Available online at www.sciencedirect.com
SCIENCE DIRECT®

Nonlinear Analysis: Real World Applications ■■■ (■■■) ■■■–■■■
www.elsevier.com/locate/na

Existence of steady freefall of rigid bodies in a second order fluid with applications to particle sedimentation

Ashwin Vaidya*,[1]

Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, PA 15261, USA

Received 10 September 2004; accepted 11 March 2005

Abstract

We study the slow motion of rigid bodies of arbitrary shape sedimenting in a quiescent viscoelastic liquid under the action of gravity. The liquid is modeled by the full second order fluid equations with arbitrary material parameters $\alpha_1 + \alpha_2$. We show existence of steady state solutions for small Weissenberg numbers and zero Reynolds numbers and apply our equations to study the steady state orientations of symmetric bodies sedimenting in viscoelastic fluids.

Keywords: Existence; Sedimentation; Second order fluid

1. Introduction

It is well known that bodies freely falling in fluids eventually acquire a constant translational and angular velocity of descent, referred to as the *terminal velocity* of the body [3,4,6,22,20,24,26]. There is plenty of experimental work on sedimentation in both Newtonian and in non-Newtonian fluids and this subject has also received ample mathematical attention [10–13,15–17]. We use the term *terminal state* to collectively signify properties such as velocity and orientation of the sedimenting body in its steady state.

---

* Tel.: +1 412 268 6133; fax: +1 412 268 6380.
E-mail address: avaidya@andrew.cmu.edu.

[1] Present address: Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA.

1468-1218/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.nonrwa.2005.03.028

There are several issues of mathematical interest associated with the phenomenon of sedimentation of particles. But the primary mathematical question that needs to be investigated is whether the set of steady state solutions to the corresponding governing equations is non-empty. In this regard, much of the work has already been done for sedimentation in the Stokes approximation of a Newtonian fluid, the Navier–Stokes model and to some extent for non-Newtonian fluids. We refer the readers to [13] and to the references therein for more details. In the case, of non-Newtonian fluids, however, only the case of translating bodies have been considered for the exterior domain problems. Therefore, in this paper, we fill this gap in the literature by showing the existence of steady solutions for a rigid body which is translating and rotating with a prescribed motion in an incompressible, second-order fluid. We extend the results of our earlier work [34] where a similar problem was considered but with the restriction upon the material parameters of the second order fluid model, $\alpha_1 + \alpha_2 = 0$. With this restriction, we could employ the Giesekus Theorem to our advantage, whereby, the velocity field could be considered identical to the Stokes flow field. As a result, the entire problem could be studied with the velocity field restricted to the Stokes velocity field and thus allowing us to take advantage of the symmetries of this flow. In this paper, however, we have no such advantage since we take $\alpha_1 + \alpha_2$ to be arbitrary. We make use of the second order model since it is the simplest case of a viscoelastic, non-Newtonian fluid model and since even for this model, there remain several open questions. It must be pointed out that even though this model can be considered as a first order approximation of a *simple fluid*, the problem that we tackle is non-trivial and the techniques employed in the proof are quite sophisticated and elegant.

The complexity of this problem is three-fold. Firstly, we are considering bodies of arbitrary shapes while the literature is primarily focused on the treatment of flow past symmetric bodies such as prolate or oblate spheroids or spheres. The second important consideration is the inclusion of rotational motion for the sedimenting body which gives rise to the term $\omega \times x$ in the rigid body motion equation. Special attention must be therefore given to the asymptotic behavior of this term as $|x| \to \infty$. The third point of significance in this paper is the nonlinearity that comes from the constitutive model. The treatment of the problem in the case of Navier–Stokes model has only recently been completely understood [15]. There have been no efforts as of yet to study the combined translational and rotational motion of rigid bodies in viscoelastic models. For these reasons, we see this paper as a valuable contribution to the literature in the motion of rigid bodies in fluids. The advances made here indicate the difficulty involved in studying such problems and also point to possible approaches in considering more sophisticated fluid models in the future.

A second motivation for this study comes from the experimental work of Chiba et al. [3], Cho et al. [4], Liu and Joseph [26] and Vaidya [33] on the *terminal orientation* of bodies in Newtonian and non-Newtonian fluids. It is well known that elongated bodies falling in a Newtonian fluid eventually orient themselves with their long side perpendicular to the direction of fall while in viscoelastic fluids, the broad side becomes perpendicular to the direction of fall, in the terminal state [22,25,26]. Furthermore, the work of Chiba et al. [3], Cho et al. [4] and Liu et al. [26] shows that in polymers of certain concentrations, intermediate orientations, called the *tilt angle* also occurs which varies continuously with the polymeric concentration [3]. Besides experimental observations, there is also ample theoretical verification of this work, mostly based on the principle of equilibrium of torques [16,17,34,35] in the steady state. We employ this simple physical idea to analyze for our model, the possible steady orientations that rigid bodies of certain symmetries can adopt. This argument is much like the one made in Vaidya [34]. The particular difficulty of this paper comes from our assumption of arbitrary $\alpha_1 + \alpha_2$ values which does not permit us the convenience of the Giesekus theorem. However, we circumvent this difficulty by writing the velocity field $u = u_s + w$, where u is the velocity field pertaining to the motion of the second order fluid, u_s is the Stokes flow field and w is the remnant term which depends upon *We*. With this definition, we are able to show that the net torque, $\mathscr{M}$, imposed on the body by the fluid can be written as

$$\mathscr{M}(u) = \mathscr{M}(u_s) + \mathscr{N}(w),$$

where $\mathscr{N}$ can be shown to be of $O(We^2)$. Since experimental observations are conducted at very small *We*, we may ignore the higher order dependence upon *We* and choose to work with merely $\mathscr{M}(u_s)$. Setting this term to zero then reveals the possible terminal states of a rigid, sedimenting body.

The outline of the paper is as follows. Sections 2 and 3 deal with the notation and formulation of the relevant equations for our problem. In Section 4, we establish the existence of solutions to the problem by splitting the problem into a coupled Stokes and a transport problem and by use of the contraction mapping principle. Section 5 is devoted to the application of the governing equations to understanding the terminal orientations adopted by purely translational bodies particles during sedimentation.

2. Notation and definitions

By $\mathbf{R}^3$, we denoted the three-dimensional Euclidean space and $\Omega \subset \mathbf{R}^3$ represents an exterior domain, i.e. an open and connected set, exterior to the body $\mathscr{B}$, which is a compact, connected subset of R^3. By Σ we refer to the boundary of Ω and n is the outer unit normal to Σ. The term B_R is defined by the set $\{y \in \mathbf{R} : |x - y| < R\}$ and ∂B_R denotes the boundary of this set.

For $\gamma_i \geqslant 0$ $(i = 1, 2, 3)$, with $|\gamma| = \sum_i \gamma_i$, we define

$$\operatorname{grad}^k u = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_k}u_m$$

and also

$$D^\gamma u = \frac{\partial^{|\gamma|}u}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_2}\partial x_3^{\gamma_3}}.$$

The symbol $\mathscr{C}^k$ with integer $k \geqslant 0$, is used to represent the Banach space of continuously differentiable functions upto the boundary in Ω, with norm

$$\|u\|_{\mathscr{C}^k(\hat{\Omega})} = \max_{0 \leqslant |\gamma| \leqslant k} sup|D^\gamma u|.$$

By $W^{m,q}$, $m \geqslant 0$, $1 \leqslant q \leqslant \infty$, we denote the usual Sobolev space with norm

$$\|u\|_{W^{m,q}} = \left(\sum_{k=0}^{m} \int_{\Omega} |D^k u|^q \right)^{1/q}$$

and

$$\|u\|_{W^{m,\infty}} = \max_{0 \leqslant |k| \leqslant m} \mathit{esssup}\, |D^k u|.$$

If $m = 0$, then $W^{0,q}(\Omega) = L^q(\Omega)$ and

$$\|u\|_{W^{0,q}} = \|u\|_q.$$

The space $D^{m,q}(\Omega)$ denotes a *homogeneous Sobolev Space*, defined as

$$D^{m,q}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) : D^l u \in L^q(\Omega), |l| = m\}$$

with the seminorm

$$\|u\|_{D^{m,q}} = |u|_{m,q} = \left(\sum_{|l|=m} \int_{\Omega} |D^l u|^q \right)^{1/q}.$$

We define the Banach spaces X, in which our existence results will be established as the space

$$\tilde{D}^{2,t}(\Omega) \cap \left[\bigcap_{m=0}^{k} D^{m+2,q}(\Omega) \right] \times D^{1,t}(\Omega) \cap \left[\bigcap_{m=0}^{k} D^{m+1,q}(\Omega) \right]$$

with the norm

$$\|u\|_X + \|\pi\|_X = \|u\|_{\frac{3t}{3-2t}} + |u|_{1,\frac{3t}{3-t}} + |u|_{2,t} + \|\pi\|_{\frac{3t}{3-t}} + |\pi|_{1,t} + \sum_{n=0}^{k+1} (|u|_{n+2,q} + |\pi|_{n+1,q}),$$

where $1 < t < \frac{3}{2}$ and $q > 3$.

