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On some generalizations of the second grade fluid model

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Abstract

In this article, we provide a brief review of some generalizations of the second grade fluid model. We discuss certain similarities between these fluids and fluids of higher grades, while also describing certain limitations of these models. The new models that we put forth are based upon some interesting experimental results which suggest that not only can normal stress coefficients depend upon the shear rate, but that this dependency is in fact not the same rate as that of shear viscosity variation with shear rate. We then discuss some steady flows of these generalized second grade fluid models.
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1. A brief review

Until 1945 and to a great extent up to the present day, characterizing the behavior of fluids, i.e. measuring their rheological properties to a large extent, surmounted to simple shear test and obtaining shear rate versus shear stress diagnosis. For fluids not obeying the linear relationship, often referred to as non-Newtonian fluids, engineers and rheologists proposed simple generalizations such as the power-law model, whereby now in addition to viscosity, another parameter, namely the power-law index (or exponent) is needed to characterize the fluid. Many of these models were not frame invariant and were not based on rigorous and rational techniques of modern continuum mechanics. With the pioneering works of Reiner [49], Rivlin [50], Oldroyd [39] and Noll [38], a new era in constitutive modeling of nonlinear fluids began. Walters [65], in an important paper, presented the relation between the different constitutive relations used for elasto-viscous liquids, such as the models due to Coleman–Noll, Rivlin, Ericksen, Green–Rivlin and Oldroyd (see [2] for a brief review).

There are several constitutive relations which are used to model the non-Newtonian (non-linear) characteristics exhibited by some fluids. Integral models, such as the K-BKZ model, which take the past history of deformation into account are used when memory effects are important. However, for many fluids, only a very short part of the history of the deformation influences the stress. The constitutive relations for these fluids may be expressed as a function of the deformation gradient and its derivatives evaluated at the current time. A material in which the stress depends only on a finite number of these time derivatives is called a material of differential type (see [61]). The stress in a fluid of

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differential type of complexity j depends upon the velocity gradient and its first $(j-1)$ derivatives. The Newtonian fluid and the second grade fluid are the simplest fluids of the differential type when the dependence upon the velocity gradient is given a more specific form.

Perhaps the simplest non-Newtonian model that can represent the normal stress effects is that of an incompressible fluid of second grade where the stress tensor $\mathbf{T}$ is given by [51]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where p is the indeterminate part of the stress tensor due to the constraint of incompressibility, μ is the coefficient of viscosity and α_1 and α_2 are material moduli which are usually referred to as the normal stress coefficients. The kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}}, \tag{2}$$

$$\mathbf{A}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_1, \tag{3}$$

$$\mathbf{L} = \mathrm{grad}\,\mathbf{u}, \tag{4}$$

where $\mathbf{u}$ is the velocity of the fluid, *grad*, the gradient operator and $\mathrm{d}/\mathrm{d}t$, the material derivative defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \mathbf{u}\cdot\mathrm{grad}(\cdot), \tag{5}$$

where $\partial/\partial t$ represents the partial derivative with respect to time.

The thermodynamics and stability of the second grade fluids have been studied in detail by Dunn and Fosdick [12] who showed that if the fluid is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid be a minimum in equilibrium, then

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \alpha_1 + \alpha_2 = 0. \tag{6}$$

It is known that for many non-Newtonian fluids which are assumed to obey Eq. (1), the experimental values reported for α_1 and α_2 do not satisfy the restriction $(6)_{2,3}$. In an important paper, Fosdick and Rajagopal [15] show that irrespective of whether $\alpha_1 + \alpha_2$ is positive, the fluid is unsuitable if α_1 is negative. In particular, they showed that if it is assumed that

$$\mu > 0, \quad \alpha_1 < 0, \quad \alpha_1 + \alpha_2 \neq 0, \tag{7}$$

which as many experiments have reported to be the case "*for those fluids which the experimentalists assume to be constitutively determined by (1), at least sufficiently well as a second order approximation*" [15, p. 147], then certain anomalous results follow. Fosdick and Rajagopal [15] proved a theorem which indicates that if $(7)_{2,3}$ hold, then an unusual behavioral property, not to be expected for any rheological fluid occurs, namely, "*that the larger the viscosity, keeping everything else fixed, the faster that initial data is amplified in motions which take place under strict isolation.*" For further details on this and other relevant issues in fluids of differential type, we refer the reader to the review article by Dunn and Rajagopal [13].

Criminale et al. [11] obtained an expression for $\mathbf{T}$, valid for any laminar shear flow:

$$\mathbf{T} = -p\mathbf{I} + \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \beta_3(\mathbf{A}_1^2 - \tfrac{1}{2}\mathbf{A}_2), \tag{8}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined above and β_1, β_2 and β_3 are functions of Π, where

$$\Pi = \tfrac{1}{2}\,\mathrm{tr}\mathbf{A}_1^2, \tag{9}$$

which are specifically given by

$$\beta_1 = \gamma_1 + 2\gamma_5\Pi + 4\gamma_7\Pi^2, \tag{10}$$

$$\beta_2 = \gamma_2 + 0.5\gamma_3 + 2(\gamma_4 + \gamma_6)\Pi + 4\gamma_8\Pi^2, \tag{11}$$

