Homework 1 Solutions

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1 Exercise 1.1

We want to prove that $||\vec{u}+\vec{v}|| \leq u+v$, but we want to take an alternate route. Instead, we prove an equivalent statement, namely $||\vec{u}+\vec{v}||^2 \leq (||\vec{u}||+||\vec{v}||)^2 = (u+v)^2$.

$$\begin{aligned}
||\vec{u}+\vec{v}||^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot(\vec{u}+\vec{v})+\vec{v}\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} \\
&= ||\vec{u}||^2+2||\vec{u}\cdot\vec{v}||+||\vec{v}||^2 \\
&\leq ||\vec{u}||^2+2||\vec{u}||||\vec{v}||+||\vec{v}||^2 \\
&= (||\vec{u}||+||\vec{v}||)^2 \\
&= (u+v)^2
\end{aligned}$$

The opposite direction follows in the same manner by writing $||\vec{u}|| = ||\vec{u}-\vec{v}+\vec{v}||$ and using the information from the first part of the proof.

2 Exercise 1.2

This problem is easier to explain with a diagram, but instead we note the following. First, $||\vec{u}\times\vec{v}|| = uvsin\theta$. Next, the area of a parallelogram is given by $A = b(h)$. In our case, the base is given by $||\vec{v}||$ and the height is given by $||\vec{u}||sin\theta$. Thus our area becomes:

$$A = b(h) = ||\vec{v}||||\vec{u}||sin\theta = uvsin\theta = ||\vec{u}\times\vec{v}||$$

3 Exercise 1.4

Let $\vec{u} = u_1\vec{e_1}+u_2\vec{e_2}+u_3\vec{e_3}$ and $\vec{v} = v_1\vec{e_1}+v_2\vec{e_2}+v_3\vec{e_3}$. Then we have

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (u_1\vec{e_1}+u_2\vec{e_2}+u_3\vec{e_3})\cdot(v_1\vec{e_1}+v_2\vec{e_2}+v_3\vec{e_3}) \\
&= u_1\vec{e_1}\cdot(v_1\vec{e_1}+v_2\vec{e_2}+v_3\vec{e_3})+u_2\vec{e_2}\cdot(v_1\vec{e_1}+v_2\vec{e_2}+v_3\vec{e_3})+u_3\vec{e_3}\cdot(v_1\vec{e_1}+v_2\vec{e_2}+v_3\vec{e_3})
\end{aligned}$$

Using the identities from (1.6) and carrying through the product, we are left with

$$\begin{aligned}\vec{u}\cdot\vec{v} &= u_1\vec{e_1}\cdot v_1\vec{e_1} + u_2\vec{e_2}\cdot v_2\vec{e_2} + u_3\vec{e_3}\cdot v_3\vec{e_3} \\ &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= \sum_{i=1}^{3} u_iv_i\end{aligned}$$

Now for the cross product (a few steps are omitted):

$$\begin{aligned}\vec{u}\times\vec{v} &= (u_1\vec{e_1} + u_2\vec{e_2} + u_3\vec{e_3}) \times (v_1\vec{e_1} + v_2\vec{e_2} + v_3\vec{e_3}) \\ &= u_1v_1(\vec{e_1}\times\vec{e_1}) + u_2v_1(\vec{e_2}\times\vec{e_1}) + u_3v_1(\vec{e_3}\times\vec{e_1}) + \ldots + u_1v_3(\vec{e_1}\times\vec{e_3}) + u_2v_3(\vec{e_2}\times\vec{e_3}) \\ &\quad + u_3v_3(\vec{e_3}\times\vec{e_3}) \\ &= (u_2v_3 - u_3v_2)\vec{e_1} + (u_3v_1 - u_1v_3)\vec{e_2} + (u_1v_2 - u_2v_1)\vec{e_3} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}u_jv_k\vec{e_i}\end{aligned}$$

4 Exercise 1.5

$det[A] = (A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}) - (A_{11}A_{23}A_{32} + A_{12}A_{21}A_{33} + A_{13}A_{22}A_{31})$

And since $A_{ji} = \vec{e_j}' \cdot \vec{e_i}$, convince yourselves that the first term in the determinant cannot be equal to the second. Thus, the determinant is non-zero and A is invertible. Next, we show that $A_{ir}A_{jr} = \delta_{ij}$. To do this, we make use of the identities (1.14) and (1.16).

$$A_{ir}A_{jr}\vec{e_j}' = A_{ir}\vec{e_r} = \vec{e_i}' = \delta_{ij}\vec{e_j}'$$

Thus we see our desired result.

5 Exercise 1.6

The bases considered in Example 1.1 corresponding to $\theta = \frac{\pi}{4}$ yields the following change of bases matrix.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the given vectors $(1, 1, 3)$ and $(\lambda, \sqrt{2}, 3)$ in their respective bases, we want to find $\lambda = u_1'$ (in the notation of the chapter). Thus we need to use identity (1.17) to say that

$$\begin{aligned}\lambda &= u_1' = A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ &= \frac{\sqrt{2}}{2}(1) - \frac{\sqrt{2}}{2}(1) + 0(3) \\ &= 0\end{aligned}$$

6 Exercise 1.7

This is the same as Exercise 1.4, but now you must represent the vectors $\vec{u}$ and $\vec{v}$ in both bases e_i and e'_i. The actual work is no different.