We introduce a new set of fields which we shall term the *auxiliary fields* and denoted $(h^{(i)}, p^{(i)})$ and $(H^{(i)}, P^{(i)})$ for $i = 1, 2, 3$. These auxiliary fields serve as a basis field for the Stokes translational and rotational velocity and pressure and hence, satisfy the equations:

$$\left.\begin{array}{l} \Delta h^{(i)} = \operatorname{grad} p^{(i)}, \\ \operatorname{div} h^{(i)} = 0, \\ \lim_{|x|\to\infty} h^{(i)}(x) = 0, \\ h^{(i)}(y) = e_i, \quad y \in \Sigma \end{array}\right\} \tag{1}$$

and

$$\left.\begin{array}{l} \Delta H^{(i)} = \operatorname{grad} P^{(i)}, \\ \operatorname{div} H^{(i)} = 0, \\ \lim_{|x|\to\infty} H^{(i)}(x) = 0, \\ H^{(i)}(y) = e_i \times y, \quad y \in \Sigma. \end{array}\right\} \tag{2}$$

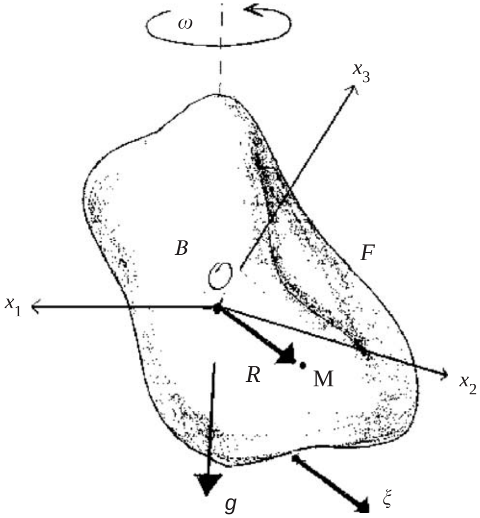

Fig. 1. Physical setting of a body, $\mathscr{B}$ freefalling in a fluid, $\mathscr{F}$.

The Einstein summation convention is used throughout this paper. Hence, for two second order tensors, A and B, the saturation is represented by the Gibbs notation, namely $A : B = A_{ij}B_{ij}$.

3. Problem formulation

The physical setting of the problem is as follows. We consider a rigid body, $\mathscr{B}$, of arbitrary shape falling in an unbounded second order fluid $\mathscr{F}$ of density ρ under the action of the acceleration due to gravity, g. In general, the body can be inhomogeneous, that is, of varying density. To formulate this problem of freefall in an exterior domain, we assume that the body-fluid system is in a steady state, that is, the translational velocity, ξ and angular velocity, ω of $\mathscr{B}$ are constants in time, in the terminal state. It is best to study the problem from a frame (F) attached to $\mathscr{B}$ so that the motion of $\mathscr{F}$ when observed from F is steady [38]. However, in this frame, the direction of g becomes an unknown. We place the origin of the frame at the centroid or geometric center of the body, which we denote by $\mathscr{O}$. The center of mass of $\mathscr{B}$ is denoted by $\mathscr{M}$. R denotes the vector from $\mathscr{O}$ to $\mathscr{M}$ (see Fig. 1). Note that for a homogeneous body, R vanishes since the center of mass now coincides with the geometric center of the body.

Therefore the governing equations for our problem can be given in two parts, the first will involve the equations for the liquid while the second part involves the equations for the body [13,31]. In this paper, the liquid is modeled by the second order fluid equations for which the total stress tensor may be decomposed as $T(v, p) = T_{\mathrm{N}}(v, p) + We\, S(v)$ where $T_{\mathrm{N}}(v, p)$ represents the Newtonian part and $S(v)$ represents the viscoelastic part of the stress tensor which is given by

$$S(v) = \alpha_1 A_2(v) + \alpha_2 A_1(v), \tag{3}$$

where

$$\begin{aligned} &A_2(u) = v\operatorname{grad} A_1(v) + A_1(v)L(v)^{\mathrm{T}} + L(v)A_1(v),\\ &A_1(v) = \operatorname{grad} v + \operatorname{grad}^{\mathrm{T}} v, \quad L(v) = \operatorname{grad} v = \partial_i v_j. \end{aligned}$$

The dimensionless parameter, *We*, namely the Weissenberg number is defined as

$$We = \frac{|\alpha_1|U}{d\eta},$$

where U is the scaling velocity, d is the diameter of $\mathscr{B}$ and η is the shear viscosity coefficient. Furthermore, the constant $\varepsilon = \alpha_2/\alpha_1$, where α_1 and α_2 are material parameters related to the first and second normal stress coefficients, Ψ_1 and Ψ_2 respectively [1].

In this paper, we study the existence of steady falls of an inhomogeneous rigid body of arbitrary shape in a second order fluid with $Re = 0$ and arbitrary ε. The governing equations for the liquid can be given by [27,28]

$$\left.\begin{aligned} &-\Delta v - We\, v\operatorname{grad}\Delta v + \operatorname{grad} p = We \operatorname{div} N(v),\\ &\operatorname{div} v = 0,\\ &v = 0 \text{ on } \partial\Omega,\\ &\lim_{|x|\to\infty} [v(x) + v_\infty(x)] = 0, \end{aligned}\right\} \tag{4}$$

where

$$N(v) = (\operatorname{grad} v)^{\mathrm{T}} A(v) + (1+\varepsilon)A(v)^2$$

and $p = \tilde{p} - xg$, is the modified pressure where we have accounted for the acceleration due to gravity. Note that v_∞ here represents the rigid body motion given by $\xi + \omega \times x$. In addition, when the motion (ξ, ω) of the body is an unknown, we must also provide the equations for the rigid, sedimenting body, which can be given in terms of the net force and torque imposed on $\mathscr{B}$ due to the liquid, namely

$$\left.\begin{aligned} &\int_\Sigma T(v,p)n = m_{\mathrm{e}} g,\\ &\int_\Sigma y \times T(v,p)n = R \times g,\\ &\omega \times x = 0, \end{aligned}\right\} \tag{5}$$

where $m_{\mathrm{e}} = (m - |\mathscr{B}|g)$ is the effective mass. The last equation above comes from the condition of steady state and claims that in its terminal state, the sedimenting body can rotate only along the direction of gravity [13].

Yet another way of expressing the equations of the body is in terms of the auxiliary fields which were introduced in the notation section (Section 2). Upon multiplying the Eq. (4)

suitably, integrating by parts and some rearrangement, we get

$$K\xi + C\omega = m_{\mathrm{e}}g + We\,\mathscr{F}_v, \tag{6}$$

$$C^{\dagger}\xi + \Omega\omega = R \times g + We\,\mathscr{M}_v, \tag{7}$$

$$\omega \times g = 0, \tag{8}$$

where the tensors K, C, Ω and the vectors $\mathscr{F}_v$ and $\mathscr{M}_v$ are defined by

$$\begin{aligned}
K_{ij} &= \int_{\Sigma} (T(h^{(i)}, p^{(i)})n)_j,\\
C_{ij} &= \int_{\Sigma} (x \times T(h^{(i)}, p^{(i)})n)_j,\\
\Omega_{ij} &= \int_{\Sigma} (x \times T(H^{(i)}, P^{(i)})n)_j,\\
\mathscr{F}_v &= \int_{\Omega} h^{(i)} \operatorname{div} S(v)\\
\mathscr{M}_v &= \int_{\Omega} H^{(i)} \operatorname{div} S(v)
\end{aligned}$$

and depend simply on the shape, size or symmetry of $\mathscr{B}$ [19]. For the details of the transformation, the reader is referred to [13,34]. When the motion of the body is an unknown, the Eqs. (6)–(8) are the relevant equations for the body that need to be analyzed. In the special case, when $\varepsilon = -1$, we have shown the existence of solutions (v, p, ξ, ω, g) to the problem for small We and $Re = 0$, in an earlier article [34]. However, in this paper, we will show existence of (v, p) when the motion of the body (ξ, ω) is prescribed. Furthermore, we use a variant of the torque Eq. (8) to investigate the terminal sedimentation behavior of the body.