$$\beta_3 = \gamma_3, \tag{12}$$

$$\gamma_m = \alpha_m(2\Pi, 0, 4\Pi^2, 8\Pi^3, 0, 2\Pi^2, 0, 4\Pi^3). \tag{13}$$

The model given by Eq. (8) is known as the CEF model. It can be seen that when $\beta_2 = 2\beta_3$, this equation reduces to the Reiner–Rivlin fluid model [49,50]. Now, since β_1, β_2 and β_3 can be assigned arbitrarily as functions of Π, then in theory, the CEF model can predict shear-thinning (or thickening) as well as normal stress effects. Coleman and Noll [8,9] introduced the concept of retarded histories and provided a mechanism whereby certain approximations can be made to the functional relationships to describe the physical assumptions that simple fluids have a fading memory. Huilgol [23], while indicating that polynomial approximations are only useful at low shear rates, derived a new constitutive relation by introducing a new tensor, related to the material derivative of the left relative Cauchy–Green tensor:

$$\mathbf{T} = -p\mathbf{I} + \eta_0\mathbf{A}_1 + \eta_1\mathbf{A}_1^2 + \eta_2\mathbf{B}_2, \tag{14}$$

where η_0, η_1 and η_2 are assumed to be constants, $\mathbf{A}_1$ is defined as before and $\mathbf{B}_2$ is given by

$$\mathbf{B}_2 = \mathbf{A}_2 + 2\mathbf{L}\mathbf{L}^{\mathrm{T}} - 2\mathbf{L}^{\mathrm{T}}\mathbf{L} + 2\mathbf{A}_1\mathbf{Z} - 2\mathbf{Z}\mathbf{A}_1, \tag{15}$$

where

$$\mathbf{Z} = \frac{\mathrm{D}}{\mathrm{D}s}\mathbf{Q}|_{s=0} \tag{16}$$

and represents the spin tensor.

Some experimental evidence seems to indicate that $\alpha_1 < 0$, which if true would mathematically lead to the rest state of the second grade fluid being unstable [10]. Slemrod [53] argued that this apparent discrepancy can be resolved if one chooses Rivlin–Ericksen constitutive relations based on a rational approximation rather than the polynomial approximation that is used to derive the grade type models. The constitutive relations, however, become substantially more complex under this expansion. The generalized rational approximation of $\mathbf{f}$, denoted by $\mathbf{f}_{m,n}$ is defined by

$$\mathbf{f}_{m,n}(\mathbf{A}_1, \mathbf{A}_2, \ldots) = \frac{\mathrm{tr}_m[\mathbf{f}_n\,\mathrm{tr}[1+\phi_n]^s]}{(1+\phi_n)^s}, \tag{17}$$

where m, n are positive integers, $s > 0$, $\mathbf{f}_n$ is the finite series given by

$$\mathbf{f}_n = \sum_{k=1}^{n} \mu^k \mathbf{S}_k, \tag{18}$$

with $\mu > 0$ and $\mathbf{S}_k$ being the kth polynomial approximation of the Rivlin–Ericksen tensor. Furthermore, in Eq. (17), the term ϕ is a function of the scalar invariants

$$\phi(\mathrm{trace}[\mathbf{A}_1^2], [\mathbf{A}_1\mathbf{A}_2], \mathrm{trace}[\mathbf{A}_1^3], \ldots), \tag{19}$$

where ϕ_n represents the truncation of the expansion of ϕ to order μ^n. Finally, tr_m refers to the truncation of a power series at order m. Though Eq. (17) does not directly describe an nth order fluid, an expansion to terms of $\mathrm{O}(\mu^4)$ in the limit as $\mu \to 0_+$ yields a constitutive relation that agrees with the second grade fluid model.

A shortcoming of the second grade fluid model is that it cannot predict 'shear-thinning' (or 'shear-thickening'), which is the decrease (or increase) in viscosity with the shear rate. Perhaps the simplest non-Newtonian model that can predict this phenomenon is the generalized power-law fluid model, which in its appropriate frame invariant form can be expressed as

$$\mathbf{T} = -p\mathbf{I} + \mu_0(\mathrm{tr}\,\mathbf{A}_1^2)^m\mathbf{A}_1, \tag{20}$$

where m is the power-law index and μ_0 is the constant shear viscosity. When $m < 0$, the fluid is shear-thinning and if $m > 0$ the fluid is shear-thickening. To overcome this shortcoming, Man and coworkers [28,29] proposed the following constitutive equations:

$$\text{Model 1:} \quad \mathbf{T} = -p\mathbf{I} + \mu\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{21}$$

$$\text{Model 2:} \quad \mathbf{T} = -p\mathbf{I} + \Pi^{m/2}(\mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2), \tag{22}$$

where m is a material parameter and Π is the second invariant of $\mathbf{A}_1$ and is given by Eq. (9). When $\alpha_1 = \alpha_2 = 0$, both models reduce to the power-law model and when $m = 0$, both models become the second grade fluid model. Even though these generalized second grade models are not derived from first principles, they are physically meaningful. The second of these models, which we henceforth refer to as GSGF Type II, has been so far neglected in the literature except for the work of Massoudi and Phouc, perhaps due to its apparent complexity. A recent study by Vaidya et al. [63] has shown that the GSGF Type II is in fact very useful in describing certain complex fluid structure interaction phenomena. The GSGF Type II is the simplest viscoelastic model that is capable of explaining the so-called 'tilt-angle' phenomenon in viscoelastic fluids which refers to the terminal orientation of a slowly sedimenting body where the angle is between the horizontal or vertical position. In fact, the authors note that it is the non-constant normal stress coefficients that help explain the sought after phenomenon which none of the standard (and relatively simple) viscoelastic or power-law models are capable of explaining.

