  \Lambda_i^m & \text{if } i \in c^J, \ i \neq m; \\
  1 - \sum_{i=1}^{m-1} \Lambda_i^m & \text{if } i \in c^J, \ i = m \\
  0 & \text{if } i \notin c^J.
\end{cases}
\]

The formateur might have significance leverage in designing the final proposal, often meeting specific demands of coalition members. However, the proto-proposal represents the benchmark by which the proto-coalition partners will measure and judge the formateur’s proposal. Whenever the proto-proposal is a detailed accounting of the offices assigned to each of the coalition members (as in the German 1979 case study), parties might reject the formateur’s re-shuffle attempt in favor of the agreed proto-proposal.

In the forth stage \( h_4 \), every legislator votes on one (or more) proto-coalition proposals (either the formateur proposal or the proto-proposal). If no proto-coalition proposals obtains the majority of votes required to be passed, the game repeats itself at the first stage up to \( T \) periods. This process continues until an allocation generates the required number of votes. If at the end of period \( T \) no proposal has been passed, a “caretaker government”, composed by non-partisan individuals\(^8\), takes place and \((s_i) = (0) \ \forall i \in N.\)

\(^8\)A formal analysis of the equilibrium strategies in case of partisan caretaker governments is provided in the on-line appendix.
I now describe the solution concept. Let $H_t \equiv \{h_1, h_2, h_3, h_4\}_{t=1}^T$ be the history of the game that contains the proposals that have been made, the identity of the proto-coalition, the identity of the proposer, and the actions taken at each of the four stages in period $t$. A strategy $\xi_i$ for player $i$ describes a sequence of actions, $a_i(H_t)_{t=1}^T$. A strategy profile $\xi \equiv \{\xi_1, \xi_2, ..., \xi_n\}$ is a sub-game perfect equilibrium (SPE), henceforth termed “equilibrium” if it constitutes a Strong Nash equilibrium in each period $t$ and it is history dependent. All moves are sequential, and there is perfect information, so the game will be solved by backward induction starting from the last period.

To understand the intuition behind the model’s main results, I examine the case in which members inside a party have the same weight.\(^9\)

**A.2** Let $u_i$ be the utility of each legislator belonging to party $i$ and $w_i$ party $i$’s size, then the utility function of party $i$ may be rewritten as $U_i = w_i u_i$

The subgame at period $t = T$ is solved backward from the last stage. All proofs of the propositions can be found in the Appendix.

\(^9\) For the sake of simplicity the model assumes every party member to have the same “weight” or “rank” inside the party. However, because the agents in the model are the parties and not the individual legislators, a more skewed membership ranking inside the party would affect only the intra-party distribution of benefits, but not the equilibrium results at the party level.
Model structure and related literature

There are two main features that distinguish my model from the existing coalition formation models in the literature: first, the endogenous nature of the formateur selection; and second, the endogenous nature of the coalition formation.

The endogenous nature of the formateur selection distinguishes my model from the foundational model of Baron & Ferejoh (1989) and all the legislative bargaining models that have originated from it. The only bargaining model with endogenous recognition of the formateur is proposed by Yildirim (2007), where every player chooses a costly effort level, which stochastically determine the proposer. The endogenous recognition rule however produces a higher “proposer advantage” than a random recognition rule as more efficient players (with lower effort costs) are more likely to become proposers and allocate the benefits to less efficient players. The endogenous formateur selection in my model is not the outcome of a contest among the parties, but the result of a mechanism by which parties exchange their right of being formateur for a share of benefits.

The endogenous nature of the coalition formation sets my model apart from the non-cooperative models in an important aspect. While the extant legislative bargaining models are concerned with predicting coalitions that allow individual coalition parties to maximizes their own utility, given which party is selected as formateur, the objective of my model is to predict coalitions that emerge endogenously and that constitute the optimal outcome for each of them. In this respect, it is also fundamentally different from the “demand bargaining” model of Morelli
(1999) in which coalitions constitute compatible, but not necessarily optimal, outcomes. In this model, parties foresee the utility of coalescing with others and seek to form a coalition with the parties that allow them to maximize their utility.

Carroll & Cox (2007) study a similar endogenous coalition formation process in systems where parties can form pre-election pacts. In their article, during the campaign stage, parties can form pre-election pacts, which can be viewed as attempts to pre-negotiate portfolio allocations.

Recently, other studies have focused on the importance of long-term stability in coalition formation. Indriðason (2010) proposes a legislative bargaining model where concerns on the coalition’s survival lead the formateur to allocate higher than predicted shares of portfolios to the coalition parties. Penn (2009) considers a dynamic voting game where players face a trade-off between the immediate value of a proposal and the long-run stability of the emerging coalition, finding that, in the long run, proposers are better off by allocating shares of benefits to their coalition partners that are large enough to deter them from deviating to alternative proposals. In my model, stability is not induced by the threat of new proposals in the long term, but by the endogenous nature of the coalition formation. Players can freely communicate and propose to coalesce with other players. Thus, a coalition proposal needs to be immune to deviations to alternative coalition proposals by both individual players and sub-coalitions of players. The immunization to all possible deviations makes the coalition stable.
Stage 4: Legislature voting stage

In the last stage of the game, a coalition proposal is put on the floor and voted by the legislature. If more than one proto-coalitions have formed in the first stage, the largest proto-coalition proposal (in terms of voting weight) is put on the floor and voted. If the proposal does not obtain the majority of votes required to be passed, the game repeats itself at the first stage up to \( T \) periods. This process continues until an allocation generates the required number of votes. If at the end of period \( T \) no proposal has been passed, a “caretaker government”, composed by non-partisan individuals, takes place. If this is the case, no party receives office benefits \( s_i = 0 \forall i \in N \). Parties vote in favor of a proposal if \( U_i \geq 0 \).\(^{10}\) Therefore, the unique subgame perfect equilibrium indicates that the largest proto-coalition’s proposal is passed on the very first stage of the game.