4. Existence and uniqueness with prescribed (ξ, ω)

The specific objective of this section is to show the existence of solutions to the Eq. (4) for arbitrary ε and sufficiently small We. The motion of the body, i.e. ξ and ω are now prescribed. The strategy that we employ involves splitting this problem into a Stokes problem and a transport problem by a map $\mathscr{A}$, such that (see [14,27,28])

$$\mathscr{A} : \psi \to (v, \pi) \to z. \tag{9}$$

Here, (v, π) solve the Stokes problem

$$\left.\begin{array}{l}
-\Delta v + \operatorname{grad} \pi = \operatorname{div} \psi,\\
\operatorname{div} v = 0,\\
v = 0 \text{ on } \partial\Omega,\\
\lim_{|x|\to\infty} [v(x) + v_{\infty}(x)] = 0,
\end{array}\right\} \tag{10}$$

where the modified pressure π is related to the original pressure by $p = \pi + We\, v \mathrm{grad}\, \pi$. Furthermore, z solves the equation

$$\left.\begin{aligned} &z + We(v \mathrm{grad}\, z - z \mathrm{grad}^{\mathrm{T}}\, v) = We\, N(v, \pi), \\ &N(v, \pi) = N(v) - \pi \mathrm{grad}\, v. \end{aligned}\right\} \tag{11}$$

Therefore, if we replace ψ by z in the above problem we get back the equations for the second order fluid (see [28]). Existence of solutions for the Eqs. (4) is then proved by showing that $\mathscr{A}$ is a contraction in space X (see Section 2 for definition) for sufficiently small Weissenberg numbers.

We subdivide the following subsection into two parts. In the first part we shall obtain preliminary results for the Stokes and Transport equations in appropriate Sobolev spaces. In the second part, we prove the existence of solutions to the Eqs. (4) and (5) with prescribed ξ and ω.

4.1. Preliminary results

Lemma 1. *Let Ω be an exterior domain of class $C^{k+2}(\Omega)$, $k \geqslant 0$. Let also, $\psi \in W^{k+1,q}$, $u_* \in W^{k+2-1/q,q}(\partial\Omega)$, div $\psi \in L^t(\Omega)$ and $u_* \in W^{2-1/t,t}(\partial\Omega)$ for $1 < t < \frac{3}{2}$ and $3 < q < \infty$. Then, there exists a unique solution, (u, π) to the Stokes problem (10) such that*

$$u \in \tilde{D}^{2,t}(\Omega) \cap \left[\bigcap_{m=0}^{k} D^{m+2,q}(\Omega)\right],$$

$$\pi \in D^{1,t}(\Omega) \cap \left[\bigcap_{m=0}^{k} D^{m+1,q}(\Omega)\right].$$

Also (u, π) satisfies the estimate

$$\begin{aligned} &\|grad\ u\|_{C^k} + \|u\|_s + |u|_{1,r} + |u|_{2,t} + \|\pi\|_r + |\pi|_{1,t} \\ &\quad + \sum_{m=0}^{k} (|u|_{m+2,q} + |\pi|_{m+1,q}) \leqslant c(\|div\,\psi\|_t + \|\psi\|_{k+1,q} + |\xi| + |\omega|) \end{aligned} \tag{12}$$

with $r = 3t/(3-t)$, $s = 3t/(3-2t)$, $v = u + v_\infty$ and $c = c(q, t, k)$.

Proof. See [11, Theorem V.4.3]. □

Lemma 2. *Let Ω be an exterior domain of class C^{k+5}, $k \geqslant 0$. Moreover, let v be such that grad $v \in C^k$ with $vn = 0$ and $\hat{F} \in W^{k+1,q}(\Omega)$, $1 < q < \infty$. Then there exists a $\delta > 0$ such that if $We\|grad\, v\|_{C^k(\Omega)} < \delta$, then the transport problem*

$$z + We(v grad\, z - z grad^{\mathrm{T}} v) = \hat{F}$$

has a unique solution $z \in W^{k+1,q}(\Omega)$, such that

$$\|z\|_{k+1,q} \leqslant c\|\hat{F}\|_{k+1,q},$$

where c is a constant depending only upon k, q and t.

Proof. It is sufficient to find suitable a priori estimates for the transport equation (11). Firstly, multiplying Eq. $(11)_1$ by $z|z|^{q-2}$ and integrating over Ω, we obtain

$$\begin{aligned}\|z\|_q^q &\leqslant We\|\operatorname{grad} v\|_{C^0(\Omega)}\|z\|_q^q + \|\hat{F}\|_q\|z\|_q^{q-1} \quad \Rightarrow \quad \|z\|_q \\ &\leqslant \frac{1}{(1 - We\|\operatorname{grad} v\|_{C^0(\Omega)})}\|\hat{F}\|_q. \end{aligned} \tag{13}$$

Next, we take the gradient of Eq. $(11)_1$, multiply through by $\operatorname{grad} z|\operatorname{grad} z|^{q-2}$ and integrate over Ω to get

$$\|\operatorname{grad} z\|_q \leqslant \frac{1}{(1 - We\|\operatorname{grad} v\|_{C^1(\Omega)})}\|\operatorname{grad} \hat{F}\|_q. \tag{14}$$

To show this estimate in general for arbitrary n, we take the $(n+1)$th derivative of the Transport equation. Therefore, we get

$$\begin{aligned}&\operatorname{grad}^{n+1} z + We\sum_{r=0}^{n+1}\binom{n+1}{r}(\operatorname{grad}^{n+1-r} v)(\operatorname{grad}^r \operatorname{grad} z) \\ &- We\sum_{r=0}^{n+1}\binom{n+1}{r}(\operatorname{grad}^{n+1-r} z)(\operatorname{grad}^r \operatorname{grad} v) = \operatorname{grad}^{n+1}\hat{F}. \end{aligned} \tag{15}$$

As earlier, multiply by $\operatorname{grad}^{n+1} z|\operatorname{grad}^{n+1} z|^{q-2}$ and integrate over Ω to get

$$\|\operatorname{grad}^{n+1} z\|_q \leqslant \frac{1}{(1 - We\|\operatorname{grad} v\|_{C^{n+1}(\Omega)})}\|\operatorname{grad}^{n+1}\hat{F}\|_q. \tag{16}$$

We obtain the desired estimate by an induction argument. To this end, we consider the case of $k = 0$ and $k = 1$, which upon addition yields

$$(1 - c_1 We\|\operatorname{grad} v\|_{C^1(\Omega)})\|z\|_{1,q} \leqslant \|\hat{F}\|_{1,q}. \tag{17}$$

Also assume that the result holds for $k = n$. Hence

$$(1 - c_n We\|\operatorname{grad} v\|_{C^n(\Omega)})\|z\|_{n,q} \leqslant \|\hat{F}\|_{n,q}. \tag{18}$$

Then adding the two estimates above corresponding to cases $k = 1$ and $k = n$, we obtain the estimate

$$(1 - c_{n+1} We\|\operatorname{grad} v\|_{C^{n+1}(\Omega)})\|z\|_{n+1,q} \leqslant \|\hat{F}\|_{n+1,q}. \tag{19}$$

Once we have obtained the a priori estimates, the existence of such a z can be shown by an argument similar to Galdi and Rajagopal [14] (also see [30]). ☐

Additionally, we also have similar estimates for $\operatorname{div} z$. It is easily verified from the transport equation that

$$\operatorname{div} z + We\, v \operatorname{grad} \operatorname{div} z = \operatorname{div} \hat{F} \tag{20}$$

Lemma 3. *Let Ω be an exterior domain of class C^{k+3}, $k \geqslant 1$ and let v satisfy grad $v \in C^{k-1}$ and $vn = 0$. Also, let $q > 3$ and $1 < r < \infty$. Then there exists a $\delta > 0$ such that if We$\|\mathrm{grad}\, v\|_{C^{k-1}} < \delta$ then there exists a solution to the equation* (20) *satisfying the following estimates*:

$$\|div\, z\|_{k,q} \leqslant c_1 \|div\, \hat{F}\|_{k,q} \quad for\ k \geqslant 1 \tag{21}$$

and

$$\|div\, z\|_r \leqslant c_2 \|div\, \hat{F}\|_r, \tag{22}$$

where c_1, c_2 depend on k, q, t and c_1 additionally depends also on We.

Proof. Proof of this lemma is similar to that of Lemma 3. Also see [14,30]. ☐

The next lemma concerns the property of the extra stress tensor, $S(v)$ for the second order fluid model which is written as

$$S(v) = v\mathrm{grad}\, A(v) + \varepsilon A(v)A(v) + (\mathrm{grad}\, v)^{\mathrm{T}} A(v) + A(v)\mathrm{grad}\, v.$$

We make the following observation.