Experimental studies with polymers [1,3,20,68], suspensions [27] and liquid crystals [59] seem to indicate that for several fluids, one does observe a substantial variation in normal stress effects with the shear rate. In fact, Harris [20] even argues that this dependence is of a power-law nature. Furthermore, experimental evidence points to the fact that at least for the fluids that have been studied, the tangential stress varies at a different rate with the shear-rate than does the normal stress. With this in mind, we suggest a further variation of the second grade model, namely,

$$\text{Model 3:} \quad \mathbf{T} = -p\mathbf{I} + \Pi^{m_1/2}(\mu\mathbf{A}_1) + \Pi^{m_2/2}(\alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2), \tag{23}$$

where m_1 and m_2 now represent the power-law indices for the viscosity and normal stress coefficients, respectively. This model has the further advantage that it can also reduce to a *Boger fluid* (a viscoelastic fluid which displays normal stress effects but does not have noticeable shear-thinning or thickening [4], when $m_1 = 0$). Therefore Eqs. (21) and (22) are actually special cases of Eq. (23). In fact,

$$\text{Model 1} \subset \text{Model 2} \subset \text{Model 3}. \tag{24}$$

Gupta and Massoudi [19] have also generalized these models (models 1 and 2) even further by assuming a temperature-dependent viscosity of the type

$$\mu(\theta) = \mu_0\, \mathrm{e}^{-M\theta}, \tag{25}$$

where θ is the temperature, μ_0 a constant viscosity and M a material parameter. This equation is generally known as the Reynolds viscosity model and is used very often in lubrication applications [56]. This is by no means the only accepted form of the temperature-dependent viscosity (see for example [17,7]). Massoudi and Phouc [36] also suggested a constitutive model of the type

$$\mathbf{T} = -p\mathbf{I} + \mu(\theta, \phi)\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{26}$$

where $\mu(\theta, \phi)$ is given based on the Einstein–Roscoe relation:

$$\mu(\theta, \phi) = \mu_0(1 - \phi/\phi_{\max})^{-2.5}\mathrm{e}^{\gamma(\theta-\theta_0)}, \tag{27}$$

where ϕ is the volume fraction and $\phi_{\max}$ is the maximum crystal fraction at which the flow can occur, θ_0 and μ_0 are reference values and γ is a constant.

It must be emphasized that among all the models discussed so far, only the second grade fluid model, given by Eq. (1), is rigorously defined where its stability and thermodynamics have been studied. In a very important paper, Rajagopal and Wineman [48] obtained certain necessary and sufficient conditions on α_1 and $\mathrm{d}\alpha_1/\mathrm{d}\theta$ if the constitutive relation for the second grade fluid, Eq. (1), is to meet the requirement that the specific internal energy be a minimum when the fluid is locally at rest. This seems to be the only rigorous study where the dependence of normal stress coefficients on temperature is studied. The generalization due to Man and co-workers and Massoudi and co-workers are all phenomenological and ad hoc. The model denoted by Eq. (21) has been used to study some engineering problems in a variety of contexts [19,35,37]. The work of Gupta and Massoudi [19] deals with the flow of a generalized second grade fluid, with a temperature-dependent viscosity, for flows between horizontal/parallel plates heated from below. Massoudi and Phouc [36] have considered the very same model for flow down an inclined plane and more recently, Massoudi et al. [37] have considered the flow of a generalized second grade fluid between vertical, heated walls. The importance of variable viscosity and viscous dissipation in non-Newtonian fluids have been discussed, for example, by Soh and Mureithi [54], Khan et al. [25], Rajagopal and Na [47], Szeri and Rajagopal [57] and Massoudi and Christie

[33,34]. It is easily seen that a further generalization of the second grade model can be obtained by combining models (23) and (26). We refrain from continuing these ad hoc generalizations due to lack of sufficient experimental evidence, at this point, regarding the exact dependence of normal stress coefficients upon θ and ϕ.

We need to mention here that the method adopted by Man and later by Massoudi and coworkers to extend the range of applicability of standard second grade fluid by generalizing certain material coefficients is not the only method that has been employed. Using the method of fractional calculus, many researchers [40,52,58] have defined $\mathbf{A}_2$ as

$$\mathbf{A}_2 = D_t^{\beta}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_1, \tag{28}$$

where D_t^{β} is the Riemann–Liouville fractional calculus operator defined by

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-\beta} f(\tau)\,\mathrm{d}\tau, \quad 0<\beta<1, \tag{29}$$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(\xi+1) = \int_0^{\infty} \mathrm{e}^{-x} x^{\xi}\,\mathrm{d}x, \tag{30}$$

and when $\beta = 1$, we obtain the standard second grade fluid model.

