Homework 2 solutions

2.1. ① T symmetric $\iff u\cdot T\cdot v = v\cdot T\cdot u$

$\forall u, v \in \mathcal{V}$

$\Rightarrow \quad u\cdot T\cdot v = (T^t\cdot u)\cdot v$

$= v\cdot(T^t\cdot u)$

$= v\cdot T\cdot u$

$\Leftarrow \quad u = e_i \quad , v = e_j$

$e_i\cdot T\cdot e_j = e_j\cdot T\cdot e_i$

$\rightarrow T_{ij} = T_{ji} \rightarrow T$ symmetric

② The proof is the same for T skew-symmetric.

2.2. ⓐ Let $T \in \mathcal{T}$ and $A = T + T^t$

Thus $A_{ij} = T_{ij} + T_{ji}$ and

$A_{ji} = T_{ji} + T_{ij} = T_{ij} + T_{ji} = A_{ij}$

$\rightarrow A$ is symmetric

ⓑ $A = T - T^t$, thus $A_{ij} = T_{ij} - T_{ji}$ and

$A_{ji} = T_{ji} - T_{ij} = -(T_{ij} - T_{ji}) = -A_{ij}$

$\rightarrow$ A is skew-symmetric

2.3. ① Let σ be a symmetric tensor and let $T \in \mathcal{T}$. $S = \frac{1}{2}(T + T^t)$

$\sigma : T = \sigma_{ir} T_{ir}$

$\sigma : S = \sigma_{ir} \frac{1}{2}(T_{ir} + T_{ri})$

$= \frac{1}{2}\sigma_{ir} T_{ir} + \frac{1}{2}\sigma_{ir} T_{ri}$

$= \frac{1}{2}\sigma_{ir} T_{ir} + \frac{1}{2}\sigma_{ri} T_{ri}$

$= \frac{1}{2}(\sigma : T) + \frac{1}{2}(\sigma : T)$

$= \sigma : T$

② Let α be an ~~anti~~ skew-symmetric tensor and $A = \frac{1}{2}(T - T^t)$.

$\alpha : T = \alpha_{ir} T_{ir}$

$\alpha : S = \alpha_{ir} \frac{1}{2}(T_{ir} - T_{ri})$

$= \frac{\alpha_{ir}}{2} T_{ir} - \frac{\alpha_{ir}}{2} T_{ri}$

$$= \frac{1}{2}\alpha_{ir}T_{ir} + \frac{1}{2}\alpha_{ri}T_{ri}$$

$$= \frac{1}{2}(\alpha : T) + \frac{1}{2}(\alpha : T)$$

$$= \alpha : T$$

2.4 $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}$ $S = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 4 \end{bmatrix}$

for T: $|T - \lambda I| = (1-\lambda)(\lambda^2+4) = 0$

$\rightarrow \lambda = \{1, -2i, 2i\}$

for S: $|S - \lambda I| = (4-\lambda)\left[(\frac{5}{2}-\lambda)^2 - \frac{1}{4}\right]$

$= 0$

$\rightarrow \lambda = \{2, 3, 4\}$

$\lambda = 2$: $S - 2I = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4-2 \end{bmatrix}$

$\rightarrow u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\lambda = 3 \qquad S - 3I = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\lambda = 4 \qquad S - 4I = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u \cdot v = 1 \cdot 1 + 1 \cdot (-1) + 0 \cdot 0 = 1 - 1 = 0$$

$$u \cdot w = 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$$

$$v \cdot w = 1 \cdot 0 + (-1) \cdot 0 + 0 \cdot 1 = 0$$

$\rightarrow \{u, v, w\}$ are mutually orthogonal.

3.3 $\quad u = -x_2 e_1 + x_1 e_2 \qquad v = x_2 e_1$

$w = (1 - x_2^2) e_1$

$$\operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 + 0 + 0 = 0$$

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 + 0 + 0 = 0$$

$$\operatorname{div} w = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} = 0+0+0=0$$

$$\operatorname{curl} u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) e_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) e_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) e_3$$

$$= 0 \cdot e_1 + 0 \cdot e_2 + 2 \cdot e_3 = (0,0,2)$$

$$\operatorname{curl} v = (0,0,-1)$$

$$\operatorname{curl} w = (0,0,2x_2)$$

$$\operatorname{grad} u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ skew-symmetric

$$\operatorname{grad} v = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{symmetric}} + \underbrace{\begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{skew-symmetric}}$$

$$\operatorname{grad} w = \begin{bmatrix} 0 & 0 & 0 \\ -2x_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -x_2 & 0 \\ -x_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x_2 & 0 \\ -x_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.4 $\operatorname{div} T = (x_2, 0, 0) \longrightarrow$

$$\operatorname{div} T \big|_{(1,0,2)} = (0,0,0)$$