Lemma 4. *If we write $u = v + v_\infty$, then*

$$S(v) = v_\infty grad\, A(u) + \tilde{S}(u),$$

where

$$\tilde{S}(u) = u grad\, A(u) + \varepsilon A(u)A(u) + (grad\, u)^{\mathrm{T}} A(u) + A(u) grad\, u.$$

Proof. It is easy to see that $A(v) = A(u)$. Therefore the only thing that remains to be shown is that the final two terms of $S(v)$ are invariant under the transformation of $v \to u$. We have that

$$\begin{aligned}(\mathrm{grad}\, u)^{\mathrm{T}} A(u) + A(u)\mathrm{grad}\, u &= (\mathrm{grad}\, u)^{\mathrm{T}} A(v) + A(v)\mathrm{grad}\, u \\ &= (\mathrm{grad}\, v)^{\mathrm{T}} A(v) + A(v)\mathrm{grad}\, v \\ &\quad + (\mathrm{grad}\, v_\infty)^{\mathrm{T}} A(v) + A(v)\mathrm{grad}\, v_\infty \\ &= (\mathrm{grad}\, v)^{\mathrm{T}} A(v) + A(v)\mathrm{grad}\, v \\ &\quad + (\mathrm{grad}\, \omega \times x)^{\mathrm{T}} A(v) + A(v)(\mathrm{grad}\, \omega \times x).\end{aligned}$$

However

$$\begin{aligned}&(\mathrm{grad}\, \omega \times x)^{\mathrm{T}} A(v) + A(v)(\mathrm{grad}\, \omega \times x) \\ &\quad = (\varepsilon_{ikm} + \varepsilon_{imk})\omega_i \partial_k v_j + (\varepsilon_{ikm} + \varepsilon_{imk})\omega_i \partial_j v_k = 0.\end{aligned}$$

Therefore it follows that $S(v) = v_\infty \mathrm{grad}\, A(u) + \tilde{S}(u)$. ☐

4.2. Existence results

In this section, we establish the existence of solutions to the Eq. (4) using the Banach fixed point theorem. We define the Banach space $B=\{\psi : \|\psi\|_{k+1,q}+\|\operatorname{div}\psi\|_t<\infty\}$ and also the subspace G_D of B, such that

$$G_D := \{\psi : \|\psi\|_{k+1,q}+\|\operatorname{div}\psi\|_t \leqslant D\}.$$

Lemma 5. *Let Ω be an exterior domain of class C^{k+5}, $k\geqslant 0$ and q,t be as defined in Lemma* 1. *Also, let the map $\mathscr{A}$ be as defined in Eq.* (9). *Then, $\mathscr{A}$ maps G_D to G_D.*

Proof. We define G_D as above and recall that $v=u+v_\infty$. Then it follows from Lemma 1 that

$$\begin{aligned}&\|u\|_s+|u|_{1,r}+|u|_{2,t}+\|\pi\|_r+|\pi|_{1,t}\\&+\sum_{m=0}^{k}(|u|_{m+2,q}+|\pi|_{m+1,q})\leqslant c(D+|\xi|+|\omega|)\end{aligned} \tag{23}$$

with $r=3t/(3-t)$, $s=3t/(3-2t)$ and $c=c(q,t,k,\Omega)$. We can also show without difficulty and using the estimate (23), that

$$\begin{aligned}\|\operatorname{div}N(v,\pi)\|_t&\leqslant c_1(\|Dv\|_{C^0}+\|\pi\|_\infty)(\|D^2v\|_t+\|\operatorname{grad}\pi\|_t)\\&\leqslant c_2(D+|\xi|+|\omega|)^2,\end{aligned}$$

where we use the fact that $\|Dv\|_{C^0}\leqslant\|Du\|_{C^0}+|\omega|$. Similarly, using the Sobolev inequality [11, Eq. (2.7), p. 32] and Eq. (23),

$$\begin{aligned}\|N(v,\pi)\|_q&\leqslant c_3\|\operatorname{grad}v\|_{C^0}(\|A(v)\|_q+\|\pi\|_q)\\&\leqslant c_4\|\operatorname{grad}v\|_{C^0}(\|D^2v\|_p+\|\operatorname{grad}\pi\|_p),\end{aligned} \tag{24}$$

where $3/2<p=3q/(3+q)<3$. Furthermore, it is easily verified that

$$\|\operatorname{grad}N(v,\pi)\|_{k,q}\leqslant c_5(\|Dv\|_{C^0(\Omega)}+\|\pi\|_\infty)(\|D^2v\|_{k,q}+\|\operatorname{grad}\pi\|_{k,q}). \tag{25}$$

Hence, combining Eq. (24), (25) and (23), we have

$$\begin{aligned}\|N(v,\pi)\|_{k+1,q}&\leqslant c_6(\|Dv\|_{C^0(\Omega)}+\|\pi\|_\infty)(\|D^2v\|_{\frac{3q}{3+q}}+\|\operatorname{grad}\pi\|_{\frac{3q}{3+q}}\\&\quad+\|D^2v\|_{k,q}+\|\operatorname{grad}\pi\|_{k,q})\\&\leqslant c_7(D+|\xi|+|\omega|)^2.\end{aligned} \tag{26}$$

In order to fulfill the assumptions of Lemmas 2 and 3, we require that

$$We\|\operatorname{grad}v\|_{C^k}\leqslant\delta$$

and also that

$$We(\|\operatorname{div}z\|_t+\|z\|_{k+1,q})\leqslant D$$

which follow from the observations that

$$\begin{aligned}We\|\operatorname{grad} v\|_{C^k} &\leqslant We(\|\operatorname{grad} u\|_{C^k} + |\omega|)\\ &\leqslant We(D + |\xi| + |\omega|) \leqslant \delta \end{aligned}\tag{27}$$

and

$$We(\|\operatorname{div} z\|_t + \|z\|_{k+1,q}) \leqslant \tilde{c} We(D + |\xi| + |\omega|)^2 \leqslant D, \tag{28}$$

respectively. Hence, for the choice, $D = \beta(|\xi| + |\omega|)$, for $\beta > 0$, the two conditions are satisfied if $We(|\xi| + |\omega|) < \delta/(\beta + 1)$ and $We(|\xi| + |\omega|) < \beta/(\beta + 1)^2$, respectively. Consequently, for $We(|\xi| + |\omega|) < \min(\delta/(\beta + 1), \beta/(\beta + 1)^2)$, we have that $z \in G_D$ and hence $\mathscr{A}$ maps G_D to G_D. □

Lemma 6. *The mapping* $\mathscr{A}$ *is a contraction in* G_D.

Proof. Let z_1 and z_2 be two different solutions to the transport Eq. (11) and let the pairs (v_1, π_1) and (v_2, π_2) be two different solutions to the Stokes problem with corresponding translational and rotation components (ξ, ω). Let us also define

$$\psi := \psi_1 - \psi_2, \;\; v := v_1 - v_2, \;\; \pi := \pi_1 - \pi_2, \;\; z := z_1 - z_2.$$

Then subtracting the Eq. (10) corresponding to the pairs (v_1, π_1) and (v_2, π_2), we have

$$\left.\begin{array}{l} -\Delta v + \operatorname{grad}\pi = \operatorname{div}\psi, \\ \operatorname{div} v = 0, \\ v = 0 \text{ on } \partial\Omega, \\ \lim_{|x|\to\infty} v(x) = 0, \end{array}\right\} \tag{29}$$

with corresponding estimate,

$$\begin{aligned}&\|v\|_s + |v|_{1,r} + |v|_{2,t} + \|\pi\|_r + |\pi|_{1,t} + \sum_{m=0}^{k}(|v|_{m+2,q} + |\pi|_{m+1,q})\\ &\quad\leqslant c(\|\operatorname{div}\psi\|_t + \|\psi\|_{k+1,q}),\end{aligned}\tag{30}$$

where $r = 3t/(3 - t)$, $s = 3t/(3 - 2t)$ and $c = c(q, t, k)$. Similarly, subtracting the Transport Eq. (11) and its divergence, Eq. (20), corresponding to z_1 and z_2, we get upon simplification,

$$\begin{aligned}&z + We(v_1 \operatorname{grad} z - z(\operatorname{grad} v_1)^{\mathrm{T}}) - We(z_2(\operatorname{grad} v)^{\mathrm{T}} - v\operatorname{grad} z_2)\\ &\quad= We(N(v_1, \pi_1) - N(v_2, \pi_2))\end{aligned}\tag{31}$$

and

$$\begin{aligned}&\operatorname{div} z + We\, v\operatorname{grad}(\operatorname{div} z_1) - We\, v_2\operatorname{grad}(\operatorname{div} z)\\ &\quad= We\operatorname{div}(N(v_1, \pi_1) - N(v_2, \pi_2))\end{aligned}\tag{32}$$

respectively, where

$$\begin{aligned}N(v_1, \pi_1) - N(v_2, \pi_2) = {}&(\operatorname{grad} v)^{\mathrm{T}} A(v_1) + (\operatorname{grad} v_2)^{\mathrm{T}} A(v) + (1 + \varepsilon)A(v)A(v_1)\\ &+ (1 + \varepsilon)A(v_2)A(v) - \pi\operatorname{grad} v_1 - \pi_2 \operatorname{grad} v.\end{aligned}\tag{33}$$