Stage 3: Bargaining over cabinets

Once a formateur is selected, the bargaining process proceeds with the formateur as the proposing party and the coalition partners as the receiving parties. The reversion point is either the proto-proposal \( \tilde{S}^m \) that the formateur \( m \) proposed to the coalition partners in the previous stage, or a vector of zeros if the coalition

\(^{10}\) In a vote of investiture or confidence, party discipline is imposed (legislators who votes against their own party are severely sanctions or even expelled. Thus parties, and not legislators, are the players of this game.
partners decide to terminate the proto-coalition’s negotiation. If the formateur proposal \( S^m \) is accepted by the coalition partners, then \( S^m \) will be put on the floor and voted by the legislature. Otherwise, either the proto-proposal \( \tilde{S}^m \) will be put on the floor and voted by the legislature, or - since it is the last period of the game, no proposal is put on the floor.

Assuming that each member of a party has equal weight inside the party, the share of benefits offered by the formateur to party \( i \in c^J \) may be rewritten as \( \Lambda^m_i = w_i \lambda^m \), where \( w_i \) is party \( i \)'s size and \( \lambda^m \in [0, 1] \) is the benefit each member of party \( i \) enjoys. The proto-proposal vector may be rewritten as:

\[
\tilde{S}^m = \left( (\lambda^m w_1), (\lambda^m w_2), \ldots, (\lambda^m w_{m-1}), (1 - \lambda^m \sum_{i=1}^{m-1} w_i), 0, \ldots, 0 \right)
\]

The proposer’s maximization problem is to propose \( S^m = (s_1, s_2, \ldots, s_n) \), such that \( U^m \) is maximized and the proposal is approved by the coalition partners and by the legislature (at the voting stage):

\[
\max_{s_i \geq 0} U^m(s_m) \quad \text{subject to} \quad \sum_{i=1}^{n} s_i \leq 1
\]

\[
U_i(s_i) \geq U_i(\lambda^m w_i) \quad \forall i \in c^J, i \neq m
\]

\[
U_i(s_i) \geq 0 \quad \forall i \in c^J, i \neq m
\]

The first constraint ensures feasibility, the second constraint ensures that each partner will voluntarily choose \( s_i \) over \( \lambda^m w_i \), the third constraint ensures that each partner will voluntarily choose \( s_i \) over terminating the proto-coalition negotiations. Notice that the third constraint is never binding for any \( \lambda^m \geq 0 \). Hence, any solution must satisfy the second constraint with equality, since the proposer maximizes its own utility by choosing an allocation \( (s_i) \) that makes
the partners indifferent with the offer $\lambda^m w_i$. Since cabinet portfolios are an homogeneous bundle normalized to 1, and $\lambda^m \in [0, 1]$, the equilibrium proposal is $S^* = \tilde{S}^m = (\lambda^m w_1, ..., \lambda^m w_{m-1}, 1 - \lambda^m \sum_{i=1}^{m-1} w_i, 0, ..., 0)$.

**Stage 2: Formateur selection**

The coalition parties make an offer to the partners in order to be the formateur of the government coalition.\textsuperscript{11} This process may be described as an auction in which each party $i$ bids a share $\Lambda^i$ to form a government coalition. This mechanism can be interpreted as a market in which each party pays a share of cabinet portfolios to the partners in order to form a government coalition.

**Proposition 1** In equilibrium the bid vector of each party is equivalent to the vector of parties’ weight multiplied by a constant:

$\Lambda^i = [\Lambda^i_1, ..., \Lambda^i_{i-1}, \Lambda^i_{i+1}, ..., \Lambda^i_n] = \lambda^i[w_1, ..., w_{i-1}, w_{i+1}, ..., w_n]$

In equilibrium, each party offers to the coalition partners a share of the benefits that is proportional to their party’s size. Parties will accept the offers that give them the highest "per capita" value. Hence, the winner of the auction is the party that bids the highest $\lambda^i$. The winning party pays the auction price by signing a proto-proposal $\tilde{S}$, which allocates to the partners the winning bid.

\textsuperscript{11}This mechanism is similar to the procedure proposed by Crawford (1979) that generates equilibrium allocations which are both Pareto efficient and equitable.
Each party has a well-defined reservation price for the role of formateur because they know the level of utility they would enjoy as formateur for every price $\lambda^i$. Furthermore, there exists a unique price that makes every party indifferent to adopting the role of formateur or receiving party. This means that the final outcome does not depend on the choice of auction.

**Proposition 2** *The equilibrium bid for each proto-coalition is unique and identical for every party $(\hat{\lambda}_{c,J})$. In equilibrium, therefore, the formateur is selected inside the proto-coalition by a tie-breaking rule that has been agreed upon by the parties.*

This is one of the innovative results of my model: because parties compete for the role of formateur, the “first-mover” advantage traditionally found in other legislative bargaining models is completely eliminated.

From the solution of the third stage, we know that the allocation $S^*$ generated by this process gives each party $i$ in the proto-coalition $c^J$ the same utility of $\tilde{S}$:

$$U(S^*) = (U_1(S), ..., U_{m-1}(S), U_m(S), U - m + 1(S), ..., U_n(S))$$

$$= (\hat{\lambda}_{c,J} w_1, ..., (\hat{\lambda}_{c,J} w_{m-1}), (\hat{\lambda}_{c,J} w_m), 0, ..., 0)$$

Thus, the equilibrium allocation is unique no matter which party is charged to form the government. Furthermore, because of the feasibility constraint, which holds with equality, we can derive a closed form solution for the equilibrium bid $\hat{\lambda}_{c,J} = 1/\sum_{i \in c^J} w_i$, where $\sum_{i \in c^J} w_i$ is the size of the proto-coalition. This leads us to our next theoretical result:

**Proposition 3** *If parties are office-motivated and the cabinet portfolios is an homogeneous and perfectly divisible good, the share that each party in the government*
coalition is allocated in equilibrium is perfectly proportional to the nominal voting weight.

\[ S^* = \left( \frac{w_1}{\sum_{i \in c} w_i}, \ldots, \frac{w_{m-1}}{\sum_{i \in c} w_i}, \frac{w_m}{\sum_{i \in c} w_i}, 0, \ldots, 0 \right) \]

**Stage 1: Coalition selection**

In the coalition formation stage each party proposes a proto-coalition. Every party \( i \)'s solve its maximization problem for every possible proto-coalition \( c_i \):

\[
\max_{c_i \in C_i} U_i(s_i | c_i, \lambda_i, \lambda_{-i}) = \max_{c'_i \in C_i} U_i \left( \frac{w_i}{\sum_{j \in c'_i} w_j} \right)
\]

subject to

\[
\sum_{j \in c'_i} w_j \geq \sum_{j \in c'' \setminus i} w_j
\]

The coalition formation stage can be represented by a simultaneous game, where each party proposes a government coalition. However, since players can communicate prior to and during play, they will try to reach an agreement (although non-binding) to coordinate their actions in a mutually beneficial way. As a consequence, we are not completely satisfied with strategy profiles that are immune only to individual deviations, but instead we would need equilibrium strategies that are also immune to deviations by groups of players (sub-coalitions) who can coordinate their actions.

