Homework 4 Solutions

J.D. Martindale

March 2, 2009

1 Section 12.3 Problem 14

The one-dimensional wave equation (1) is given by

$$\frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2}$$

(a) We now posit a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} G_n(t) sin\left(\frac{n\pi x}{L}\right)$$

and substitute this in to equation (1). We first need to know what the time and spacial derivatives are. These are found to be as follows:

$$\frac{\delta^2 u}{\delta t^2} = \sum_{n=1}^{\infty} \frac{\delta^2 G_n(t)}{\delta t^2} sin\left(\frac{n\pi x}{L}\right)$$
$$\frac{\delta^2 u}{\delta x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 G_n(t) sin\left(\frac{n\pi x}{L}\right)$$

Thus the wave equation becomes

$$\sum_{n=1}^{\infty} \left(\frac{\delta^2 G_n(t)}{\delta t^2} + \left(\frac{n\pi}{L}\right)^2 G_n(t)\right) sin\left(\frac{n\pi x}{L}\right) = 0$$

The only way for this series to be identically equal to 0 is for each inner term to be equal to 0. Thus this is equivalent to the equation

$$\ddot{G_n} + \lambda_n^2 G = 0, \qquad \lambda_n = \frac{cn\pi}{L}$$

From part (b), we know that forced vibrations under an external force $P(x,t)$ are governed by the PDE

$$u_{tt} = c^2 u_{xx} + \frac{P}{\rho}$$

(c) We now assume a sinusoidal force $P = A\rho sin\,(\omega t)$ so that if $\lambda_n^2 \neq \omega^2$, our solution is given by

$$G_n(t) = B_n cos\,(\lambda_n t) + B_n^* sin\,(\lambda_n t) + \frac{2A(1-cos(n\pi))}{n\pi(\lambda_n^2-\omega^2)} sin\,(\omega t)$$

(The substitutions in the problem are straight forward, and the solution is found by guessing a solution of the form $G_n(t) = B_n cos\,(\lambda_n t) + B_n^* sin\,(\lambda_n t) + C\frac{2A}{n\pi}(1-cos(n\pi))sin\,(\omega t)$ and solving for C). Now we must satisfy the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$.

$$\begin{aligned} G_n(0) &= B_n \\ \left.\frac{\delta G_n(t)}{\delta t}\right|_{t=0} &= \lambda_n B_n^* + \frac{2A\omega(1-cos(n\pi))}{n\pi(\lambda_n^2-\omega^2)} \end{aligned}$$

Thus our first initial condition tells us

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right) = f(x)$$

Thus we must choose the B_n's so that $u(x,0)$ becomes the Fourier sine series of $f(x)$:

$$B_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right)$$

Our second initial condition tells us

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{\infty}\left(\lambda_n B_n^* + \frac{2A\omega(1-cos(n\pi))}{n\pi(\lambda_n^2-\omega^2)}\right) sin\left(\frac{n\pi x}{L}\right) = 0 \\ \Rightarrow B_n^* &= -\frac{2A\omega(1-cos(n\pi))}{\lambda_n n\pi(\lambda_n^2-\omega^2)} \end{aligned}$$

(d) If $\lambda_n = \omega$, we now have a different situation. We instead guess a solution of the form

$$G_n(t) = B_n cos\,(\omega t) + B_n^* sin\,(\omega t) + Ct\frac{2A}{n\pi}(1-cos(n\pi))cos\,(\omega t)$$

The linearity in our guess stems from the obvious fact that when $\lambda_n = \omega$, our solution in part (c) fails to exist. Substituting this guess in to the equation

$$\ddot{G_n} + \omega^2 G_n = \frac{2A}{n\pi}\left(1-cos(n\pi)\right) sin\,(\omega t)$$

(and omitting much of the busy work simplification) yields our constant C:

$$C = -\frac{1}{2\omega}$$

Thus our solution is

$$G_n(t) = B_n cos\,(\omega t) + B_n^* sin\,(\omega t) - \frac{At}{n\pi\omega}\left(1-cos(n\pi)\right) cos\,(\omega t)$$

2 Section 12.3 Problem 15

Small free vertical vibrations of a uniform elastic beam are modeled by the fourth-order PDE

$$\frac{\delta^2 u}{\delta t^2} = -c^2 \frac{\delta^4 u}{\delta x^4}$$

We posit a seperable solution of the form $u = F(x)G(t)$, whose derivatives are computed as:

$$\begin{aligned} \frac{\delta^2 u}{\delta t^2} &= F(x)\ddot{G(t)} \\ \frac{\delta^4 u}{\delta x^4} &= F^{(4)}(x)G(t) \end{aligned}$$

Thus our PDE becomes

$$\begin{aligned} F\ddot{G} &= -c^2 F^{(4)} G \\ \Rightarrow \frac{F^{(4)}}{F} &= -\frac{\ddot{G}}{c^2 G} = \beta^4 = const \end{aligned}$$

Thus we can solve to find a general solution for both $F(x)$ and $G(t)$:

$$\begin{aligned} F^{(4)} &= \beta^4 F \\ \Rightarrow F(x) &= Acos\,(\beta x) + Bsin\,(\beta x) + Ccosh\,(\beta x) + Dsinh\,(\beta x) \\ \ddot{G} &= -c^2\beta^4 G \\ \Rightarrow G(t) &= acos\left(c\beta^2 t\right) + bsin\left(c\beta^2 t\right) \end{aligned}$$

3 Section 12.4 Problem 3

It is an easy calculation to find that the mass per unit length $\rho = \frac{weight\ per\ unit\ length}{g} = \frac{0.9}{2g}\frac{nt}{m}$ where $g = 9.80\frac{m}{s}$, thus

$$\begin{aligned} c^2 &= \frac{T}{\rho} = \frac{300nt}{0.9nt} \cdot 2m \cdot 9.80\frac{m}{s^2} = 80.83^2\frac{m^2}{s^2} \\ \Rightarrow c &= 80.83\frac{m}{s} \end{aligned}$$

4 Section 12.4 Problem 10

The Tricomi equation is given by

$$yu_{xx} + u_{yy} = 0$$

If we relate this equation back to the general quasilinear form (14), we see that $A = y$, $B = 0$, and $C = 1$. This equation is of mixed type due to the coefficient y in front of the u_{xx} term. The defining condition becomes $AC - B^2 = y$, and thus the Tricomi equation is elliptic in the upper half-plane and hyperbolic in the lower half-plane.

We now seek a seperable solution $u(x, y) = F(x)G(y)$. Substituting this in to the Tricomi equation yields

$$\begin{aligned} yF_{xx}G + FG_{yy} &= 0 \\ \Rightarrow \frac{F''}{F} = -\frac{G''}{yG} &= -k \end{aligned}$$

Setting $k = 1$, we retrieve the Airy equation:

$$G'' - yG = 0$$