Then we have the estimates

$$\begin{aligned}\|\operatorname{div}[N(v_1,\pi_1)-N(v_2,\pi_2)]\|_t \leqslant c(&\|\operatorname{grad} v_1\|_{C^0}+\|\pi_2\|_\infty+\|\operatorname{grad} v_2\|_{C^0}\\ &+\|D^2v_1\|_t+\|D^2v_2\|_t+\|\operatorname{grad}\pi_2\|_t)\\ &\times(\|\operatorname{grad} v\|_{C^0}+\|\pi\|_\infty+|D^2v\|_t+\|\operatorname{grad}\pi\|_t)\end{aligned} \tag{34}$$

and

$$\begin{aligned}\|[N(v_1,\pi_1)-N(v_2,\pi_2)]\|_{k+1,q} \leqslant c(&\|\operatorname{grad} v_1\|_{C^0}+\|\operatorname{grad} v_2\|_{C^0}+\|\pi_2\|_\infty\\ &+\|D^2v_1\|_{k,q}+\|D^2v_2\|_{k,q}+\|\operatorname{grad}\pi_2\|_{k,q}\\ &+\|D^2v_1\|_{\frac{3q}{3+q}}+\|D^2v_2\|_{\frac{3q}{3+q}}+\|\operatorname{grad}\pi_2\|_{\frac{3q}{3+q}})\\ &\times(\|\operatorname{grad} v\|_{C^0}+\|\pi\|_\infty+\|D^2v\|_{k,q}\\ &+\|D^2v\|_{\frac{3q}{3+q}}+\|\operatorname{grad}\pi\|_{k,q}+\|\operatorname{grad}\pi\|_{\frac{3q}{3+q}})\end{aligned} \tag{35}$$

upon suitable application of the Sobolev inequality [11].

The estimates for the Transport Eq. (31), employing also Eq. (35) is then given by

$$\begin{aligned}\|\operatorname{div} z\|_t+\|z\|_{k+1,q}\leqslant cWe(&\|\operatorname{grad} v_1\|_{C^0}+\|\operatorname{grad} v_2\|_{C^0}+\|\pi_2\|_\infty+\||D^2v_1\|_{k,q}\\ &+\|D^2v_2\|_{k,q}+\|\operatorname{grad} v_2\|_{k,q}+\|\pi_2\|_\infty+\|D^2v_1\|_t\\ &+\|D^2v_2\|_t+\|\operatorname{grad}\pi_2\|_t+\|D^2v_1\|_{\frac{3q}{3+q}}+\|D^2v_2\|_{\frac{3q}{3+q}}\\ &+\|\operatorname{grad}\pi_2\|_{\frac{3q}{3+q}})(\|\operatorname{grad} v\|_{C^0}+\|\pi\|_\infty+|D^2v\|_t\\ &+\|\operatorname{grad}\pi\|_t+\|\operatorname{grad} v\|_{C^0}+\|\pi\|_\infty+\|D^2v\|_{k,q}\\ &+\|\operatorname{grad}\pi\|_{k,q}+\|D^2v\|_{\frac{3q}{3+q}}+\|\operatorname{grad}\pi\|_{\frac{3q}{3+q}})\end{aligned}$$

which upon further simplification can be shown to be

$$\begin{aligned}&\leqslant cWe(\|\operatorname{grad} v_1\|_{C^0}+\|\operatorname{grad} v_2\|_{C^0}+\|\pi_2\|_\infty+\|D^2v_1\|_{k,q}+\|D^2v_2\|_{k,q}\\ &\quad+\|\operatorname{grad}\pi_2\|_{k,q}+\|\pi_2\|_\infty+\|D^2v_1\|_t+\|D^2v_2\|_t+\||\operatorname{grad}\pi_2\|_t+\|D^2v_1\|_{\frac{3q}{3+q}}\\ &\quad+\|D^2v_2\|_{\frac{3q}{3+q}}+\|\operatorname{grad}\pi_2\|_{\frac{3q}{3+q}})(\|\operatorname{div}\psi\|_t+\|\psi\|_{k+1,q})\\ &\quad\leqslant cWe(\|\operatorname{div}\psi_1\|_t+\|\psi_1\|_{k+1,q}+\|\operatorname{div}\psi_2\|_t+\|\psi_2\|_{k+1,q}+2|\xi|\\ &\quad+2|\omega|)(\|\operatorname{div}\psi\|_t+\|\psi\|_{k+1,q})\\ &\quad\leqslant\hat{c}We(D+|\xi|+|\omega|)(\|\operatorname{div}\psi\|_t+\|\psi\|_{k+1,q}).\end{aligned} \tag{36}$$

Therefore, combining the results of Eqs. (37) and (36), and recognizing that $\hat{c}We(D+|\xi|+|\omega|)<1$, from the results of Lemma 5, we have that $\mathscr{A}$ is a contraction. □

We are now in a position to prove the main result of this section.

Theorem 1 (*Main Existence Theorem*). *Let Ω be an exterior domain of class C^{k+5}, $k \geqslant 0$ and ξ, $\omega \in R^3$. Also, let $q > 3$ and $1 < t < \frac{3}{2}$. Then, there is a $c > 0$, such that if We$(|\xi| + |\omega|) < c$, the problem* (4) *has a unique solution (v, π) where*

$$u \in \tilde{D}^{2,t}(\Omega) \cap \left[\bigcap_{m=0}^{k} D^{m+2,q}(\Omega)\right],$$

$$\pi \in D^{1,t}(\Omega) \cap \left[\bigcap_{m=0}^{k} D^{m+1,q}(\Omega)\right],$$

satisfying the estimate,

$$\|u\|_s + |u|_{1,r} + |u|_{2,t} + \|\pi\|_r + |\pi|_{1,t} + \sum_{m=0}^{k} (|u|_{m+2,q} + |\pi|_{m+1,q}) \leqslant C(|\xi| + |\omega|), \tag{37}$$

where $r = 3t/(3-t)$, $s = 3t/(3-2t)$, $C = C(q, t, k)$ and $v = u + \xi + \omega \times x$. Furthermore, the extra-stress tensor $S(v)$, defined in Eq. (3), *satisfies the estimate,*

$$\|div\, S(v)\|_t + \|S(v)\|_{k+1,q} \leqslant C_1(|\xi| + |\omega|)^2, \tag{38}$$

where $C_1 = C_1(q, t, k)$.

Proof. The proof of this theorem follows from the above Lemmas 1, 2 and 6 and the Banach Fixed Point theorem. The estimate (38), on the extra-stress tensor for the second order liquid, follows from the Eqs. (23) and (28) in Lemma 5. ☐

5. Application to particle sedimentation

5.1. Formulation of problem to first order in *We*.

In the previous parts of this paper, we have established the existence of solutions (v, p) to the problem (4) corresponding to the freefall of a rigid body of arbitrary shape in a second order liquid. In this section, we shall specialize the torque equations to study the terminal orientation of rigid bodies in the second order fluid. We shall follow the argument outlined in [34,13] and consider, for our rigid body, several different symmetries. We employ the heuristic idea proposed by Joseph and Feng [21] that the terminal orientation of elongated bodies in liquids are a result of competing inertial and viscoelastic torques acting on the body. Since our analysis assumes a zero Reynolds number flow, we will analyze the problem when the viscoelastic contribution to the torque is in equilibrium. But first, we need to obtain our torque equations upto first order in *We*. We therefore write

$$v = v_s + \tilde{v},\ \pi = \pi_s + \tilde{\pi},$$

where (v_s, π_s) are the solutions to the Stokes problem and which asymptotically approaches a rigid body motion. Additionally we also define $u = u_s + \tilde{u}$ and it follows that $v_s = u_s + v_\infty$. Hence substituting this formulation in Eq. (4) we have that $(\tilde{v}, \tilde{\pi})$ satisfies

$$\left.\begin{array}{l} -\Delta\tilde{v} + \operatorname{grad}\tilde{\pi} = We \operatorname{div} S(v, \varepsilon), \\ \tilde{v}|_\Sigma = 0, \\ \operatorname{div}\tilde{v} = 0, \\ \lim_{|x|\to\infty}\tilde{v} = 0. \end{array}\right\} \tag{39}$$