2. A few comments on the GSGF models

Upon taking a closer look at the GSGF models of both types, we make the following observations:

1. The main feature of the GSGF Type I model is that it accounts for normal stress effects as well as shear-thinning and shear-thickening properties. Let us consider the well-known third grade fluid model where

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \mathbf{S}_2 + \mathbf{S}_3, \tag{31}$$

where

$$\mathbf{S}_2 = \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{32}$$
$$\mathbf{S}_3 = \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\mathrm{tr}\,\mathbf{A}_2)\mathbf{A}_1, \tag{33}$$

and $\mathbf{A}_3$ is given by

$$\mathbf{A}_3 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_2 + \mathbf{A}_2\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_2. \tag{34}$$

Fosdick and Rajagopal [16] showed that for an incompressible thermodynamically compatible fluid of grade three, this model reduces to

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta[\mathrm{tr}\,\mathbf{A}_1^2]\mathbf{A}_1, \tag{35}$$

where

$$\mu > 0,$$
$$\alpha_1 \geqslant 0,$$
$$|\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta},$$
$$\beta \geqslant 0.$$

If we re-write this equation as

$$\mathbf{T} = -p\mathbf{I} + [\mu + \beta\,\mathrm{tr}\,\mathbf{A}_1^2]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{36}$$

it can be seen that this equation can also be considered as a generalization of the standard second grade fluid model with an effective viscosity, μ_{eff}, given by

$$\mu_{\mathrm{eff}} = \mu + \beta\,\mathrm{tr}\,\mathbf{A}_1^2. \tag{37}$$

This was pointed out by Massoudi and Christie [33]. In fact, Mansutti and Rajagopal [30] and Mansutti et al. [31] have used the power-law model, generalized second grade model and the thermodynamically compatible third grade model to study the steady flows past a porous plate and also between intersecting plates. It is readily seen from this equation that the third grade fluid model can not only predict the normal stress but also the shear-thickening effect, which emerges from the $\mathbf{S}_3$ portion of the tensor. This particular model has been widely studied and has been found to be useful in the analysis of several classes of viscoelastic flows. Hence, it becomes immediately obvious upon comparing the two models that the GSGF Type I is in fact similar to the third grade fluid in terms of possessing a general viscosity power-law function.

2. Similarly, the GSGF Type II model can be compared to the fourth grade fluid model which is represented by the stress tensor

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4, \tag{38}$$

where $\mathbf{S}_2$ and $\mathbf{S}_3$ are as given above and $\mathbf{S}_4$ is given by

$$\mathbf{S}_4 = \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\ + \gamma_5(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1^2 + \gamma_7(\operatorname{tr}\mathbf{A}_3)\mathbf{A}_1 + \gamma_8(\operatorname{tr}\mathbf{A}_2\mathbf{A}_1)\mathbf{A}_1, \tag{39}$$

and $\gamma_1 \ldots \gamma_8$ are material constants. The tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are defined above while $\mathbf{A}_4$ is written as

$$\mathbf{A}_4 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_3 + \mathbf{A}_3\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_3. \tag{40}$$

Once again, we may consider the GSGF Type II as a special case of the fourth grade fluid model where the term $\gamma_5(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_2$ in the fourth grade fluid model may be considered a special case of the varying normal stress. The convenience of the GSGF Type I and II models lies in the fact that they are able to generalize in some sense the features of the grade-type fluids while remaining relatively simple to handle computationally. It is not difficult to see that the third and fourth grade fluids become very complicated to work with and as a result one sees relatively little work with these models, especially the fourth grade fluid model (see [21,67] for some such recent papers on the fourth grade fluid model). Hence there is much to be said for these generalized second grade models especially for problems where a qualitative solution is sought.

3. Steady flows of GSGF models

In this section we will look at two simple flows of the GSGF models. The governing equations of motion are the conservation equations for mass, linear momentum and energy which are given, respectively by, assuming incompressibility

$$\operatorname{div}\mathbf{u} = 0, \tag{41}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \tag{42}$$

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \mathbf{T}:\mathbf{L} - \operatorname{div}\mathbf{q} + \rho r, \tag{43}$$

where $\mathbf{b}$ is the body force vector, $\mathbf{T}$ is the Cauchy stress tensor and $\mathrm{d}/\mathrm{d}t$ is the material time derivative, ε is the specific internal energy, $\mathbf{q}$ is the heat flux vector and r is the radiant heating. The specific internal energy ε is related to the specific Helmholtz free energy through

$$\varepsilon = \psi + \theta\eta = \varepsilon(\theta, \mathbf{A}_1, \mathbf{A}_2). \tag{44}$$

Example 1 (*Convective motion between vertical walls*). If we consider flow of the generalized second grade fluid of Type III, between two vertical walls (see Fig. 1a),

$$\mathbf{u} = (0, u(x), 0), \tag{45}$$

$$\theta = \theta(x), \tag{46}$$

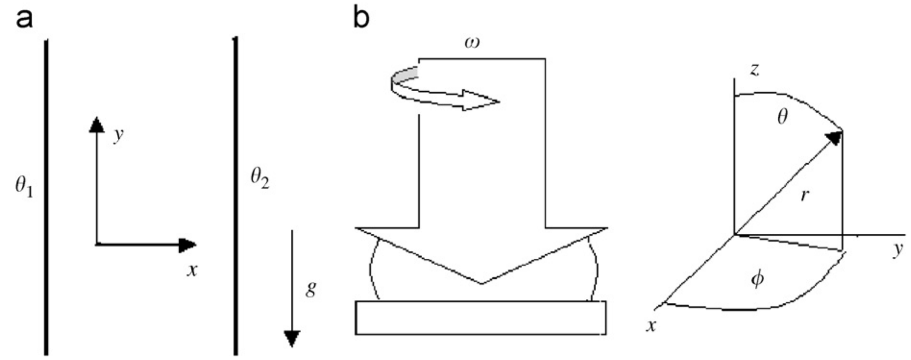

Fig. 1. Flow between (a) vertical walls and in (b) a cone-plate.

