Homework 5 Solutions

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1 Section 12.9 Problem 3

With the Laplacian given by $\nabla^2 u = (\delta_x^2 + \delta_y^2)u$, equations (3) and (4) tell us that

$$\begin{aligned} u_{xx} &= \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta \\ u_{yy} &= \frac{y^2}{r^2}u_{rr} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta \end{aligned}$$

If $u_\theta = 0$, the Laplacian reduces greatly to

$$\begin{aligned} u_{xx} + u_{yy} &= \frac{x^2+y^2}{r^2}u_{rr} + \frac{x^2+y^2}{r^3}u_r \\ &= u_{rr} + \frac{u_r}{r} \end{aligned}$$

2 Section 12.9 Problem 5

The only solution to $\nabla^2 u = 0$ depending only on r relies on the previous problem. We know that in this case

$$\nabla^2 u = u_{rr} + \frac{u_r}{r} = 0$$

Thus we can say

$$\begin{aligned} \frac{d}{dr}(u_r) &= -\frac{u_r}{r} \\ \rightarrow ln(u_r) &= -ln(r) + C_1 \\ \rightarrow u_r &= \frac{C_2}{r} \end{aligned}$$

Integrating this in r yields

$$u = a\ ln(r) + b$$

3 Section 13.2 Problem 4

$y = \frac{1}{2} + \frac{1}{4}\pi i$. To represent this in polar coordinates, we need to find r and θ:

$$\begin{aligned} r &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\pi}{4}\right)^2} = \sqrt{\frac{1}{4} + \frac{\pi^2}{16}} \\ \theta &= tan^{-1}\left(\frac{\frac{\pi}{4}}{\frac{1}{2}}\right) = tan^{-1}\left(\frac{\pi}{2}\right) \end{aligned}$$

Thus our representation becomes

$$r\left(cos(\theta) + i\ sin(\theta)\right)) = \sqrt{\frac{1}{4} + \frac{\pi^2}{16}}\left(cos\left(tan^{-1}\left(\frac{\pi}{2}\right)\right) + i\ sin\left(tan^{-1}\left(\frac{\pi}{2}\right)\right)\right)$$

4 Section 13.2 Problem 20

We see that $r = 12$ and $\theta = \frac{3\pi}{2}$, so that

$$\begin{aligned} x &= r\ cos(\theta) = 12cos(\frac{3\pi}{2}) = 0 \\ y &= r\ sin(\theta) = 12sin(\frac{3\pi}{2}) = -12 \end{aligned}$$

Thus $x + i\ y = -12i$.

5 Section 13.2 Problem 21

First note that $-i = e^{i\left(\frac{3\pi}{2}+2\pi k\right)}$ so that $\sqrt{-i} = e^{i\left(\frac{3\pi}{4}+\pi k\right)} = cos\left(\frac{3\pi}{4} + \pi k\right) + i\ sin\left(\frac{3\pi}{4} + \pi k\right)$. Thus the first two roots are where $k = -1, k = 0$, namely

$$\begin{gathered} cos\left(-\frac{\pi}{4}\right) + i\ sin\left(-\frac{\pi}{4}\right) \\ cos\left(\frac{3\pi}{4}\right) + i\ sin\left(\frac{3\pi}{4}\right) \end{gathered}$$

6 Section 13.2 Problem 24

Let $\theta = \frac{1}{3}tan^{-1}\left(\frac{4}{3}\right)$, and note that $3 + 4i = 5e^{i(3\theta+2\pi k)}$. Thus $\sqrt[3]{3+4i} = \sqrt[3]{5}e^{i\left(\theta+\frac{2\pi}{3}k\right)}$. The three roots are

$$\begin{gathered} \sqrt[3]{5}\left(cos\theta + i\ sin\theta\right) \\ \sqrt[3]{5}\left(cos(\theta + \frac{2\pi}{3}) + i\ sin(\theta + \frac{2\pi}{3})\right) \\ \sqrt[3]{5}\left(cos(\theta + \frac{4\pi}{3}) + i\ sin(\theta + \frac{4\pi}{3})\right) \end{gathered}$$

7 Section 13.2 Problem 27

We start with the equation

$$z^2 - (8-5i)z + 40 - 20i = 0$$

We can use the quadratic equation to find the solutions:

$$\begin{aligned} z_\pm &= \frac{(8-5i) \pm \sqrt{(8-5i)^2 - 4(40-20i)}}{2} = \frac{(8-5i) \pm \sqrt{-121}}{2} \\ &= 4 - \frac{5i}{2} \pm \frac{11i}{2} \end{aligned}$$

Thus our two solutions are $z = 4 + 3i$ and $z = 4 - 8i$.