So, in order to obtain the relevant equation at first order in *We*, we evaluate the freefall equations at v_s and establish that there are several remnant terms which are of $O(We^2)$ and hence can be ignored in our analysis. More specifically, we must show the following properties:

Lemma 7. *Let $\mathscr{B}$ be a body of class C^3. Then there exist positive numbers $We_0 = We_0(\mathscr{B}, \varepsilon)$, $C_i = C_i(\mathscr{B}, D, k, q, t)$ $(i = 1, 2, 3)$, such that for any $0 < We < We_0$, $\frac{3}{2} < q_1 < \infty$, $k \geqslant 0$, $1 < t < \frac{3}{2}$ and $q > 3$, we have*

(1) $\|v - v_s\|_X \leqslant C_1 We$,
(2) $\|\tilde{S}(u) - \tilde{S}(u_s)\|_{q_1} \leqslant C_2 We$,
(3) $|\mathscr{N}_1 + \mathscr{N}_2| \leqslant C_3 We$,

where $\mathscr{N}_1 = \mathscr{F}_v(v) - \mathscr{F}_v(v_s)$ and $\mathscr{N}_2 = \mathscr{M}_v(v) - \mathscr{M}_v(v_s)$ and

$$\tilde{S}(u) = u \operatorname{grad} A(u) + \varepsilon A(u)A(u) + (\operatorname{grad} u)^T A(u) + A(u) \operatorname{grad} u.$$

Proof of Property 1. From the Stokes estimates that we had earlier, we have

$$\begin{aligned} &\|\tilde{v}\|_s + \|\tilde{\pi}\|_m + \|\operatorname{grad}\tilde{v}\|_m + \|\operatorname{grad}\tilde{\pi}\|_t + \|D^2\tilde{v}\|_t \\ &\quad + \|D^2\tilde{v}\|_{k,q} + \|\operatorname{grad}\tilde{\pi}\|_{k,q} \leqslant cWe(\|\operatorname{div} S(v)\|_t + \|\operatorname{div} S(v)\|_{k,q}). \end{aligned} \tag{40}$$

The final estimates on $S(v)$ follows from our arguments earlier in this paper regarding estimates of the Transport equation. Based on the results of Lemma 3 and Theorem 1, it is therefore easily verified that

$$\begin{aligned} &\|\operatorname{div} S(v)\|_t + \|\operatorname{div} S(v)\|_{k+1,q} \leqslant c_1(\|\operatorname{grad} v\|_{C^0} + \|\pi\|_\infty + \|D^2 v\|_{k,q} + \|D^2 v\|_t \\ &\quad + \|D^2 v\|_{\frac{3q}{3+q}})(\|\operatorname{grad} v\|_{C^0} + \|\pi\|_\infty + |D^2 v\|_t + \|\operatorname{grad}\pi\|_t \\ &\quad + \|\pi\|_\infty + \|D^2 v\|_{k,q} + \|\operatorname{grad}\pi\|_{k,q} + \|D^2 v\|_{\frac{3q}{3+q}} + \|\operatorname{grad}\pi\|_{\frac{3q}{3+q}}) \\ &\quad \leqslant c_2(\|u\|_X + \|\pi\|_X + |\omega|)^2 \leqslant c_3(|\xi| + |\omega|)^2 \leqslant C_1(\mathscr{B}, D, k, q, t). \end{aligned} \tag{41}$$

Proof of Property 1 of Lemma 7 follows. □

Proof of Property 2. Let us write $\tilde{S}(u)=\tilde{S}_0(u)+\tilde{S}_1(u)$, where $\tilde{S}_0(u)=u\,\mathrm{grad}\,A(u)$ and $\tilde{S}_1(u)$ is the remaining term. Similarly, we may write $\tilde{S}(u_s)=\tilde{S}_0(u_s)+\tilde{S}_1(u_s)$. Then,

$$\begin{aligned}\tilde{S}_0(u)-\tilde{S}_0(u_s)&=\tilde{u}\,\mathrm{grad}\,A(u)+u_s\mathrm{grad}\,A(\tilde{u})\\ &\Rightarrow \|S_0(u)-S_0(u_s)\|_q \leqslant \|\tilde{u}\,\mathrm{grad}\,A(u)\|_q+\|u_s\mathrm{grad}\,A(\tilde{u})\|_q\\ &\leqslant \|D^2u\|_q\|\tilde{u}\|_\infty+\|D^2\tilde{u}\|_q\|u_s\|_\infty.\end{aligned}$$

Similarly, we can obtain estimates for the remaining terms. Hence,

$$\begin{aligned}\|\tilde{S}_1(u)-\tilde{S}_1(u_s)\|_q &\leqslant \|\varepsilon A(u)A(\tilde{u})+\varepsilon A(\tilde{u})A(u)+(\mathrm{grad}\,\tilde{u})^{\mathrm{T}}A(u)\\ &\quad+(\mathrm{grad}\,u)^{\mathrm{T}}A(\tilde{u})+\mathrm{grad}\,u\;A(\tilde{u})+\mathrm{grad}\,\tilde{u}\;A(u)\|_q\\ &\leqslant c_3(\|Du\|_{C^0}+\|D^2u\|_{\frac{3q}{3+q}})(\|D\tilde{u}\|_{C^0}+\|D^2\tilde{u}\|_{\frac{3q}{3+q}}).\end{aligned}$$

Finally combining the above estimates, we have

$$\|\tilde{S}(u)-\tilde{S}(u_s)\|_q \leqslant c_4\|\tilde{u}\|_X \leqslant C_2(\mathscr{B},D,k,q,t)We$$

with $q>\frac{3}{2}$, upon using the results of Property 1. □

Proof of Property 3. For proof of the final property, we need to make some preliminary definitions. We define

$$\mathscr{N}_1=\mathscr{F}_v(v)-\mathscr{F}_v(v_s)=-\int_\Omega[S(v)-S(v_s)]:D(h^{(i)})$$

and

$$\mathscr{N}_2=\mathscr{M}_v(v)-\mathscr{M}_v(v_s)=-\int_\Omega[S(v)-S(v_s)]:D(H^{(i)}),$$

where $i=1,2,3$. Let us also define

$$\tilde{H}^{(i)}=\begin{cases}h^{(i)}, & i=1,2,3,\\ H^{(i)}, & i=4,5,6.\end{cases}\tag{42}$$

It then follows that

$$\begin{aligned}|\mathscr{N}_1+\mathscr{N}_2| &\leqslant \int_\Omega|S(v)-S(v_s)||D(\tilde{H}^{(i)})|\\ &\leqslant \int_\Omega|\tilde{S}(u)-\tilde{S}(u_s)||D(\tilde{H}^{(i)})|+\int_\Omega|v_\infty\mathrm{grad}\,A_1(\tilde{u})||D(\tilde{H}^{(i)})|\\ &\leqslant (1+|\xi|)\|\tilde{S}(u)-\tilde{S}(u_s)\|_2\|D(\tilde{H}^{(i)})\|_2+c_1|\omega|\left\|\frac{1}{x}\right\|_3\|D^2\tilde{u}\|_{\frac{3}{2}}\\ &\leqslant c_2\|\tilde{u}\|_X \leqslant C_3(\mathscr{B},D,k,q,t)We,\end{aligned}\tag{43}$$

where we use the fact that $\|\mathrm{grad}\,\tilde{H}^{(i)}\|_s<\infty$ for $s>\frac{3}{2}$ from the Stokes estimates and also Property 2. Therefore, we have our desired result. □

In order to analyze the steady state orientations of the body we write the net torque imposed by the second order fluid upon the body can be shown to be of the form [17]

$$\mathscr{M} = -K\xi - C^{\dagger}\omega + We\,\mathscr{M}_v(v) + R \times g. \tag{44}$$

However, as explained in the introduction, we are interested in the first order effects of the viscoelastic parameters alone. We therefore write the total torque as

$$\begin{aligned}\mathscr{M} &= -K\xi - C^{\dagger}\omega + We\,\mathscr{M}_v(v_{\mathrm{s}}) + We(\mathscr{M}_v(v) - \mathscr{M}_v(v_{\mathrm{s}})) + R \times g \\ &= -K\xi - C^{\dagger}\omega + We\,\mathscr{M}_v(v_{\mathrm{s}}) + We\,\mathscr{N}_2 + R \times g.\end{aligned}$$

