Homework 3 solutions

1. $\int_0^1 \sin(\pi m x)\sin(\pi n x)\,dx$

$$= \int_0^1 \frac{1}{2}\left[\cos((m-n)\pi x) - \cos((m+n)\pi x)\right] dx$$

this integral is equal to 0 if $m \neq n$.

If $m = n$, the integral reduces to

$$\frac{1}{2}\int_0^1 (1-\cos(\pi 2 n x))\,dx = \frac{1}{2} - \int_0^1 \cos(\pi 2 x n)\,dx$$

$$= \frac{1}{2}$$

3. $u_t + u_x = 0 \qquad -\infty < x < \infty \;,\; t > 0$

$u(x,0) = \cos(x) \qquad x \in \mathbb{R}$

This is a linear transport equation with speed $= 1$. Thus we perform the method of characteristics to obtain the solution

$u(x,t) = \cos(x-t)$

4. $x u_x + t u_t + 2u = 0 \qquad t > 1$

$u(x,1) = \sin(x)$

rearrange this equation to get

$\frac{x}{t} u_x + u_t + \frac{2}{t} u = 0$

We seek a solution of the form

$u(x,t) = f\left(\frac{x}{t}\right) g(t)$

$u_x = \frac{g(t)}{t} f'\left(\frac{x}{t}\right) \qquad u_t = f\left(\frac{x}{t}\right) g'(t) - \frac{x}{t^2} f'\left(\frac{x}{t}\right) g(t)$

$$\rightarrow \frac{x}{t}\cancel{\left(\frac{1}{t}g(t)f'\left(\frac{x}{t}\right)\right)} + f\left(\frac{x}{t}\right)g'(t)$$

$$\cancel{-\frac{x}{t^2}f'\left(\frac{x}{t}\right)g(t)} + \frac{2}{t}f\left(\frac{x}{t}\right)g(t) = 0$$

$$\rightarrow f\left(\frac{x}{t}\right)g'(t) + \frac{2}{t}f\left(\frac{x}{t}\right)g(t) = 0$$

$$\rightarrow g'(t) = -\frac{2}{t}g(t)$$

Thus $g(t) = \frac{1}{t^2}$. We know that

$u(x,1) = f\left(\frac{x}{1}\right)g(1) = f(x) = \sin(x)$.

Thus our solution becomes

$$u(x,t) = \frac{1}{t^2}\sin\left(\frac{x}{t}\right)$$

HW 3 Problem 2

(1) $u_t = u_{xx}$ $\quad 0 < x < 1, \quad t > 0$

$u(0,t) = 0$

$u(1,t) = 1$

$u(x,0) = x^2$

Let $u(x,t) = \cancel{(\quad)}\, x + v(x,t)$

(1) becomes
$$\left.\begin{aligned} v_t &= v_{xx} \\ v(0,t) &= 0 \\ v(1,t) &= 0 \\ v(x,0) &= x^2 - x \end{aligned}\right\} \quad (2)$$

The problem (2) can now be solved by standard methods. In particular if we use Fourier Transform then

$$\hat{v}_t \cancel{+} s^2 \hat{v} = 0 \qquad \hat{v}(s,t) = \mathcal{F}(v(x,t))$$

$$\hat{v}(s,0) = \hat{v}_0(s) = \mathcal{F}(x^2 - x^2)$$

$$\Rightarrow \quad \hat{v}(s,t) = c_1(s)\, e^{-s^2 t} = \hat{v}_0(s)\, e^{-s^2 t}$$

$$\Rightarrow \boxed{v(x,t) = \mathcal{F}^{-1}\left(\hat{v}(s,t)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{v}_0(\lambda)\, e^{-s^2 t}\, e^{-i\lambda x}\, e^{isx}\, d\lambda\, ds}$$

$u(x,t) = x + v(x,t)$ is the final solution to (1)

HW #3 Problem 5

Solve using eigenfunction expansion method :

(1) $u_t = u_{xx} + \sin(\pi x) \qquad 0 < x < 1, \quad t > 0$

(2) $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$

(3) $u(x,0) = 1$

Here $f(x,t) = \sin(\pi x)$

Let $u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$ (4)

Put (4) in (1) to get

$$\sum_{n=1}^{\infty} \left(T_n'(t) \sin(n\pi x) - T_n(t)(n\pi)^2 \sin(n\pi x) - \gamma_n \sin(n\pi x) \right) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} T_n' - (n\pi)^2 T_n - \gamma_n = 0 \quad (5) \quad n = 1, 2, \ldots$$

$$\gamma_n = 2 \int_0^1 \sin(\pi x) \sin(n\pi x)\, dx = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

$\Rightarrow \gamma_1 = 1, \quad \gamma_2 = \gamma_3 \ldots = 0$

Similarly the IC can be given by

$n = 1, 2$

$$T_n(0) = 2 \int_0^1 \sin(n\pi \xi)\, d\xi = -\frac{2}{n\pi} \cos n\pi \xi \Big|_0^1$$

$$= \frac{-2}{n\pi} \left(\cos n\pi - \cos 0 \right) = \begin{cases} \frac{4}{n\pi} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

So the problem to solve is

$$T_n' - (n\pi)^2 T_n - \gamma_n = 0$$

$$T_n(0) = \frac{4}{n\pi} \ (n=1,3,\dots), \ 0 \ (n=2,4\dots)$$

<u>n=1</u>/ $$T_1' - \pi^2 T_1 - 1 = 0$$
$$T_1(0) = \frac{4}{\pi}$$

$$\int \frac{dT_1}{\pi^2 T_1 + 1} = \int dt = t + c_0$$

let $\eta = \pi^2 T_1 + 1 \Rightarrow d\eta = \pi^2 dT_1$

$$\Rightarrow \frac{1}{\pi^2}\int \frac{d\eta}{\eta} = t + c_0 \Rightarrow \ln \eta = \pi^2 t + \tilde{c}_0$$

$$\eta = c_1 e^{\pi^2 t} \quad \text{where} \quad c_1 = \frac{4}{\pi}$$

$$\Rightarrow \eta(t) = \frac{4}{\pi} e^{\pi^2 t} \Rightarrow \pi^2 T_1(t) + 1 = \frac{4}{\pi} e^{\pi^2 t}$$

$$\boxed{T_1(t) = \frac{1}{\pi^2}\left(\frac{4}{\pi} e^{\pi^2 t} - 1\right)}$$

$\underline{n=2}$

$$T_2' - 4\pi^2 T_2 = 0$$

$$T_2(0) = 0$$

$$T_2(t) = c_0 e^{-4\pi^2 t} = 0$$

$\underline{n=3}$

$$T_3' - 9\pi^2 T_3 = 0$$

$$T_3(0) = \frac{4}{3\pi}$$

$$\Rightarrow \quad T_3(t) = \frac{4}{3\pi} e^{9\pi^2 t}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{\pi^2}\left(\frac{4}{\pi} e^{\pi^2 t} - 1\right)\sin(\pi x) + \frac{4}{3\pi} e^{9\pi^2 t} \sin(3\pi x) + \frac{4}{5\pi} e^{25\pi^2 t} \sin(5\pi x) + \cdots}$$