There are several equilibrium refinements that take into account group deviations, the most relevant being the Strong Nash Equilibrium (Aumann, 1959), the Coalition-Proof Nash Equilibrium (Bernheim, 1987), and the Largest Consis-
tent Set (Chew, 1994). While the Strong Nash Equilibrium requires immunity to all possible coalitional deviations, the Coalition-Proof Nash Equilibrium restricts attention to a limited class of “self-enforcing” coalitional deviations, that is, ones that are themselves robust against further “self-enforcing” deviations by sub-coalitions. On this respect, Chwe (1994) proposes the most farsighted refinement, according to which an outcomes is in the “consistent set” if it dominates - for all players - the final outcomes of all the possible deviations (but not necessary the transitory outcomes). The notion of self-enforceability provides a useful mean of distinguishing coalitional deviations that are viable from those that are not resistant to further deviations, while the notion of consistency provides a way to distinguish ultimately relevant outcomes from temporary ones. However, since one can argue that in many circumstances non viable deviations have successfully upset potential agreements, we will not restrict our attention to the set of strategies that are consistent or robust against self-enforcing coalition deviations, but we will adopt the more comprehensive Strong Equilibrium refinement (Aumann, 1959) to take account of all potential deviations.

**Proposition 4** If communication is allowed, unlimited, but non-binding, then there exists a Strong Equilibrium coalition which is the smallest minimal winning proto-coalition.

**Lemma 5** Suppose that cabinet portfolios are an homogeneous and perfectly divisible good, that the utility function of the parties satisfy A.2. Then, there exists an equilibrium coalition $c^*$, such that:
- $c^*$ has the smallest winning size.

- Every party $i \in c^*$ offers to the partners $j \in c^*$ a share equal to $\hat{\lambda}w_j = \frac{w_j}{\sum_{k \in c^*} w_k}$.

- The formateur’s proposal allocates $s_j = \hat{\lambda}w_j$ to $j \in c^*$ and 0 to $j \notin c^*$.

The equilibrium of the game is unique as long as there exists a unique smallest minimal winning coalition, otherwise at least one party will be indifferent to the choice between coalitions, which will be equally likely to be proposed.

**Multi-period SPE**

Assuming that the number of periods is finite, the game may be solved by backward induction starting from the last period. There exists a sub-game perfect equilibrium in which each party makes the same coalition proposal, the same bid to be formateur, and the same proposal to allocate the cabinet portfolios. The equilibrium strategies are thus unique.

**Proposition 6** A strategy configuration is a sub-game perfect equilibrium for a four-stage, $T$-period, $n$-party legislature, with endogenous formateur, if and only if the following properties attain:

1. In every period, each party $i$ proposes the smallest minimal winning coalition, which includes itself: $c_i$ such that $\sum_{j \in c_i} w_j = \min \sum_{j \in C} w_j, \forall C \geq L/2 + 1$.

2. In every period, each party $i$ in the proto-coalition bids (offers to the members of the partners $j \in c^*$) a share of ministries equal to $\hat{\lambda} = 1/\sum_{j \in c^*} w_j$;
3. In every period, every party $i$ selected as formateur makes a proposal that allocates $s_j = \hat{\lambda}w_j$ to every partner in the proto-coalition ($j \in c^*$), $s_i = 1 - \sum_{j \in c^*} \hat{\lambda}w_j = \hat{\lambda}w_i$ to itself, and 0 to every party outside the proto-coalition.

4. In every period $t < T$, every party in the proto-coalition $c^*$ votes for any proposal in which all parties receive at least $\delta \hat{\lambda}w_j$, and every party outside the coalition is indifferent to voting for the proposal or going to the next period.

5. In the first period, the formateur’s proposal allocates to every party in the proto-coalition $s_j = \hat{\lambda}w_j \forall j \in c^*$, which is greater or equal than $\delta \hat{\lambda}w_j$. The proposal in the first period is thus passed, and the legislature adjourns.

In such a framework, at every period of the game each party proposes the same proto-coalition; each party in the proto-coalition offers the same share $\hat{\lambda}$ to be the formateur; and each formateur proposes the same equilibrium allocation $S^*$.\textsuperscript{12} The equilibrium strategies are, thus, unique.

**Heterogenous preferences over portfolios**

In the previous section we have seen how the baseline model with parties with homogeneous preferences over the cabinet portfolios predicts equilibrium allo-

\textsuperscript{12}If the number of periods is infinite, the qualitative properties of the equilibria in the finite-period case hold for subgame perfect equilibria that are stationary, i.e., if the continuation values for structurally equivalent subgames are the same.
cations that are perfectly proportional to the nominal voting weight of the coalition parties. However, parties’ preferences over cabinet portfolios are often not homogenous. Parties often show intense preferences for particular portfolios reflecting the relative importance different parties give to different issues and policies pertaining to different ministries.\(^\text{13}\) This is especially relevant for small parties which link their political campaign to specialized policy issues. For example, consider the “Green” party in several parliamentary legislatures. In view of stated interests, one may think that it would obtain a greater utility from holding the “Environment” ministry than other comparable ministries.

Qualitative allocation of portfolios in most parliamentary legislatures shows some continuity. For example, in Denmark, the Social Democrats always hold the Ministry of Labour and almost always the Ministry of Social Welfare; the Liberals, advocating farmers’ interests, always hold the Ministry of Agriculture; and the Conservatives, linked to the business interests, always hold the Ministry of Trade and Industry. In Germany, either the Christian Democratic Union

\(^{13}\) Notice however that portfolios’ preferences cannot be interpreted as a proxy for policy preferences. Since portfolio preferences lack to identify the location of a party’s position on a certain (salient) issue, they might be able to describe the intensity of policy preferences but not the direction. For example, in Germany, both CDU and SPD show intense preferences for the Department of Family Affairs, Labour, and Social Affairs, however the direction of these preferences is opposite. While SPD has established itself as a lefty-socialist party, CDU has adopted more liberal economic policies, especially in the past 30 years.
or the Social Democrats control the Department of Family Affairs and Labour and Social Affairs, while the Free Democratic Party usually controls the Foreign Ministry or the Ministry of Economic Affairs (Müller & Strøm, 2000). Bäck et al (2011) analyze the impact of the saliency that government coalition parties attach to specific policy areas on the qualitative characteristics of the cabinet portfolio they control. They find that parties which emphasize certain policy areas in their political manifestos are more likely to be allocated portfolios with jurisdiction over those policy areas.