where θ is the temperature, it can be seen that with Eq. (45), the incompressibility condition is automatically satisfied. Additionally, the linear momentum equation reduces to

$$0=\frac{\partial}{\partial x}\left(-p+(2\alpha_1+\alpha_2)\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^{m_2+2}\right), \tag{47}$$

$$0=\left(\frac{\partial}{\partial x}\left(\mu(\theta)\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^{m_1}\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\frac{\partial}{\partial y}\left(-p+\alpha_2\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^{m_2+2}\right)-\rho_0(1-\gamma(\theta-\theta_m))g\right). \tag{48}$$

If we take $\hat{p}\equiv p-(2\alpha_1+\alpha_2)|\mathrm{d}u/\mathrm{d}x|^{m_2+2}$, then it follows that

$$\frac{\partial}{\partial x}(\hat{p})=0, \tag{49}$$

and since

$$\frac{\partial}{\partial y}(\hat{p})=\frac{\partial}{\partial y}(p), \tag{50}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(\theta)\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^{m_1}\frac{\mathrm{d}u}{\mathrm{d}x}\right)-\rho_0(1-\gamma(\theta-\theta_m))g=\frac{\mathrm{d}}{\mathrm{d}y}(\hat{p})$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}y}(\hat{p})=c. \tag{51}$$

If we further choose this constant c to be equal to $-\rho_0 g$, then the momentum equation in the x-direction reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(\theta)\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^{m_1}\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\rho_0\gamma(\theta-\theta_m)g=0. \tag{52}$$

For the velocity and temperature fields assumed in our problems, it then follows that (see Eqs. (45) and (46))

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=0. \tag{53}$$

With these assumptions, the energy equation becomes

$$k\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}+\mu(\theta)\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^{m_1+2}=0. \tag{54}$$

The boundary conditions for the problem are

$$u(\pm L) = 0, \tag{55}$$
$$\theta(-L) = \theta_1, \quad \theta(L) = \theta_2, \tag{56}$$

where $\theta_1 > \theta_2$.

It can be seen that these equations are in fact identical to the governing equations for the flow of a GSGF Type I which have been solved by the authors recently [37]. That is, Model 1 and Model 3 give the same equations since the effects of α_2 are absorbed in the modified pressure term, $\hat{p}$. Therefore, the same conclusion can be drawn for Model 2 as well if we set $m_1 = m_2$. Hence, Model 3, Model 2 and Model 1 yield identical velocity and temperature profiles for this problem. In fact, perhaps a more significant observation here is that *the governing equations are the same as those for a power-law model*, where $\alpha_1 = \alpha_2 = 0$, since the normal stress coefficients are absorbed in the pressure term.

Example 2 (*Flow between a cone and plate*). Another important geometry which is used to experimentally determine the viscometric functions is based upon the flow between a cone and plate (see Fig. 1b). The velocity profile, expressed in spherical coordinates[1] is given by

$$\mathbf{u} = (u_r, u_\theta, u_\phi) = (0, 0, u(r, \theta)) \tag{57}$$

with boundary conditions

$$u(r, \pi/2) = 0, \quad u(r, \pi/2 - \beta) = \omega r \cos\psi, \tag{58}$$

where $\mathbf{u}$ as defined in Eq. (57) automatically satisfies the incompressibility condition. The problem of flow of a second grade fluid in a cone and plate geometry has been studied by Fetecau and Fetecau [14] and an exact solution is obtained by separation of variables

$$u(r, \theta) = \sum_n \psi_n(\theta) w_n(r)$$

with the solution given by

$$u(r, \theta) = \frac{1}{r^{2/3}} \sum_{n=1}^{\infty} A_n r^{\mu_n} \psi_n(\theta), \tag{59}$$

where A_n $(n = 1, 2, \ldots)$ are constants and $\psi_n(\theta)$ represent the eigenfunctions corresponding to the problem

$$\psi_n''(\theta) + 3ctg(\theta)\psi_n'(\theta) = -\lambda_n \psi_n(\theta). \tag{60}$$

The solution, is seen to be completely independent of the normal stress material parameters. In fact the authors note in the conclusion that the solution to the problem is the same as the solution to the flow of a Newtonian fluid.

Let us now analyze the flow field induced in the GSGF models due to the cone-plate rotation. As a result of Eq. (57), the velocity gradient tensor becomes

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & -\dfrac{u}{r} \\ 0 & 0 & -\cot\theta\dfrac{u}{r} \\ \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} & 0 \end{bmatrix}, \tag{61}$$

[1] Note that θ, in this example refers to an angular variable and not temperature, as in example 1.

and the components of the stress tensor **T** can be written as

$$T_{rr} = -p + \Pi^{m_2/2}(2DA\alpha_1 + \alpha_2 A^2), \tag{62}$$
$$T_{r\theta} = T_{r\theta} = \Pi^{m_2/2}(\alpha_1 FA + \alpha_1 DB + \alpha_2 AB), \tag{63}$$
$$T_{r\phi} = T_{\phi r} = \Pi^{m_1/2}\mu A, \tag{64}$$
$$T_{\theta\theta} = -p + \Pi^{m_2/2}(2\alpha_1 FB + \alpha_2 B^2), \tag{65}$$
$$T_{\theta\phi} = T_{\phi\theta} = \Pi^{m_1/2}\mu B, \tag{66}$$
$$T_{\phi\phi} = -p + \Pi^{m_2/2}(2\alpha_1 CA + 2\alpha_1 EB + \alpha_2 A^2 + \alpha_2 B^2), \tag{67}$$

where

$$\Pi = (A^2 + B^2)^2,$$
$$A = \frac{\partial u}{\partial r} - \frac{u}{r},$$
$$B = \frac{1}{r}\left(\frac{\partial u}{\partial \theta} - u\right),$$
$$C = -\frac{u}{r}, \quad D = \frac{\partial u}{\partial r},$$
$$E = -\frac{u \cot\theta}{r}, \quad F = \frac{1}{r}\frac{\partial u}{\partial \theta}.$$