In light of Lemma 7, we realize that $|\mathscr{N}_2| = O(We)$, therefore at first order in *We*, we may effectively ignore this term. Also, the full form of the viscoelastic contribution to the torque is given as

$$\begin{aligned}\mathscr{M}_v(v_{\mathrm{s}}) &= \int_{\Omega} \{v_{\mathrm{s}}\,\mathrm{grad}\,\Delta v_{\mathrm{s}} + \mathrm{div}\,[(\mathrm{grad}\,v_{\mathrm{s}})^{\mathrm{T}} A(v_{\mathrm{s}})] + (1+\varepsilon)\mathrm{div}\,[A(v_{\mathrm{s}})A(v_{\mathrm{s}})]\} \cdot H^{(i)} \\ &= \mathscr{M}_1(v_{\mathrm{s}}) + (1+\varepsilon)\mathscr{M}_2(v_{\mathrm{s}}).\end{aligned} \tag{45}$$

Using the result from our earlier paper [34], we can show that, writing $v_{\mathrm{s}} = \xi_i h^{(i)} + \omega_i H^{(i)}$, we have

$$\begin{aligned}\mathscr{M}_v(v_{\mathrm{s}}) = {} & \xi_i \xi_j (A_{Rk}^{(i,j)} + (1+\varepsilon)U_{Rk}^{(i,j)}) + \omega_i \omega_j (B_{Rk}^{(i,j)} + (1+\varepsilon)V_{Rk}^{(i,j)}) \\ & + \xi_i \omega_j (C_{Rk}^{(i,j)} + (1+\varepsilon)W_{Rk}^{(i,j)}),\end{aligned} \tag{46}$$

where we define the *torque coefficients* as

$$\begin{aligned}\mathbf{A}_T^{(i,j)} &= \frac{1}{2}\int_{\Sigma} Z_1^{(i)} Z_1^{(j)} n, \\ \mathbf{B}_T^{(i,j)} &= \frac{1}{2}\int_{\Sigma} (Z_2^{(i)} Z_2^{(j)} - 4Z_{2j}^{(i)}) n, \\ \mathbf{C}_T^{(i,j)} &= \int_{\Sigma} (Z_1^{(i)} Z_2^{(j)} - 2Z_{1j}^{(i)}) n, \\ \mathbf{A}_R^{(i,j)} &= \frac{1}{2}\int_{\Sigma} Z_1^{(i)} Z_1^{(j)} y \times n, \\ \mathbf{B}_R^{(i,j)} &= \frac{1}{2}\int_{\Sigma} (Z_2^{(i)} Z_2^{(j)} - 4Z_{2j}^{(i)}) y \times n, \\ \mathbf{C}_R^{(i,j)} &= \int_{\Sigma} (Z_1^{(i)} Z_2^{(j)} - 2Z_{1j}^{(i)}) y \times n,\end{aligned} \tag{47}$$

where

$$Z_1^{(i)} = \mathrm{curl}\,h^{(i)}, \quad Z_2^{(i)} = \mathrm{curl}\,H^{(i)}$$

and also,

$$\begin{aligned}
\mathbf{U}_T^{(i,j)} &= \int_\Omega h^{(k)} \mathrm{div}\,[A(h^{(i)})A(h^{(j)})],\\
\mathbf{V}_T^{(i,j)} &= \int_\Omega h^{(k)} \mathrm{div}\,[A(H^{(i)})A(H^{(j)})],\\
\mathbf{W}_T^{(i,j)} &= \int_\Omega h^{(k)} \mathrm{div}\,[A(h^{(i)})A(H^{(j)}) + A(H^{(i)})A(h^{(j)})],\\
\mathbf{U}_R^{(i,j)} &= \int_\Omega H^{(k)} \mathrm{div}\,[A(h^{(i)})A(h^{(j)})],\\
\mathbf{V}_R^{(i,j)} &= \int_\Omega H^{(k)} \mathrm{div}\,[A(H^{(i)})A(H^{(j)})],\\
\mathbf{W}_R^{(i,j)} &= \int_\Omega H^{(k)} \mathrm{div}\,[A(h^{(i)})A(H^{(j)}) + A(H^{(i)})A(h^{(j)})]. \qquad (48)
\end{aligned}$$

It is easily seen that when $\varepsilon = -1$, we revert back to the case studied in [34]. With the results at hand, we can write the final, simplified form of the net torque as

$$\begin{aligned}
\mathscr{M} = &- K_{kj}\xi_j - C^{\dagger}_{kj}\omega_j + (R \times g)_k + We(\xi_i\xi_j(A_{Rk}^{(i,j)} + (1+\varepsilon)U_{Rk}^{(i,j)})\\
&+ \omega_i\omega_j(B_{Rk}^{(i,j)} + (1+\varepsilon)V_{Rk}^{(i,j)}) + \xi_i\omega_j(C_{Rk}^{(i,j)} + (1+\varepsilon)W_{Rk}^{(i,j)})). \qquad (49)
\end{aligned}$$

5.2. Viscoelastic contribution to Torque under different symmetries

In this section, we analyze the special forms that Eq. (49) can take for different shapes of the sedimenting body $\mathscr{B}$. We will follow closely the argument used in [19] and [34]. We begin with the observation that the torque coefficients on the right-hand side of Eq. (49) are third order tensors and we define two classes of third order tensors, namely

$$\mathscr{U} = \{\mathbf{A}_T^{(i,j)}, \mathbf{B}_T^{(i,j)}, \mathbf{C}_R^{(i,j)}, \mathbf{U}_T^{(i,j)}, \mathbf{V}_T^{(i,j)}, \mathbf{W}_R^{(i,j)}\}$$

and

$$\mathscr{V} = \{\mathbf{A}_R^{(i,j)}, \mathbf{B}_R^{(i,j)}, \mathbf{C}_T^{(i,j)}, \mathbf{U}_R^{(i,j)}, \mathbf{V}_R^{(i,j)}, \mathbf{W}_T^{(i,j)}\}$$

such that elements of $\mathscr{U}$ are distinguished from those of $\mathscr{V}$ by the respective transformation laws

$$Q_{ijk} = |a| a_{il} a_{jm} a_{kn} Q_{lmn} \qquad (50)$$

or

$$Q_{ijk} = -|a| a_{il} a_{jm} a_{kn} Q_{lmn}. \qquad (51)$$

See [33,34] for proof of this result. In light of this result, we may observe that the elements of the two classes $\mathscr{U}$ and $\mathscr{V}$ can be greatly simplified for shapes such as spheres, cubes, rectangular blocks, ellipsoids and several others (see [34, Sections 6,7] for more details). We see that the symmetry properties of elements of $\mathscr{U}$ are similar. However, the term $\mathbf{U}_T^{(i,j)}$

does not remain identical to itself under exchange of variables i and j as in the case of the term $\mathbf{A}_T^{(i,j)}$. Similarly, the term $\mathbf{U}_R^{(i,j)}$ also does not remain identical to itself under exchange of variables i and j while $\mathbf{A}_R^{(i,j)}$ does. The symmetry properties of these terms can be read off from [34].

In order to analyze the orientation of bodies of specific shapes in their steady states, we need only consider the *spin free state* of the body. This is done by prescribing $\omega = 0$ in the Eq. (49), employing the specific symmetries of the body to simplify the equations and looking for the appropriate terminal angles. Our motivation for considering the purely translational motion comes from experimental observations where in the case of most long sedimenting bodies such as rectangular blocks, cylinders and ellipsoids, no rotational motion is observed. We make some additional assumptions as well. We consider the analysis for homogeneous bodies, i.e. those with $R=0$. This second assumption is made on the basis that inhomogeneities in the body can contribute to its final orientation as well. We are however, only interested in the effect that the second order fluid has upon the body.

Example 1. In the case of a homogeneous isotropic body (having *three symmetry planes and three axes of rotational symmetry*), all the torque coefficients vanish. We also have $C = 0$ [19], therefore, the Eq. (49) is trivially satisfied suggesting that any orientation is allowed. Examples of such bodies include spheres.

Example 2. For an orthotropic body (i.e. a body with *three planes of reflection symmetry*) we have $C = 0$. Then, employing the results of [34, Lemma 5], the Eq. (49) reduces to

$$\begin{aligned}&(\xi_2\xi_3(A_{R1}^{(2,3)} + (1+\varepsilon)U_{R1}^{(2,3)}) + \xi_2\xi_3(A_{R1}^{(2,3)} + (1+\varepsilon)U_{R1}^{(3,2)}))\hat{\mathbf{e}}_1\\ &\quad + (\xi_1\xi_3(A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(1,3)}) + \xi_1\xi_3(A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(3,1)}))\hat{\mathbf{e}}_2\\ &\quad + (\xi_2\xi_1(A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(2,1)}) + \xi_2\xi_1(A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(1,2)}))\hat{\mathbf{e}}_3 = 0. \qquad (52)\end{aligned}$$

Analysis of the possible scenarios yields the same results as in [34] and is in agreement with experiments. So when the body is homogeneous, Eq. (52) is satisfied if either (a) each of the torque coefficients are zero or the sums $A_{R1}^{(2,3)} + (1+\varepsilon)U_{R1}^{(2,3)} = A_{R1}^{(2,3)} + (1+\varepsilon)U_{R1}^{(3,2)} = 0$, $A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(1,3)} = A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(3,1)} = 0$ and $A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(2,1)} = A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(1,2)} = 0$, (b) $\xi_2 = \xi_3 = 0$, (c) $\xi_1 = \xi_3 = 0$, or (d) $\xi_1 = \xi_2 = 0$. Cases (b), (c) and (d) imply that motion must occur only along the x_1, x_2 or x_3 directions respectively, as indicated in Fig. 2.

