MATH 529 TEST 1 SOLUTIONS

Problem #1 (a) Component form of

$$\rho\left(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}\right) = \mu\Delta\vec{u} + \vec{\nabla p} + \vec{f}.$$

Then
$$\vec{u}_t = \left\langle \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right\rangle$$

$$\vec{u}\cdot\nabla\vec{u} = u_i\partial_i u_j = \Big\langle u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\Big\rangle$$

$$\Delta\vec{u} = \langle \Delta u, \Delta v, \Delta w\rangle = \langle \partial_{xx}u + \partial_{yy}u + \partial_{zz}u,\ \partial_{xx}v + \partial_{yy}v + \partial_{zz}v,\ \partial_{xx}w + \partial_{yy}w + \partial_{zz}w\rangle$$

$$\vec{\nabla p} = \left\langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right\rangle,\quad \vec{f} = \langle f_1, f_2, f_3\rangle$$

Combining these we get the 3 components:

$$\boxed{\begin{aligned}
\rho\left(\frac{\partial u}{\partial t} + uu_x + vu_y + wu_z\right) &= \mu(u_{xx} + u_{yy} + u_{zz}) + p_x + f_1\\
\rho\left(\frac{\partial v}{\partial t} + uv_x + vv_y + wv_z\right) &= \mu(v_{xx} + v_{yy} + v_{zz}) + p_y + f_2\\
\rho\left(\frac{\partial w}{\partial t} + uw_x + vw_y + ww_z\right) &= \mu(w_{xx} + w_{yy} + w_{zz}) + p_z + f_3
\end{aligned}}$$

(b) If $u = u(y)$ then we get

$$\mu u_{yy} + p_x = 0$$

$$\left.\begin{aligned} p_y &= 0 \\ p_z &= 0 \end{aligned}\right\} \Rightarrow p = p(x)$$

so $\mu u''(y) = p_x(x) = c$ (constant)

$$\Rightarrow \quad \mu u'(y) = cy + c_1$$

$$\mu u(y) = \frac{cy^2}{2} + c_1 y + c_2$$

or $$\boxed{u(y) = \frac{c}{2\mu} y^2 + \frac{c_1}{\mu} y + \frac{c_2}{\mu}}$$

Test 1 Solutions

MATH 529

Pbm 2 (a) $u_{xx} + 5u_x + 6u = 0$ ——— (1)

Linear constant coefficient equation.

$A = 1,\ B = 0,\ C = 0 \Rightarrow B^2 - AC = 0 \Rightarrow$ PARABOLIC

Note that if $u = c_1 e^{-2x} + c_2 e^{-3x}$

$$\Rightarrow \quad u_x = -2c_1 e^{-2x} - 3c_2 e^{-3x}$$

$$u_{xx} = 4c_1 e^{-2x} + 9c_2 e^{-3x}$$

so $u_{xx} + 5u_x + 6u = -6c_1 e^{-2x} + 6c_2 e^{-3x} + 6\left(c_1 e^{-2x} + c_2 e^{-3x}\right)$

$$= 0$$

so $\boxed{u(x) = c_1 e^{-2x} + c_2 e^{-3x}}$

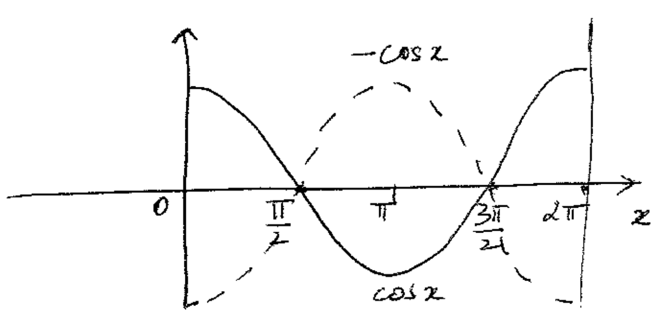
solves the equation (1).

(b) $u_{xx} + \cos(x)\, u_{yy} = 0$ ——— (2)

$A = 1,\ B = 0,\ C = \cos x$

so $B^2 - AC = -\cos x = \begin{cases} > 0 & \frac{\pi}{2} < x < \frac{3\pi}{2} \Rightarrow \text{Hyperbolic} \\ = 0 & x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \text{Parabolic} \\ < 0 & 0 \leq x < \frac{\pi}{2},\ \frac{3\pi}{2} < x < 2\pi \end{cases}$

