A Model of Endogenous Government

Formation

On line Appendix

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Equilibria simulations for a 3-party legislature

In this section of the Appendix, I develop numerical simulations of the government formation model with parties with heterogeneous preferences in a setting where a three-party legislature, none with a majority of the seats, is composed by an arbitrary number of legislators (set at $L=19$). Simulations are run for all possible permutations of parties’ size (with the size of every party ranging from 1 to 9) and preferences (the preference parameter space is discretized into nine points: $\alpha_i \equiv \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \forall i \in N$), for a total of 7290 cases\(^1\). Equilibria were calculated by assuming a Cobb-Douglas utility function over two types of ministerial posts as described in section “Heterogenous preferences over portfolios” of the paper. Figure 1 plots the equilibrium allocations for the predicted equilibrium coalition as a function of the government party’s size.

![Figure 1 about here]

The resemblance with the figure 1 of the paper is striking. Even when parties have different preferences over cabinet portfolios, the size effect seems to still drive most of the equilibrium strategies in equilibrium.

![Table 1 about here]

\(^1\)We computed 10 legislatures with different sized parties summing to 19 legislators (ten combinations of size 3 from a set $S = 1, 2, 3, 4, 5, 6, 7, 8, 9$ summing to 19) with $9^3$ preference profiles each.
Table 1 compares regression analysis employing both actual and simulated data. The first three columns report the estimated coefficients from an analysis using actual observations from fifteen West European countries from 1945 to 2002 (using data from STA, 2005). The second three columns report the coefficients using the simulated data.

The simulated data provide almost a perfect match to the real-world data, both in terms of sign of the coefficients and magnitude. As a first example, consider the most discussed question in the literature of whether there exists a one-to-one correspondence between Portfolio allocation and Seat share contribution (as suggested by Gamson, 1961). Column 1 and 4 show how the same analysis conducted with actual and simulated data reaches very similar results: the Seat share contribution to the government explains 90% and 97% of the Portfolios allocation, respectively.

Column 2 and 5 report the results from a model in which we add a dummy variable (Major party) for the party with the largest number of seats in the legislatures (which, in most countries, is the formateur). In this model, the Seat share contribution explains 93% and 98% of the Portfolios allocation, respectively; while being the Major party affects the allocation of benefit in a negative and significant way in both actual and simulated data.

In addition, columns 3 and 6 include a variable capturing the absolute size of each party (Raw seat share), showing once more how the actual and the simulated data come close: in this model the Seat share contribution explains 96% and 98% of the Portfolios allocation.
The degree of concurrence between equilibrium simulation and real-world data is remarkable, particularly considering that the theoretical model is highly simplified and that it abstracts away from a number of determinants that may affect the allocations of portfolios such as party’s policy preferences, possible pre-election agreements, non perfect divisibility of cabinet portfolios, or specific conventions that affect the government formation process from the selection of the formateur to the kind of parties that can possibly coalesce together.

**Incumbent caretaker governments**

In this section of the Appendix, I develop an extension of the baseline model which accounts for partisan caretaker governments.

In many parliamentary legislatures, the incumbent government serves as a temporary government until the new government is formed or as a caretaker government until new elections are called or political crisis are solved. In these cases, if the government formation is unsuccessful at the end of period T, the utility for the members of the incumbent governments is positive rather than zero. This non-zero reservation price for the incumbent government parties affects the equilibrium strategies of the players.

**Stage 4: Legislature voting stage**

Solving the game backward, in the last stage of the game incumbent parties \((i \in c^I, \text{ with } c^I \text{ denoting the incumbent government coalition})\) vote in favor of a
proposal if $U_i \geq \delta \bar{U}_i$, where $\bar{U}_i$ is the utility of continuing to hold portfolios until a new government is appointed, and $\delta \in [0, 1]$ is a discount factor that accounts for the caretaker government *interim* term and for its limited powers:

$$v_i = 1 \quad \text{if} \quad \begin{cases} U_i \geq \delta \bar{U}_i & \forall i \in c^l; \\ U_i \geq 0 & \forall i \notin c^l. \end{cases}$$

Let assume that the reservation price for the incumbent parties is proportional to their seat contribution in the incumbent government coalition - that is, the portfolio allocation of the incumbent government is defined by Proposition 7 of the baseline model:

$$\bar{U}_i = \frac{\widetilde{w}_i}{\sum_{l \in c^l} \widetilde{w}_l} \quad \forall i \in c^l$$

(1)

where $\widetilde{w}_i$ denotes the weight of party $i$ in the former legislature.\(^2\)

**Stage 3: Bargaining over cabinets**

In bargaining stage, the formateur of the proto-coalition $c^l$ incorporates this new vector of constraints in the maximization problem, proposing $S^m = (s_1, s_2, \ldots, s_n)$.