In this section I relax the assumption of cabinet portfolios being considered as a single good that sum to 1, allowing parties to extract different utilities from different positions. That is, parties decide how to allocate a perfectly divisible bundle composed of \( k \) ministerial positions. Units are chosen so that the \( k \)-vector of goods to be allocated sums to 1.

Thus, \( S \equiv (s_1, s_2, ..., s_n) \) denotes the set of feasible allocations, where \( s_i \) is the \( k \)-vector consumption bundle of party \( i \). Let \( U_i = f_i(s_i) \) denote party \( i \)'s continuous and strongly monotonic utility function for \( i = 1, ..., n \).

Government formation proceeds in four stages as described in the previous section. The only difference is that each party inside the proto-coalition (say \( c^j \)) bids some scalar multiple \( \Lambda^i \) of a numéraire \( \tau \geq 0 \). The numéraire \( \tau \geq 0 \) can be any bundle of goods which satisfy a “desirability” property, which is, any bundle that every party weakly prefer to the bundle 1.\(^{14} \) The party bidding the highest

\(^{14}\)The desirability assumption is satisfied if the utility function \( U_i \) is strongly monotonic for every strictly positive component of \( \tau \).
share of the numéraire, says \( m \), wins the privilege of being formateur and signs a proto-proposal \( \tilde{S}^m \) which assigns \( (\Lambda_i^m \tau) \) to the coalition partners \( i \in c^J \), the remaining share of the cabinet posts \( (1 - \sum_{i \in c^J} \Lambda_i^m \tau) \) to the formateur \( m \), and \( (0) \) the parties outside the proto-coalition \( i \notin c^J \). Assuming again that each member of a party has equal weight inside the party, \( (\Lambda_i^m \tau) \) maybe re-written as \( (\lambda^m \tau w_i) \).

**Equilibrium solution**

**Stage 3: Cabinet portfolio allocation**

At the third stage, the proposer party maximizes its payoff by proposing an allocation \( S^m = (s_1, s_2, ..., s_n) \), such that:

\[
\max_{s_i \geq 0} \quad U^m(s_m) \quad \text{subject to} \quad \sum_{i=1}^{n} s_i \leq 1
\]

\[
U^i(s_i) \geq U^i(\lambda^m \tau w_i) \quad \forall i \in c^J, i \neq m
\]

\[
U^i(s_i) \geq 0 \quad \forall i \in c^J, i \neq m
\]

As in the baseline model, any solution must satisfy the second constraint with equality, however, the equilibrium proposal \( S^m = (s_1, s_2, ..., s_n) \) is not necessarily equal to the proto-proposal \( \tilde{S}^m = [(\lambda^m \tau w_1), (\lambda^m \tau w_2), ..., (1 - \lambda^m \sum_{i=1}^{m-1} \tau w_i), 0, ..., 0) \). There may exist multiple allocation vectors that makes the partners indifferent with the proto-proposal, depending on the functional form of the parties’ utility function. The proposer party maximizes its utility by choosing the allocation vector that assigns its most preferred cabinet posts to itself, leaving the rest to the partners. Thus, the utility of the coalitions parties is higher when they have complementary
Proposition 7 If the bundle of cabinet portfolios is composed by n-ministries and parties have heterogeneous preferences on different cabinet positions, then the equilibrium allocation depends on the party sizes and their preferences. The equilibrium allocation is perfectly proportional to the relative size of the coalition parties if and only if their utility functions are identical \( f_i = f_j, \quad \forall i, j \in c' \). 

To see how coalition parties’ preferences affect the equilibrium allocation of the cabinet portfolios, assume a legislature composed of three parties, none with a majority of seats. Assume that cabinet positions are of two different types: “Interior Affairs” (IA) and “Foreign Affairs” (FA) and that parties have Cobb Douglas utility functions over these two types of ministerial positions, with \( \alpha_i \) identifying the preference parameter over the Interior posts:

\[
U_i = (IA_i)^{\alpha_i}(FA_i)^{1-\alpha_i}, \quad \forall i \in N
\]

Figure 2 plots the equilibrium allocations for a coalition government composed by party \( i \) and \( j \), where the size and preferences of party \( j \) are fixed to 5 legislators and \( \alpha_j = 0.5 \) respectively, and the size and preference of party \( i \) vary.\textsuperscript{15}

[Figure 2 about here]

For every parameter \( \alpha_i \), the larger party \( i \)'s size \( (w_i) \) is, the higher is the share of cabinet posts that party \( i \) is allocated to. However, the number of cabinet posts

\textsuperscript{15}Party \( i \)'s nominal voting weight \( w_i \) varies between 1/6 (when party \( i \) has 1 legislators) to 2/3 (when party \( i \) includes 10 legislators) and \( \alpha_i \in [0.1, 0.9] \).
increases at the same rate as the number of legislative seats contributed to the
government (being perfectly proportional) only when the two coalition parties
have the same preferences over the two types of cabinet posts \((\alpha_i = \alpha_j = 0.5)\).
When party \(i\) prefers one type of posts over the other (either \(\alpha_i > 0.5\) or \(\alpha_i < 0.5\)),\(^{16}\) party \(i\) will be better off by obtaining few posts of the preferred type than many of
the less-preferred ones. Thus, the more skewed party’s \(i\) preference is, the less
than proportional is the share of portfolio that party \(i\) is allocated in equilibrium.

**Stage 2: Formateur selection**

Each party has a well-defined reservation price for the role of formateur. Fur-
thermore, if the numéraire is *desirable*, there exists a unique price that makes
every player indifferent to holding the role of proposer or receiver.