$\Downarrow$ Elliptic



(c) Show curl(grad ϕ) = 0

$$\text{grad } \phi = \frac{\partial \phi}{\partial x_i} \underline{e_i} \qquad (i=1,2,3)$$

$$\text{curl}(\text{grad } \phi) = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i} \underline{e_i}\right)$$

$$= \varepsilon_{ijk}\, \partial_j (\partial_i \phi\, \underline{e_i})$$

$$= \sum_{i,j=1}^{3} \varepsilon_{ijk}\, \partial_j \partial_i \phi\, \underline{e_i} = \sum_{i=1}^{3} \left(\varepsilon_{i1k} \partial_1 \partial_i \phi + \varepsilon_{i2k} \partial_2 \partial_i \phi + \varepsilon_{i3k} \partial_3 \partial_i \phi\right)$$

$$= \cancelto{0}{\varepsilon_{11k}} \partial_1 \partial_1 \phi + \varepsilon_{21k} \partial_1 \partial_2 \phi + \varepsilon_{31k} \partial_1 \partial_3 \phi + \varepsilon_{12k} \partial_2 \partial_1 \phi + \cancelto{0}{\varepsilon_{22k}} \partial_2 \partial_2 \phi$$

$$+ \varepsilon_{32k} \partial_2 \partial_3 \phi + \varepsilon_{13k} \partial_3 \partial_1 \phi + \varepsilon_{23k} \partial_3 \partial_2 \phi + \cancelto{0}{\varepsilon_{33k}} \partial_3 \partial_3 \phi$$

<u>when k = 1</u>, the only terms that survive are

$$\varepsilon_{321} \partial_2 \partial_3 \phi + \varepsilon_{231} \partial_3 \partial_2 \phi = -\partial_2 \partial_3 \phi + \partial_3 \partial_2 \phi = 0$$

<u>when k = 2</u>, the only surviving terms are

$$\varepsilon_{312} \partial_1 \partial_3 \phi + \varepsilon_{132} \partial_3 \partial_2 \phi = 0$$

and similarly <u>when k = 3</u>, the only surviving terms are

$$\varepsilon_{213} \partial_1 \partial_2 \phi + \varepsilon_{123} \partial_2 \partial_1 \phi = 0$$

so $\boxed{\text{curl}(\text{grad } \phi) = 0}$

(c) The short proof of curl grad $\phi = 0$

Let $\vec{u} = \text{grad}\,\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle = \langle u, v, w \rangle$

$$\text{curl}\,\vec{u} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = (\partial_y w - \partial_z v)\,\hat{e}_1 + (\partial_z u - \partial_x w)\,\hat{e}_2 + (\partial_x v - \partial_y u)\,\hat{e}_3$$

$$\text{curl grad}\,\phi = \left(\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial y} \right) \hat{e}_1 + \left(\frac{\partial}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial}{\partial x}\frac{\partial \phi}{\partial z} \right) \hat{e}_2 + \left(\frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} \right) \hat{e}_3$$

But since $\frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x}$ (and also for other variables)

we have curl grad $\phi = 0$.

Problem 3 (a) $u_t + Q u_{xxxx} = 0$ ——— (3)

Take Fourier Transform of eqn. (3)

$$\mathcal{F}(u_t) + Q\,\mathcal{F}(u_{xxxx}) = 0$$

$$\boxed{U(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx}$$

$$\Rightarrow \quad U_t + Qk^4 U = 0$$

so $\boxed{U(k,t) = U(k,0)\, e^{-Qk^4 t}}$

Then the inverse relation is given by

$$u(x,t) = \mathcal{F}^{-1}(U(k,t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)\, e^{ikx}\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,0)\, e^{-Qk^4 t}\, e^{ikx}\, dk$$

$$\Rightarrow \quad \boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,0)\, e^{-Qk^4 t + ikx}\, dk}$$

(b) $0 = u_t + Q u_{xxxx} = X(x)T'(t) + Q X^{(4)}(x)\, T(t)$

$$\Rightarrow \quad \frac{-T'}{QT} = \frac{X^{(4)}}{X} = k^4$$

$\nearrow$ $T' + k^4 Q T = 0 \Rightarrow T = T_0\, e^{-Qk^4 t}$

$\searrow$ $X^{(4)} - k^4 X = 0 \Rightarrow$ Note that $X(x) = e^{+ikx}, e^{-ikx}, e^{kx}, e^{-kx}$ are all solutions

i.e. $X(x) = Ae^{ikx} + Be^{-ikx} + Ce^{kx} + De^{-kx}$

Since u must be bounded $C = D = 0$, so

$$X = \hat{A}\cos kx + \hat{B}\sin kx = Ae^{ikx} + Be^{-ikx}$$

$$\boxed{u(x,t) = (\hat{A}\cos kx + \hat{B}\sin kx)\, T_0\, e^{-Qk^4 t}}$$

which has the same general form as the solution from part (a)