\(^2\)If a government formation process follows a government crisis within the same legislature cycle (known also as government reshuffle), parties’ weight do not change and hence the equilibrium prediction remain the same. Thus, the existence of partisan caretaker government affect the equilibrium of the game only if the government formation process follows new elections.
such that utility $U^m$ is maximized and such that the proposal is approved by the proto-coalition parties:

$$\max_{s_i \geq 0} U^m(s_m) \quad \text{subject to} \quad \sum_{i=1}^{n} s_i \leq 1$$

$U^i(s_i) \geq U^i(\lambda^m w_i) \quad \forall i \in c^I, i \notin c^f, i \neq m$

$U^i(s_i) \geq \delta U_i \quad \forall i \in c^f \cap c^I, i \neq m$ (2)

The first constraint ensures feasibility, the second constraint ensures that each partner will voluntarily choose $s_i$ over $\lambda^m w_i$, and the third constraint ensures that each partner will voluntarily choose $s_i$ over terminating the proto-coalition negotiations. Any solution must satisfy either the second or the third constraint with equality, depending on which is binding. If the second constraint is binding, the proposer maximizes its own utility by choosing an allocation $(s_i)$ that makes the partners indifferent with the offer $\lambda^m w_i$, exactly as in the baseline model. Elsewhere, the proposer maximizes its utility by allocating to an incumbent party a share that makes it indifferent with its reservation price $\tilde{w}_i / \sum_{l \in c^l} \tilde{w}_l$.

**Lemma 1** If the members of a proto-coalitions are not incumbent, or if the share of benefits that each incumbent government party is allocated is larger than their reservation price ($\lambda^m w_i > \delta \sum_{i \in c^l} \tilde{w}_i$), the equilibrium allocation is the same as in the baseline model.
Stage 2: Formateur selection

The coalition parties make an offer to the partners in order to be the formateur of the government coalition. $U^m(s_m)$ represents the value of becoming formateur in any given proto-coalition. If the third constraint of the maximization problem is binding, the value of being formateur is bounded by the reservation price of the incumbent government parties. Otherwise, the value of being formatuer is unaffected by the reservation price of the incumbent parties.

**Lemma 2** If the members of a proto-coalitions are not incumbent, or if the share of benefits that each incumbent government party is allocated is larger than their reservation price ($\lambda^m w_i > \frac{\sum_{j \in c^j} w_j}{\sum_{i \in c^j} w_i}$):

1. the equilibrium bid is the same as in the baseline model $\lambda^m = \frac{1}{\sum_{i \in c^j} w_i}$;
2. the share of cabinet portfolio each party is allocated in equilibrium is perfectly proportional to the nominal voting weight:

$$S^* = \left( \frac{w_1}{\sum_{i \in c^j} w_i}, ..., \frac{w_{m-1}}{\sum_{i \in c^j} w_i}, \frac{w_m}{\sum_{i \in c^j} w_i}, 0, ..., 0 \right)$$

Notice that, if $\sum_{i \in c^j} w_i > \sum_{j \in \tau} w_j$, the incumbency status of some members does not affect the equilibrium allocation of benefits. Intuitively, if either the incumbent gains legislative seats in the new election cycle ($w_1 > \bar{w}_j$) or if it gains relative weight inside a proto-coalition, its portfolio allocation will not be affected by its reservation price.
Proposition 3 If there exists one proto-coalition member $k$, belonging to the incumbent government coalition ($k \in c^J \cap \tau$), for which the reservation price is larger than the equilibrium offer of the formateur ($\sum_{i \in c^I} w_i < \delta \sum_{i \in E} w_i$):

1. the equilibrium bid for each party in the proto-coalition except $k$ is:

$$\hat{w}_{c^J} = \frac{1}{\sum_{i \in c^J \setminus k} w_i} \left[ 1 - \delta \frac{\tilde{w}_k}{\sum_{i \in c^I} w_i} \right] \quad \forall i \in c^J \setminus k$$

(4)

2. the share of cabinet portfolio each party is allocated in equilibrium is

$$S^* = \begin{cases} \frac{\delta \tilde{w}_k}{\sum_{i \in \pi} w_i} & \text{s.t. } \sum_{i \in \pi} w_i < \sum_{i \in c^I} w_i \\ \frac{\delta \sum_{i \in c^J \cap c^I} \tilde{w}_k}{\sum_{i \in c^J \setminus k} w_i} \sum_{i \notin k \in c^J} \frac{w_i}{w_i} & \forall j \in c^J \setminus k, \\ 0 & \forall j \notin c^J \end{cases}$$

Thus, the incumbent party $k$ enjoys an “incumbency premium” $\Delta_{k}^{J}$, while the partners suffer an “incumbency loss” $\Upsilon_{k}^{J, k}$. 
Stage 1: Coalition selection

In the coalition formation stage each party proposes the proto-coalition that maximizes its utility. If incumbent parties have a reservation price different than zero, the qualitative properties of the Strong equilibria depend on the reservation price for the incumbent parties and the parties’ weight.