**Proposition 8** For any numéraire bundle there exists a unique price \((\hat{\lambda})\) that makes
every coalition party indifferent to being the formateur or a receiving party.

The equilibrium bid for each proto-coalition is unique and identical for every
party \((\hat{\lambda}_{c,j})\). However, from the solution of the third stage, the allocation \((S^m)\) is no
longer necessarily equal to the proto-proposal \((\tilde{S}^m)\), and therefore the equilibrium
bid \(\hat{\lambda}_{c,j}\) is function of both the parties’ size vectors and their preferences.

**Stage 1: Coalition selection**

\(^{16}\)Given the use of a Cobb-Douglas aggregator, the equilibrium allocation is
equivalent for \(\alpha_i\) equidistant from \(\alpha_j\).
As in the homogeneous case, parties solve by backward induction the maximization problem for every possible proto-coalition and propose the coalition that secures the highest utility. However, since the equilibrium bid $\hat{\lambda}_{c,i}$ for every party $i$ is no longer equal to party $i$’s nominal voting weight, it is not necessarily the case that parties maximize their utility by choosing the smallest minimal coalition.

**Proposition 9** Suppose that cabinet portfolios are perfectly divisible bundles of goods, and that parties have heterogeneous preferences on the goods composing the bundles. Then, there exists an equilibrium coalition $c^*$ that is minimal winning.

To see how heterogeneous preferences affect the formation of the government coalition in equilibrium, assume again to have a three-party legislature composed by parties $i$, $j$, and $k$ that have Cobb-Douglas utility functions over “Interior Affairs” (IA) and “Foreign Affairs” (FA) cabinet posts with $\alpha_i$, $\alpha_j$, and $\alpha_k$ representing their preference parameters. Assume now that the sizes of the three parties are fixed, and that party $i$ controls three legislative seats, party $j$ controls four seats, and party $k$ controls two seats ($w_i = 3/9$, $w_j = 4/9$, and $w_k = 2/9$).

Table 1 shows the allocations at the bargaining stage for all the possible proto-coalitions for different values of $\alpha_i$, $\alpha_j$, and $\alpha_k$. The numbers in bold represent the equilibrium bids, allocations, and utilities for the equilibrium coalition.

As shown in the first panel of Table 1, if all parties are indifferent among the types of ministerial posts ($\alpha_i = \alpha_j = \alpha_k = 0.5$), the equilibrium allocations in the
bargaining stage, for each possible proto-coalition, are equivalent to the homogeneous case (they are perfectly proportional to the party nominal voting weight). The strong equilibrium coalition is the smallest minimal winning coalition: both party $i$ and party $j$ would propose to form a coalition with party $k$, while party $k$ would be better off by proposing a coalition with party $i$.

The second panel shows the equilibrium allocations when party $i$ holds more extreme preferences than party $j$ and $k$ ($\alpha_i = 0.2, \alpha_j = \alpha_k = 0.5$). In such a case, party $i$ would obtain a larger utility than in the case it were indifferent, because of the complementarity of its preferences with the other parties. The strong equilibrium of the game (in bold) would still be the coalition between party $i$ and party $k$ but the share of cabinet posts that they obtain ($IA + FA$) is no longer perfectly proportional to their nominal voting weight (party $i$ would receive 54.5% of the seats and party $k$ would receive 45.5%, with their nominal voting weights being equal to 0.6 and 0.4, respectively).

The third panel shows that, when the preferences of the three parties are sufficiently different from each other ($\alpha_i = \alpha_k = 0.2, \alpha_j = 0.9$), the complementarity benefit may lead to a strong equilibrium coalition (in bold) that is not the smallest winning coalition (but still minimal winning). In this case, the complementarity benefit would be exploited by a coalition composed by party $j$ and $k$. Party $k$ would receive a larger utility by forming a coalition with a larger party (party $j$) with different preferences than by forming a coalition with a smaller party (party $i$) with similar preferences.

Numerical simulations of the equilibrium predictions for all possible permu-
Conclusion

Previous empirical studies on government formation have shown that portfolio allocations can be well explained by a “rule of thumb” that assigns offices in proportion to the votes each party contributes to the coalition’s total, an empirical regularity called “Gamson’s law.” For their part, formal theorists have generally analyzed the process as a bargaining game over the resources of government, predicting in equilibrium that proposers - or “formateurs”- should be able to exploit their privileged position for their own benefit, receiving a disproportionately larger share of resources (cabinet portfolios) than their coalition partners. The expectations of these theories, however, are contradicted by the empirical evidence.

This article proposes a theoretical foundation for one of the strongest empirical regularities in political science – Gamson’s law. My theoretical model departs from the existing literature by analyzing the government formation problem in a framework where the proposer is not designated exogenously but is selected by the parties that have agreed to coalesce, and where government coalitions are not unilaterally proposed by a formateur but emerge endogenously as parties maximize their utility.

This article break new ground by theorizing the endogenous formation of the government coalition. It also provides a more realistic perspective on the way
that coalition governments form in parliamentary democracies. In fact, in both fully structured and free-style systems, parties engage in negotiations that aim at both skimming through potential viable government coalitions and identifying prospective formateurs, making the official nomination of the formateur a mere ratification of a decision that the coalition parties have already reached.

The model I develop here makes another important contribution: it is able to explain proportional allocations as functions of parties’ sizes and preferences with regard to cabinet portfolios. In this respect, the model represents an advancement that improves on the non-cooperative literature, closing the gap between the theoretical literature and the empirical findings.

The model can also account for how the composition of the equilibrium government coalition changes as a function of the size and preferences of the parties, and it can be fruitfully extended to account for other institutional constraints that can affect the government formation process in different parliamentary legislatures. Using this baseline model and assuming parties that are both office-seeking and policy-pursing, we should be able to explain why minority and super-majority governments form. The formation and the stability of these governments, like the strong proportionality of the allocation of cabinet portfolios, constitutes an important and unexplained paradox in the literature on coalition theory in parliamentary legislatures.
Appendix

Proof of Proposition 1 Assume to have a coalition composed by three parties \((c = \{1, 2, 3\})\) with \(\Lambda^j_i\) denoting \(i\)'s offer to partner \(j\) (with \(j \in c \setminus i\), \(\forall i \in c\). Suppose by contradiction that the vector bid can be denoted as \(\Lambda^i = \lambda^i[(w_1 + x_1), \ldots, (w_{i-1} + x_{i-1}), (w_{i+1} + x_{i+1}), \ldots, (w_n + x_n)]\) with \(x_i \in \mathbb{R}\) such that \(\sum_i x_i = 0\).