As a result the equations of linear momentum can be given along the $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ directions, respectively, as follows:

$$0 = \frac{\partial}{\partial r}\left[-p + \Pi^{m_2/2}(2DA\alpha_1 + \alpha_2 A^2)\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\Pi^{m_2/2}(\alpha_1 FA + \alpha_1 DB + \alpha_2 AB)\right]$$
$$+ \frac{\Pi^{m_2}}{2r}\left[2\alpha_1 DA - 2\alpha_1 CA - 2\alpha_1 B^2 + \cot\theta(\alpha_1 FA + \alpha_1 DB + \alpha_2 AB)\right], \tag{68}$$
$$0 = \frac{\partial}{\partial r}\left[\Pi^{m_2/2}(\alpha_1 FA + \alpha_1 DB + \alpha_2 AB)\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[-p + \Pi^{m_2/2}(2\alpha_1 FB + \alpha_2 B^2)\right]$$
$$+ \frac{\Pi^{m_2}}{2r}\left[3\alpha_1 FA + 3\alpha_1 DB + 3\alpha_2 AB + \cot\theta(2\alpha_1 FB - 2\alpha_1 CA - 2\alpha_1 EB - \alpha_2 A^2)\right], \tag{69}$$
$$0 = \frac{\partial}{\partial r}\left[\Pi^{m_1/2}\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[-p + \Pi^{m_1/2}\right] + \frac{\Pi^{m_2/2}}{r}[3 + 2\cot\theta] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{p}, \tag{70}$$

where we define

$$\hat{p} = -p + \Pi^{m_2/2}(2\alpha_1 CA + 2\alpha_1 EB + \alpha_2 A^2 + \alpha_2 B^2) = T_{\phi\phi}. \tag{71}$$

Eqs. (68)–(70) suggest that $\hat{p}$ can at most be a function of r and θ and not ϕ. Eq. (70) can be solved for u_ϕ, the velocity field, using separation of variables (see [14]), $u(r, \theta) = f(r)\, g(\theta)$, with $\partial\hat{p}/\partial\phi = c$ (constant). With u_ϕ given, Eqs. (68) and (69) can be used to solve for p explicitly.

We note that the solution for u_ϕ depends only upon μ and is independent of the normal stress coefficients. Therefore in the case of GSGF Type III, the flow in a cone and plate geometry is the same as that of a power-law fluid. When $m_1 = 0$ then the flow profile is given by [14] (see Eq. (59)). When $m_1 \neq 0$, the velocity and pressure profiles for a power-law fluid in the cone-plate have been solved for by Chaturani and Narasimhan [6] using a perturbation technique. A detailed study of the effect of primary and secondary flows that arise in this geometry are looked at and also the effect of the flow on the shear-thinning (or thickening) index, n, has been observed. The results in [6] should therefore hold for the GSGF models as well.

4. Viscometric flows of the GSGF models

Bird, Armstrong and Hassager define *viscometric flows* as *unidirectional shear flows* where [3, pp. 131–134] "...shear rate $\dot{\gamma}$ is independent of time at a given particle." (the reader is also referred to [60] for a comprehensive discussion of viscometry). Alternatively, a flow is referred to as *viscometric* [24, p. 41] if the deformation gradient $\mathbf{F}$ at time t is given by

$$\mathbf{F}(\mathbf{X}, t) = \mathbf{Q}(\mathbf{X}, t)[1 + t\mathbf{M}(\mathbf{X}, t)], \tag{72}$$

where t refers to time, $\mathbf{Q}(\mathbf{X}, t)$ is orthogonal, with $\mathbf{Q}(\mathbf{X}, 0) = \mathbf{I}$ and $\mathbf{M}(\mathbf{X}, t)$ is defined such that

$$\mathbf{M}(\mathbf{X}, t) \neq 0, \quad \mathbf{M}^2(\mathbf{X}, t) = 0. \tag{73}$$

Some typical examples of viscometric flows include [3,41,60]: (a) flow in a pipe, (b) couette flow, (c) torsional flow, (d) flow between a cone and plate and (e) helical flows. Motivated by the previous section, we will now analyze the viscometric properties of the three models under some such flow. We use the standard symbols, η, Ψ_1 and Ψ_2 to describe the viscosity, first and second normal stress differences, respectively, also known as the *viscometric functions*, which are given by

$$\eta = \frac{T_{xy}}{\dot{\gamma}}, \tag{74}$$

$$\Psi_1 = \frac{T_{xx} - T_{yy}}{\dot{\gamma}^2}, \tag{75}$$

$$\Psi_2 = \frac{T_{yy} - T_{zz}}{\dot{\gamma}^2}. \tag{76}$$

4.1. Simple shear flow

Let us look at the normal stresses generated by a fluid in a shear flow:

$$\mathbf{u} = (u_1, u_2, u_3) = (\dot{\gamma}\, y, 0, 0) \tag{77}$$

in standard Cartesian coordinates. The resulting stress tensor for the GSGF Type III is then given by

$$\mathbf{T} = \begin{bmatrix} -p + (2\alpha_1 + \alpha_2)|\dot{\gamma}|^{m_2+2} & \mu_0|\dot{\gamma}|^{m_1}\dot{\gamma} & 0 \\ \mu_0|\dot{\gamma}|^{m_1}\dot{\gamma} & -p + \alpha_2|\dot{\gamma}|^{m_2+2} & 0 \\ 0 & 0 & -p \end{bmatrix}. \tag{78}$$

The viscometric functions for the GSGF Type III are therefore

$$\eta = \mu_0\dot{\gamma}^{m_1}, \quad \Psi_1 = 2\alpha_1\dot{\gamma}^{m_2}, \quad \Psi_2 = \alpha_2\dot{\gamma}^{m_2}. \tag{79}$$

The corresponding functions for GSGF Type II can be retrieved by setting $m_1 = m_2$ and for Type I by letting $m_2 = 0$.