Proposition 4 If communication is allowed, unlimited, but non-binding, and smallest minimal winning coalition which does not include any incumbent party for which the reservation price is higher than the equilibrium offer of the formateur, then the smallest minimal winning coalition is the unique Strong Equilibrium of the game:

Proof. Let \( c' \) be the smallest minimal winning coalition \((\sum_{j \in c'} w_j < \sum_{j \in c''} w_j \forall c'' \neq c' \in \mathcal{C})\) which does not include any incumbent party with a reservation price higher than the equilibrium offer of the formateur \( (\exists \, k \text{ s.t. } \sum_{i \in c} w_k < \sum_{i \in c'} w_i)\). Let suppose that there exists a minimal winning coalition \( c' \) which does include an incumbent party with a reservation price higher than the equilibrium offer of the formateur.
(\(k \in \overline{c}\) s.t. \(\sum_{i \in c} w_i \geq \sum_{i \in c'} \tilde{w}_i\)). The smallest minimal winning coalition \(c'\) is the Strong Nash equilibrium strategy for party \(i\). If party \(i\) chooses \(c^J\), then it would obtain a share \(\lambda^J w_i\), which is less than the payoff it would receive by proposing \(c'\), making such strategy dominated by \(c_i = c'\)

\[
\lambda^J w_i = \left(1 - \frac{\delta \tilde{w}_k}{\sum_{j \in c'} \tilde{w}_j}\right) \frac{w_i}{\sum_{j \in c' \setminus c} w_j} < \left(1 - \frac{w_k}{\sum_{j \in c' \setminus c} w_j}\right) \frac{w_i}{\sum_{j \in c' \setminus c} w_j} = \frac{w_i}{\sum_{j \in c'} w_j}
\]

If party \(i\) chooses \(c^-\) such that \(\sum_{j \in c^-} w_j < \sum_{j \in c^*} w_j\), its payoff would be more than the payoff it would receive by proposing \(c^*\). However, if party \(i\) proposed a minority coalition, all the parties outside that coalition would have an incentive to deviate together and form a majority proto-coalition \(c^-i\) which would be voted first and be approved by the legislature. Thus the “less than majority” coalition is not immune to deviation by group of players, thus is not a “Strong equilibrium”.

**Proposition 5** If communication is allowed, unlimited, but non-binding, and there exists a smallest minimal winning coalition \((c^J\) such that \(\sum_{j \in c^J} w_j < \sum_{j \in c'} w_j \forall c' \in C\) which includes an incumbent party for which the reservation price is higher than the equilibrium offer of the formateur \((k \in c^J\) s.t. \(\sum_{i \in c^J} w_i < \sum_{i \in c'} \tilde{w}_i\)), then the smallest minimal winning coalition is the unique Strong Equilibrium of the game if:
Proof. Let $c''$ be the smallest minimal winning coalition which does not include any incumbent party with a reservation price higher than the equilibrium offer of the formateur. $c'$ is the Strong Nash equilibrium of the game if:

$$
\begin{align*}
\sum_{i \in c'} \frac{\delta \tilde{w}_i}{w_i} &\geq \max(\lambda'w_k) \quad \forall c' \in \mathcal{C} \\
\lambda^c w_i &\geq \max(\lambda'w_k) \quad \forall c' \in \mathcal{C}
\end{align*}
$$

(5)

Rearranging the terms, we have that:

$$
1 - \frac{\delta \sum_{j \in c' \setminus k} \tilde{w}_j}{\sum_{j \in c'} \tilde{w}_j} \leq \frac{\sum_{j \in c'} w_j - \sum_{j \in c' \setminus k} w_j}{\sum_{j \in c'} w_j}
$$

The left hand side of the inequality represents the loss of pie for the non-incumbent parties, the pie that is not up for bargaining if party $j$ proposes coalition $c'$. The right hand side of the equation represents the loss of efficiency that party $j$ suffers in proposing a coalition $c''$ larger than minimum weight.\[\square\]
Discussion

The non-partisan caretaker government assumption in the paper is not a completely natural assumption: in some cases, caretaker governments are partisan with Cabinet members belonging to political parties. However, assuming partisan caretaker governments would affect parties’ strategies and equilibria only if the utility of participating in the caretaker government is sufficiently large (proposition 5). However, since caretaker governments are meant to be temporary solutions to the government formation impasse, one can claim that the utility from participating in the interim caretaker government is significantly smaller than the utility from forming a full term government. Thus, the assumption of non-partisan caretaker governments - or equivalently, the assumption of sufficiently small utility of taking part of a caretaker government - provides a necessary preliminary to a more general analysis and it is reasonable in most situations where the caretaker government rules temporarily and it is supposed to only handle daily issues and budgets.
Figure 1: Equilibrium share of cabinet posts as plotted against the share of legislative seats for a legislature composed by 3 parties and 19 legislators. The figure plots the equilibrium allocations for the equilibrium government coalition for all the possible compositions of parties. The 45 degrees line identifies proportional allocation to the parties’ nominal voting weight.
Table 1: Portfolios allocation

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<th>Simulations</th>
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Notes: Values in parentheses are standard errors. *** significant at 0.01 level; ** significant at 0.05 level; * significant at 0.10 level. The real world data in the first three columns are for fifteen West European countries from 1945 to 2002. These data are from Snyder et al. (2005) dataset. The simulated data in the second three columns represent coefficients and standard errors using data from the 7,290 simulated legislatures.