Suppose that in a 3-party coalition, \((x_3 > x_2 > x_1 > 0)\) and that party 1’s bid vector is:

\[
\Lambda^1 = [\Lambda^1_2, \Lambda^1_3] = [\lambda^1 (w_2 + x_1), \lambda^1 (w_3 - x_1)]
\]  

(1)

The equilibrium bid must make both the receiving partners and the bidder better off or indifferent between winning and losing the auction. By application of the Intermediate Value Theorem, there exists a unique price \((\hat{\lambda}^j_i)\) that makes every party indifferent to the possibility of winning and losing the auction.

\[
\begin{align*}
\Lambda^1_2 &\geq 1 - (\Lambda^2_1 + \Lambda^2_3) \\
\Lambda^1_3 &\geq 1 - (\Lambda^3_1 + \Lambda^3_2) \\
1 - (\Lambda^1_2 + \Lambda^1_3) &\geq max(\hat{\lambda}^2_1, \hat{\lambda}^3_1) \\
1 - \lambda^1 (w_2 + w_3) &\geq max(\hat{\lambda}^2_2 (w_1 \pm x_2), \hat{\lambda}^3 (w_1 \pm x_3))
\end{align*}
\]

(2)

The right-hand side set of inequalities are derived by plugging the equation for \(\Lambda^1\) (6) in the left hand side set of inequalities and solving for \(\lambda^1\). Since \(\frac{1}{\sum_i w_i - x_i} < \frac{1}{\sum_i w_i + x_i}\), then party 1 must bid at least \(\hat{\lambda}^1 = \frac{1}{\sum_i w_i - x_i}\) in order to make both the partners indifferent. By the same argument, party 2 and party 3 need to bid at least the amount that make the two other parties indifferent:

\[
\hat{\lambda}^1 = \frac{1}{\sum_i w_i - x_1}, \quad \hat{\lambda}^2 = \frac{1}{\sum_i w_i - x_2}, \quad \hat{\lambda}^3 = \frac{1}{\sum_i w_i - x_3}
\]
Winning the auction is a best reply for player 1 as long as the third inequality is satisfied. However, since we have not specified the bid vectors for party 2 and party 3, we need to analyze two different cases:

**Case 1:** the arg max is greater than \( \lambda^j x_1 \)

Assume, without loss of generality that \( \max(\hat{\lambda}^2(w_1 \pm x_2), \hat{\lambda}^3(w_1 \pm x_3)) = \hat{\lambda}^3(w_1 + x_3) \). By substituting \( \hat{\lambda}^1 \) and \( \hat{\lambda}^3 \) into the third inequality of (7), we get a contradiction. Thus, party 1 never bids in equilibrium.

**Case 2:** the arg max is less than \( \lambda^j x_1 \)

Assume, without loss of generality that \( \max(\hat{\lambda}^2(w_1 \pm x_2), \hat{\lambda}^3(w_1 \pm x_3)) = \hat{\lambda}^2(w_1 - x_2) \). By substituting \( \hat{\lambda}^1 \) and \( \hat{\lambda}^2 \) into the third inequality of (7), we have a contradiction for all \( x_1 > x_2 \). Thus, by backward induction, only the party proposing the “least disproportional” allocation (lowest \( x \)) - among those parties receiving their highest utility from a less than proportional bid - bid in equilibrium.

Suppose party \( i \) bids in equilibrium. Party \( i \) maximize its utility by choosing a vector that minimizes the cost of the auction \( (x_i = 0) \):

\[
\max_{x_i} U_i = 1 - \hat{\lambda}^i \left( \sum_{j \in c \setminus i} w_j \right) = 1 - \frac{1}{\sum_{j \in c \setminus i} w_i - x_i} \left( \sum_{j \in c \setminus i} w_j \right)
\]

For the same argument, the equilibrium strategy of each party \( j \in c \) is to choose \( x_j = 0 \). In equilibrium, the bid vector is therefore equal to a “per capita” bid that multiplies the partners’ weight vector.

\[
\hat{\Lambda}^i = \hat{\lambda}^i[w_1, ..., w_{i-1}, w_{i+1}, ..., w_n]
\]

That concludes the proof. ■

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Proof of Proposition 2 By application of the Intermediate Value Theorem, there exists a unique price \( \hat{\lambda}_{i,j} \) that makes every party in the coalition \( c_{i,j} \) indifferent to the possibility of winning and losing the auction. Thus, there exists a price \( \Lambda^m (\Lambda^j) \) that makes party \( m \) (\( j \)) indifferent between winning and losing the auction.

\[
\begin{align*}
U^m (\lambda_m w_m) &= \max_{s_i \geq 0} U^m (s_m) \\
\text{subject to:} & \sum_{i=1}^{n} s_i \leq 1 \\
U^i (s_i) &\geq U^i (\lambda^m w_i) \forall i \in c^j \setminus m
\end{align*}
\]

\[
\begin{align*}
U^j (\lambda_j w_j) &= \max_{s_i \geq 0} U^j (s_j) \\
\text{subject to:} & \sum_{i=1}^{n} s_i \leq 1 \\
U^i (s_i) &\geq U^i (\lambda^j w_i) \forall i \in c^j \setminus j
\end{align*}
\]

But, from the maximization problems above, party \( m \) is indifferent between being the proposer (making the government proposal) or the receiver (agreeing to the proposal made by a coalition partner).

\[
\max_{s_i \geq 0} U^m (s_m) = U^m (\lambda^m w_m) = U^m (\lambda^j w_m)
\]

Therefore, \( \lambda^m = \lambda^j = \hat{\lambda}_{c,j} \) for every \( j \in c^j \). That concludes the proof.