Example 3. Bodies with fore-aft symmetry have *three symmetry planes and one axis of rotational symmetry*. For such bodies, $C \equiv 0$, therefore on using [34, Lemma 6], the Eq. (49) becomes

$$\begin{aligned}&(\xi_1\xi_3(A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(1,3)}) + \xi_1\xi_3(A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(3,1)}))\hat{\mathbf{e}}_2\\ &\quad + (\xi_2\xi_1(A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(2,1)}) + \xi_2\xi_1(A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(1,2)}))\hat{\mathbf{e}}_3 = 0. \qquad (53)\end{aligned}$$

Note that we have chosen the axis of rotational symmetry to lie along x_1, without any loss of generality. When $\mathscr{B}$ is homogeneous, Eq. (53) holds if (a) each of the torque coefficients

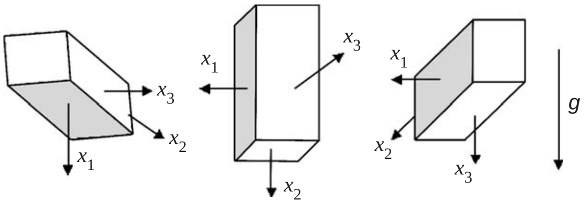

Fig. 2. Possible terminal orientations of an Orthotropic body.

ae zero or if $A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(1,3)} = A_{R1}^{(1,3)} + (1+\varepsilon)U_{R1}^{(3,1)} = 0$ and $A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(2,1)} = A_{R1}^{(2,1)} + (1+\varepsilon)U_{R1}^{(1,2)} = 0$, (b) $\xi_1 = 0$ or (c) $\xi_2 = \xi_3 = 0$. Condition (b) suggests that the body is moving along the x_2x_3 plane (i.e. in the horizontal position) and (c) says that the body moves along the x_1 direction (i.e. along its longest axis).

Among the multiple possibilities that emerge for orientation of various bodies considered, only one will be the stable one and the others will be unstable. The analysis of stability will not be dealt with in this paper in any detail. We refer the readers to [17] where a quasi-steady stability argument has been used to select the appropriate final and physically relevant orientation. This same argument can be adapted for our purposes as well. However, we recognize that a rigorous nonlinear stability for this problem remains, at this point, beyond reach. In conclusion, it must be stated that the main benefit of this argument remains its simplicity in explaining the phenomenon of freefall of rigid bodies of any shape in a second order fluid. The results of our analysis are in complete agreement with experimental observations of [3,4,24] and [22] and also consistent previous theoretical work [34,35]. It must be noted that the terminal angles predicted by using the second order fluid model include either the horizontal or the vertical state; no intermediate, tilt-angles are allowed. This remains the subject of an upcoming paper.

Acknowledgements

The author wishes to thank Dr. G.P. Galdi for his help and guidance. This work was supported by the NSF grant DMS-0103970.

References

[1] R.B. Bird, R.C. Armstrong, Dynamics of Polymeric Fluids, vol. 1, Wiley-Interscience Publications, New York, 1987.
[3] K. Chiba, K. Song, A. Horikawa, Motion of a slender body in quiescent polymer solutions, Rheol. Acta 25 (1986) 280–388.
[4] K. Cho, Y.I. Cho, N.A. Park, Hydrodynamics of a vertically falling thin cylinder in non-Newtonian fluids, J. Non-Newtonian Fluid Mech. 45 (1992) 105–145.
[6] R.G. Cox, The steady motion of particles of arbitrary shape at small Reynolds number, J. Fluid Mech. 23 (1965) 625–643.

[10] G.P. Galdi, Slow motion of a body in a viscous incompressible fluid with applications to particle sedimentation, Recent Developments in Partial Differential Equations, Quaderni Di Matematica, vol. 2, 1998, 1–35.
[11] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vol. 1, 2nd Corrected Ed., Springer, Berlin, 1998.
[12] G.P. Galdi, Slow steady fall of rigid bodies in a second order fluid, J. Non-Newtonian Mech. 93 (2000) 169–177.
[13] G.P. Galdi, On the motion of a rigid body in a viscous fluid: a mathematical analysis with applications, Handbook of Mathematical Fluid Mechanics, vol. 1, North-Holland, Amsterdam, 2002, pp. 653–791.
[14] G.P. Galdi, K.R. Rajagopal, Slow motion of a body in a fluid of second grade, Int. J. Eng. Sci. 35 (1997) 33–67.
[15] G.P. Galdi, A. Silvestre, Strong solutions to the problem of motion of a rigid body in a Navier–Stokes liquid under the action of forces and torques, Nonlinear Problems of Mathematical Physics and Related Topics, Kluwer/Plenum, New York, 2002, pp. 123–149.
[16] G.P. Galdi, A. Vaidya, Translational steady fall of symmetric bodies in Navier–Stokes liquid, with application to particle sedimentation, J. Math. Fluid Mech. 3 (2001) 183–211.
[17] G.P. Galdi, A. Vaidya, M. Pokorny, D.D. Joseph, J. Feng, Orientation of bodies sedimenting in a second order liquid at non-zero Reynolds number, Math. Models Meth. Appl. Sci. 12 (2001) 1653–1690.
[19] V. Happel, H. Brenner, Low Reynolds Number Hydrodynamics, Prentice-Hall, Englewood Cliffs, 1965.
[20] D.D. Joseph, Flow induced microstructures in Newtonian and viscoelastic fluids, Proceedings of the Fifth World Congress of Chemical Engineering, vol. 6, 1996, pp. 3–16.
[21] D.D. Joseph, J. Feng, A note on forces that move particles in a second order fluid, J. Non-Newtonian Fluid Mech. 64 (1996) 299–302.
[22] D.D. Joseph, Y.J. Liu, Orientation of long bodies falling in a viscoelastic fluid, J. Rheol. 37 (1993) 961–983.
[24] L.G. Leal, The slow motion of slender rod-like particles in a second order fluid, J. Fluid Mech. 69 (1975) 305–337.
[25] L.G. Leal, Particle motion in a viscous liquid, Ann. Rev. Fluid Mech. 12 (1980) 435–476.
[26] Y.J. Liu, D.D. Joseph, Sedimentation of particles in polymer solutions, J. Fluid Mech. 255 (1993) 565–595.
[27] A. Novotny, A. Sequeira, J.H. Videman, Existence of three dimensional flows of second-grade fluids past an obstacle, Nonlinear Anal. TMA 30 (5) (1997) 3051–3058.
[28] A. Novotny, A. Sequeira, J.H. Videman, Steady motions of viscoelastic fluids in 3-D exterior domains-existence, uniqueness and asymptotic behavior, Arch. Rat. Mech. Anal. 149 (1999) 49–67.
[30] M. Pokorny, Comportement Asymptotique des Solutions de Quelques Equations aux Derivees Partielles Decrivant L'Ecoulement de Fluides dans les Domaines Non-Bornes, Doctoral Thesis, University of Toulon and Var, Charles University, 1999.
[31] D. Serre, Chute Libre d'un Solide dans un Fluide Visqueux Incompressible, Jpn. J. Appl. Math. 4 (1987) 99–110.
[33] A. Vaidya, Orientation of rigid bodies freefalling in Newtonian and non-Newtonian liquids, Ph.D. Thesis, Department of Mechanical Engineering, University of Pittsburgh, 2004.
[34] A. Vaidya, Steady fall of bodies of arbitrary shape in a second order fluid at zero Reynolds number, Jpn. J. Ind. Appl. Math. 21 (3) (2004) 299–321.
[35] A. Vaidya,, A note on the terminal orientation of symmetric bodies in a power-law fluid, Appl. Math. Lett. (2005), to appear.
[38] H.F. Weinberger, Variational principles for a body falling in a steady Stokes flow, Proceedings of the Symposium on Continuum Mechanics and related Problems of Analysis, vol. 2, 1974, pp. 330–339.