4.2. Couette flow

Consider the couette flow motion of the fluid of the form

$$\mathbf{u} = (u_r, u_\theta, u_z) = (0, u(r), 0) \tag{80}$$

in cylindrical coordinates. The stress tensor corresponding to this flow for the GSGF Type III is given by

$$\mathbf{T} = \begin{bmatrix} -p + (\alpha_1 + \alpha_2)|\dot{\gamma}|^{m_2+2} & \mu_0|\dot{\gamma}|^{m_1}\dot{\gamma} & 0 \\ \mu_0|\dot{\gamma}|^{m_1}\dot{\gamma} & -p + (\alpha_1 + \alpha_2)|\dot{\gamma}|^{m_2+2} & 0 \\ 0 & 0 & -p \end{bmatrix}, \tag{81}$$

where $\dot{\gamma} = r(\mathrm{d}/\mathrm{d}r)(\frac{u_\theta}{r})$. The viscometric functions are hence

$$\eta = \mu_0\dot{\gamma}^{m_1}, \quad \Psi_1 = 0, \quad \Psi_2 = (\alpha_1 + \alpha_2)\dot{\gamma}^{m_2}. \tag{82}$$

We could similarly, obtain the viscometric functions corresponding to the complete set of viscometric flows described above. It is easy to see that in all such cases, a similar result will be obtained. Bird et al. [3] argue that for all viscometric flows the stress tensor is of the form

$$\mathbf{T} = \begin{bmatrix} -p+\tau_{11} & \tau_{12} & 0 \\ \tau_{12} & -p+\tau_{22} & 0 \\ 0 & 0 & -p+\tau_{33} \end{bmatrix}, \tag{83}$$

where τ is the extra-stress tensor.

5. Discussion

Perhaps the similarities in the flows of the GSGF models and the generalized Newtonian counterparts are not that surprising since the flows being analyzed are primarily shear flows. The primary task of the GSGF constitutive models is to explain normal stress effects and these models are appropriate for (a) unsteady flows, (b) non-viscometric (or non-shearing) flows and (c) in cases where the full stress tensor is important. An example of case (c) is the problem of terminal orientation of a sedimenting body in a viscoelastic fluid [63]. The terminal behavior of a freefalling body in a fluid is strongly dependent upon the fluid model used, since the steady orientation is the solution to the equation

$$\left[\int_{\partial\Omega} \mathbf{x} \times \mathbf{T}\cdot\mathbf{n}\right](\Theta) = 0, \tag{84}$$

where $\partial\Omega$ refers to the surface of the body, $\mathbf{x}$ is any point on the surface, $\mathbf{T}$ is the stress tensor, Θ is the angle at which the body falls and $\mathbf{n}$ is the unit normal acting out of the body. Hence, for this problem the more general model (Type II) is capable of capturing the essential physics which the standard second grade model is incapable of doing. Yet another interesting problem of case (c) would be the flow induced by a rotating sphere, where, through observing the streamline pattern and measuring the couple on the rotating sphere, certain conclusions can be drawn about the viscometric functions [66].

An example of case (b) is the uniaxial elongational flow, which corresponds to a motion of the form [3]

$$\mathbf{u} = \left(\dot{\varepsilon}x, \frac{-\dot{\varepsilon}}{2}y, \frac{-\dot{\varepsilon}}{2}z\right), \tag{85}$$

where $\dot{\varepsilon}$ is the strain-rate. A simple calculation then reveals that

$$\eta = \mu_0\dot{\varepsilon}^{m_1}, \quad \Psi_1 = 3\mu_0|\dot{\varepsilon}|^{m_1} + 3(\alpha_1+\alpha_2)\dot{\gamma}^{m_2}, \quad \Psi_2 = 0. \tag{86}$$

Phenomena such as *die swell* can now be explained from such flows since here, the swelling occurs in a direction orthogonal to the flow and depends on the normal stresses developed in the flow. Since the first normal stress difference depends, both, upon the viscosity and normal stress coefficients, we can clearly see that the behavior of the fluid would be different from that of a Newtonian fluid. There are several *free surface flows* where a non-zero normal stress difference can influence the velocity and result in such phenomena as rod-climbing, tubeless siphon, etc.