Proof of Proposition 3 Notice that for the feasibility constraint in the third-stage solution (which holds with equality) \( \sum_{i=1}^{n} s_i = 1 \). Since the constraints hold with equality in any equilibrium allocation \( s_i = \hat{\lambda}_{c,j} w_i \), we derive that \( \sum_{i \in c^j} \hat{\lambda}_{c,j} w_i = 1 \) and thus: \( \hat{\lambda}_{c,j} = \frac{1}{\sum_{i \in c^j} w_i} \), where \( \sum_{i \in c^j} w_i \) is the size of the proto-coalition. We can then rewrite \( S^* \) as: \( \left( w_1 / \sum_{i \in c^j} w_i, ..., w_{m-1} / \sum_{i \in c^j} w_i, w_m / \sum_{i \in c^j} w_i, 0, ..., 0 \right) \) where \( w_i / \sum_{i \in c^j} w_i \) is the nominal voting weight of party \( i \) in the proto-coalition.
Proof of Proposition 4 Let $c_i$ be party $i$’s strategy. A smallest minimal winning coalition $c^*$ is the Strong Nash equilibrium strategy for party $i$. If party $i$ chooses $c^+$ such that $\sum_{j \in c^+} w_j > \sum_{j \in c^*} w_j$ then its payoff is less than the payoff it would receive by proposing $c_J\left(\frac{w_i}{\sum_{j \in c^+} w_j} < \frac{w_i}{\sum_{j \in c^*} w_j}\right)$, making such strategy dominated by $c_i = c^+$. If party $i$ chooses $c^-$ such that $\sum_{j \in c^-} w_j < \sum_{j \in c^*} w_j$, its payoff would be more than the payoff it would receive by proposing $c^-$. However, if party $i$ proposed a minority coalition, all the parties outside that coalition would have an incentive to deviate together and form a majority proto-coalition $c_{-i}$ which would be voted first and be approved by the legislature. Thus the “less than majority” coalition is not immune to deviation by group of players, thus is not a “Strong equilibrium”.

Proof of Proposition 6 Since the game ends after period $T$, the continuation value for all null sub-games and for all parties is $v_i = 0$. In the third stage of period $T$, the formateur proposes an allocation that must make the parties in the proto-coalition better off or at least indifferent within the proto-proposal. As proved in Proposition 3, the proto-proposal allocates $\frac{w_i}{\sum_{j \in c^*} w_j}$ to the parties in the proto-coalition and 0 to the parties outside the proto-coalition. As in Proposition 4, the proto-coalition is the smallest minimal winning coalition. Thus, the continuation $v_i$ at period $T-1$ is the expectation of party $i$-th allocation in the $T$-th period $\{(v_i = \frac{w_i}{\sum_{j \in c^*} w_j} \forall i \in c^*); (v_i = 0 \forall i \notin c^*)\}$. 

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At time $T - 1$, the formateur party has to offer to the other parties at least the continuation value $\delta v_i$ in order to obtain a majority vote. However, as it happens in period $T - 1$, the formateur has to propose an allocation that must make the parties in the proto-coalition better off or at least indifferent within the proto-proposal. The constraint imposed by the future utilities on what the partners can receive, then, is not binding ($\delta w_i / \sum_{j \in c^*} w_j \leq w_i / \sum_{j \in c^*} w_j \forall \delta \in (0, 1]$). Thus, the game ends at the first period, with the formateur proposing an allocation that makes indifferent the partners in the smallest minimal winning coalition with the proto-proposal ($v_i = w_i / \sum_{j \in c^*} w_j, \forall i \in c^*$) and the other parties with their continuation value ($v_i = 0, \forall i \notin c^*$).

**Proof of Proposition 7** Let’s consider the case in which a coalition is composed by two parties with equal weight. Thus, the problem is equivalent to a two-person two-goods exchange economy where two agents have utility functions over two goods and the initial endowments (or allocations) of the two goods. Let assume that the initial endowment is the proportional allocation with respect their voting weights. Assume that the allocation $(x_1, y_1)$ denotes the share of the two goods that party 1 obtains and $(x_2, y_2)$, the share of the two goods that party 2 obtains. Assume the functional form to be a Cobb-Douglas utility function and the total endowment of each good to be one, so that $x_2 = 1 - x_1$. Then party 1’s utility can be written as $U_1 = x^\alpha y^{1-\alpha}$, and 2’s utility is $U_1 = x^\beta y^{1-\beta}$. Then a point is Pareto efficient if
Thus, solving for $y$, a point is on the contract curve (it is Pareto efficient) if:

$$y = \frac{(1-\alpha)\beta x}{(1-\beta)\alpha + (\beta - \alpha) x} = x + \frac{(1-\beta)\alpha x}{(1-\alpha)\beta x}$$

Thus, the contract curve for the Cobb-Douglas case depends on a single parameter $\left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta}\right)$. The proportional allocation (which yields $x = y$ is a Pareto efficient outcome and therefore an equilibrium of the game if and only if the ratio is equal to 1. But $\left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta}\right)$ is equal to 1 if and only if $\alpha = \beta$, or in other words when the parties have homogeneous preferences. ■

**Proof of Proposition 8** No party would strictly prefer the role as a receiving party if $\hat{\lambda} = 0$. The desirability assumption of $\tau$ ensures that for any party there exists a finite value of $\hat{\lambda}$ at which the party would prefer the role as a receiving party. Assuming that each party’s utility as proposer is a continuous function of $\hat{\lambda}$, then, by the Intermediate Value Theorem, there exists a unique price $(\hat{\lambda}_{i,j})$ that makes every party in the coalition $c_{i,j}$ indifferent to the outcome of winning or losing the auction. The rest of the proof follows proof of Proposition ???. ■

**Proof of Proposition 9** Let first prove that an equilibrium coalition cannot be larger than minimal winning. Let’s prove it by contradiction: suppose to have a coalition government $c^*$ larger than minimal winning composed by party 1, party 2, and party 3, such that $w_1 = w_2 + w_3$. By proof, we know that an allocation
\((x_1^*, y_1^*)\) is optimal if \(y_1 = \frac{x_1}{x_1 + (1 - \beta_2)\alpha(1 - x_1)}\), where \(\alpha\) is the preference parameter for party 1 and \(\beta\) is the average preference parameter of party 2 and party 3 \((\beta = \frac{\beta_2 + \beta_3}{2})\). Notice that, for any \(\alpha\), the larger the parameter \(\beta\) is, the larger the utility of party 1. Then, suppose that party 1 deviates to a smaller coalition (which is still minimal winning) composed by party 1 and party 3, such that \(w_1 + w_3 > \frac{1}{2}\), and suppose that party 3’s preference parameter \(\beta_3\) is larger than \(\beta_2\). Then:

\[
\frac{x_1}{x_1 + \left(\frac{(1 - \beta_3)\alpha}{(1 - \alpha)\beta}\right)(1 - x_1)} > \frac{x_1}{x_1 + \left(\frac{(1 - \beta_2)\alpha}{(1 - \alpha)\beta}\right)(1 - x_1)}
\]

Thus, since party 1 is better off by deviating to a smallest winning coalition, a coalition that is larger than minimal winning is not an equilibrium of the game.