We may summarize the main conclusions reached in this paper as follows:

1. *Flows of the generalized second grade type, with* (i) *the stress tensor of the general form expressed by Eq.* (83) *and* (ii) *constant* (*modified*) *pressure gradient*, $\nabla\hat{p}\cdot\mathbf{m}$, *are equivalent to their generalized Newtonian counterparts, where* $\mathbf{m}$ *is the unit vector in the direction of the flow.*
2. *The generalized second grade models are still useful and pertinent in describing various fluid phenomena where normal stress effects become important and may be used, due to their relative simplicity, over higher grade non-Newtonian models.*

Table 1
A brief summary of the second grade fluid models and their variations

	Authors (year)	Model	Comments
Polynomial approximation	Rivlin, Ericksen (1955) [51]	$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2$	Based on retarded motion expansion to second order
	Coleman, Duffin, Mizel (1965) [10]		Instability of rest state for $\alpha_1<0$
	Dunn, Fosdick (1974) [12]		Thermodynamic analysis suggests $\mu>0$, $\alpha_1\geqslant 0$, $\alpha_1+\alpha_2=0$
	Rajagopal, Wineman[a] (1981) [48]		$\alpha_1(\theta)>0$, $\alpha_1'(\theta)\geqslant 0$
Ad hoc models	Man (1987) [29]	(a) $\mathbf{T}=-p\mathbf{I}+\mu\Pi^{m/2}\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2$	No analysis performed on these equations yet
		(b) $\mathbf{T}=-p\mathbf{I}+\Pi^{m/2}(\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2)$	No analysis performed on these equations yet
	Massoudi, Gupta[a] (1993) [19]	$\mathbf{T}=-p\mathbf{I}+\mu_o\mathrm{e}^{-M\theta}\Pi^{m/2}\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2$	No analysis performed on these equations yet
	Massoudi, Phouc[a,b] (2004) [36]	$\mathbf{T}=-p\mathbf{I}+\mu(\theta,\phi)\Pi^{m/2}\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2$	No analysis performed on these equations yet
	Massoudi, Vaidya (2006) [37]	$\mathbf{T}=-p\mathbf{I}+\Pi^{m_1/2}(\mu\mathbf{A}_1)+\Pi^{m_2/2}(\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2)$	No analysis performed on these equations yet
Other models	Criminale, Ericksen, Filbey (1958) [11]	$\mathbf{T}=-p\mathbf{I}+\beta_1\mathbf{A}_1+\beta_2\mathbf{A}_2+\beta_3(\mathbf{A}_1^2-\frac{1}{2}\mathbf{A}_2)$	
	Huilgol (1968) [23]	$\mathbf{T}=-p\mathbf{I}+\eta_0\mathbf{A}_1+\eta_1\mathbf{A}_1^2+\eta_2\mathbf{B}_2(\mathbf{A}_1)$	
	Slemrod (1999) [53]	See Eq. (17)	Based on rational approximation

[a] θ represents temperature.
[b] ϕ represents the volume fraction.

The first observation, though significant, is not completely new. It has been shown previously [60,62,69] that there are flows which [60, p. 129]:

"...for a certain class of constitutive relations [if it] solves the differential equations of motion regardless of the choice of material constants or functions which distinguish one member of the class from another."

Such a velocity field has been referred to as being *universal*. In fact, Yin and Pipkin [69] argue that viscometric universal flows are of three types, namely (a) simple shear flows, (b) flow between tilted planes and (c) couette flows. It has also been shown [62] that the universal flows of second grade fluids are the same as those for the Navier–Stokes fluid, provided div $\mathbf{A}_2$ can be written as a gradient. It has also recently been shown that the minimization of the Helmholtz free energy of an upper-convected Maxwell fluid yields the same velocity profile in channel flow as that of a Newtonian fluid [18]. Truesdell and Rajagopal [62] discuss a systematic approach to identifying classes of universal flows for the grade-n type fluid. Our observation therefore stands as a corollary to these earlier results but is nevertheless significant considering the number of papers that seem to be appearing on the viscometric flows of the generalized second grade models. It seems important then to look at the nearly viscometric [26] or the non-viscometric type flows of the GSGF models.

Finally, we need to mention that another form of generalizing the grade fluid models, so far not attempted, is to assume that the material moduli can also depend on the pressure. In fact, in recent years, Rajagopal (see [43–45]) has provided a rigorous methodology where constitutive relations for a class of fluids whose viscosity depends on pressure and shear rate can be obtained. These types of fluids are encountered, for example, in lubrication industry where the fluid is under high pressure [56], and in fact this phenomenon was recognized by Stokes as early as 1845 [55]. The various types of 'grade' fluids (or fluids of differential type) would all fall in the category of explicit constitutive relations. However, for many fluids (known as the rate-dependent models such as Maxwell or Oldroyd models) the rate of the stress tensor $\mathbf{T}$ is described implicitly as a function of $\mathbf{T}$ and $\mathbf{D}$. For example, Rajagopal and co-workers [64], have proposed that

$$\mathbf{T}=-p\mathbf{I}+2\mu\mathbf{D}, \tag{87}$$

where

$$\mu = \mu(p, |\mathbf{D}|^2). \tag{88}$$

Specifically, Hron et al. [22] suggested

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p)|\mathbf{D}|^{n-2}\mathbf{D}, \tag{89}$$

where for $n = 2$ this model reduces to

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p)\mathbf{D}. \tag{90}$$

When $n \in <-1, 2)$, the fluid is shear thinning, and when $n > 2$, it is shear-thickening. Interestingly, by doing some analysis, they showed that, in the generalization of the Navier–Stokes model with pressure-dependent viscosity one observes secondary flows just as one would in the generalized second grade models.

In conclusion, it must be mentioned that in almost all problems involving the various constitutive relations of the second grade type (as shown in Table 1), the need for additional boundary conditions arises. We refer the reader to Rajagopal and Kaloni [46], Rajagopal [42], Chan Man Fong et al. [5] and Maritz and Sauer [32] for a discussion of this issue.

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