Let now prove that an equilibrium coalition can be larger than the smallest minimal winning coalition. Let’s suppose to have a coalition composed by two parties are of equal size \((w_1 = w_2)\) which is larger than the smallest minimal winning coalition. An equilibrium allocation is Pareto optimal if \(y_1\) is equal to \(\frac{x_1}{x_1 + (1 - \beta)\alpha(1 - x_1)}\). Suppose that party 1 deviates to a smaller coalition (which is the smallest minimal winning coalition) composed by party 1 and party 3, where party 3 has the same preferences than party 1 \((\beta = \alpha)\). Then the equilibrium allocation would be \(x = y = \frac{w_3}{w_1 + w_3}\). Thus, party 1 deviates to the smallest winning coalition if and only if:

\[
\frac{w_1}{w_1 + w_3} > \frac{x}{x + \left(\frac{(1 - \beta)\alpha}{(1 - \alpha)\beta}\right)(1 - x)}
\]

For values of \(\alpha\) and \(\beta\) such that the above inequality is not satisfied, a coalition that is larger than the smallest minimal winning coalition is the equilibrium coalition of the game.■
References


Figure 1: Share of cabinet portfolio plotted against share of legislative seats contributed to the government coalition, using the Warwick and Druckman (2006) dataset for 14 Western European countries from 1945 to 2000. The top panel shows the share of portfolio for all coalition governments. The bottom panels show plots for MWC only (left) and Minority and Super-majority governments (right). The 45 degrees line identifies perfect proportionality.
Party i’s nominal voting weight: \( \frac{w_i}{w_i + w_j} \)

Share of cabinet posts: \( g_{68} \)

Figure 2: Equilibrium allocation for a proto-coalition composed by party i and party j, where party j’s size is fixed to 5 legislators and party i’s size varies between 1 and 10. Party j’s preference parameter is fixed at \( \alpha_j = 0.5 \). The blue, red, and green lines describe the equilibrium allocations when \( \alpha_i = 0.5 \), \( \alpha_i = 0.7 \), and \( \alpha_i = 0.9 \) respectively.
Table 1: Portfolios allocations with homogeneous preferences

<table>
<thead>
<tr>
<th>(\alpha_i = \alpha_j = \alpha_k = 0.5)</th>
<th>(c = {i, j})</th>
<th>(c = {i, k})</th>
<th>(c = {j, k})</th>
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</thead>
<tbody>
<tr>
<td>(\tilde{\lambda}_c)</td>
<td>0.14</td>
<td><strong>0.20</strong></td>
<td>0.17</td>
</tr>
<tr>
<td>((IA_i, FA_i))</td>
<td>(0.43, 0.43)</td>
<td><strong>(0.60, 0.60)</strong></td>
<td>(0, 0)</td>
</tr>
<tr>
<td>((IA_j, FA_j))</td>
<td>(0.57, 0.57)</td>
<td><strong>(0, 0)</strong></td>
<td>(0.67, 0.67)</td>
</tr>
<tr>
<td>((IA_k, FA_k))</td>
<td>(0, 0)</td>
<td><strong>(0.40, 0.40)</strong></td>
<td>(0.33, 0.33)</td>
</tr>
<tr>
<td>(U_i(IA_i, FA_i))</td>
<td>0.43</td>
<td>0.60</td>
<td>0</td>
</tr>
<tr>
<td>(U_j(IA_j, FA_j))</td>
<td>0.57</td>
<td>0</td>
<td>0.67</td>
</tr>
<tr>
<td>(U_k(IA_k, FA_k))</td>
<td>0</td>
<td><strong>0.40</strong></td>
<td>0.33</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\alpha_i = 0.2, \alpha_j = \alpha_k = 0.5)</th>
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<th>(c = {i, k})</th>
<th>(c = {j, k})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{\lambda}_c)</td>
<td>0.15</td>
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<tr>
<td>((IA_i, FA_i))</td>
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<tr>
<td>((IA_k, FA_k))</td>
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<td>(0.33, 0.33)</td>
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<tr>
<td>(U_i(IA_i, FA_i))</td>
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<td><strong>0.63</strong></td>
<td>0</td>
</tr>
<tr>
<td>(U_j(IA_j, FA_j))</td>
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<td>0</td>
<td>0.67</td>
</tr>
<tr>
<td>(U_k(IA_k, FA_k))</td>
<td>0</td>
<td><strong>0.42</strong></td>
<td>0.33</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>(\alpha_i = \alpha_k = 0.2, \alpha_j = 0.9)</th>
<th>(c = {i, j})</th>
<th>(c = {i, k})</th>
<th>(c = {j, k})</th>
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<td><strong>0.21</strong></td>
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<td>((IA_i, FA_i))</td>
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<tr>
<td>(U_j(IA_j, FA_j))</td>
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<td>0</td>
<td><strong>0.40</strong></td>
<td><strong>0.42</strong></td>
</tr>
</tbody>
</table>

Notes - The table reports the equilibria for a legislature composed by three parties (party \(i\), party \(j\), and party \(k\)) with 3, 4, and 2 members respectively. The three panels represent three different scenarios: in the first panel, all parties are indifferent between the two types of posts \(IA\) and \(FA\) \((\alpha_i = \alpha_j = \alpha_k = 0.5)\); in the second panel, \(\alpha_i = 0.2\) and \(\alpha_j = \alpha_k = 0.5\); in the third panel \(\alpha_i = \alpha_k = 0.2\) and \(\alpha_j = 0.9\). In each panel, the table reports the equilibrium bid \(\tilde{\lambda}_c\), the equilibrium allocation vector \((IA_i, FA_i), (IA_j, FA_j), (IA_k, FA_k)\), and the utility \((U_i, U_j, U_k)\) for every possible proto-coalition (columns).