

Non-linear stability for convection with quadratic temperature dependent viscosity

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SUMMARY

In this paper, we study the non-linear stability of convection for a Newtonian fluid heated from below, where the viscosity of the fluid depends upon temperature. We are able to show that for Rayleigh numbers below a certain critical value, Ra_c , the rest state of the fluid and the steady temperature distribution remains non-linearly stable, using the calculations of Diaz and Straughan (*Continuum Mech. Thermodyn.* 2004; **16**:347–352). The central contribution of this paper lies in a simpler proof of non-linear stability, than the ones in the current literature, by use of a suitable maximum principle argument. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper we discuss the stability of convection for a Newtonian fluid in a horizontal layer heated from below, with quadratic viscosity–temperature dependence. We employ the energy stability method [1–3] to establish the stability of the basic state to non-linear perturbations. The linearized stability has been studied in much detail but the non-linear stability problem remains to be explored completely. The non-linear stability for such convection problems in different settings has been treated extensively by Capone [4], Capone and Gentile [5], Carr and Straughan [6], Diaz and Straughan [7], Straughan [8], Straughan [9], Rionero [10] and Rionero and Mulone [11]. The readers are referred to Diaz and Straughan [7] for a thorough introduction to the past literature on this subject. Furthermore, the majority of the studies have been conducted for the linear temperature dependent viscosity case. As is pointed out in earlier work, there are several fluids, for which the quadratic, rather than the linear viscosity model, is more appropriate and this is one of the motivations for this paper.

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It could be argued that this paper derives its immediate motivation from the work of Diaz and Straughan [7] which deals with the stability analysis for convection of a non-Newtonian, power-law fluid for a quadratic viscosity–temperature model. Though this work deals with a more complex non-Newtonian fluid model, a similar result for the simpler, Newtonian case cannot be shown using the same methods. The difficulty in treating the relatively simpler case comes from finding suitable estimates for terms of the form $\langle \hat{f}(\theta) |D(w)|^2 \rangle$, where θ and w represent the thermal and velocity perturbations, respectively, to the basic state under investigation and \hat{f} represents an appropriate function of θ , which arise in the energy equation. However, in this article, we are able to obtain equivalent results to those mentioned above with a simple argument for the Newtonian case, though with slightly more restrictive conditions on the initial data. The central contribution of this paper lies in the technique of the proof. We are able to invoke a maximum-principle for the convection–diffusion equation to find an alternative and simpler argument for stability than is in use in the existing literature. The sharpest critical Rayleigh number below which the steady state remains stable is obtained by a suitable variational formulation for the problem. Furthermore, though we have chosen to work with a quadratic viscosity–temperature function here, our argument could be used to study several different temperature dependent viscosity models.

The paper employs standard mathematical notation. We use the notation ∂_t to indicate the time derivate. By Ω , we refer to the domain $\mathbb{R} \times \mathbb{R} \times (0, d) \times (0, T)$, which is an open connected set with boundary denoted by $\partial\Omega$, denoting the walls of the channel. The function space of interest is the space $\mathcal{H}_1 \times \mathcal{H}_2$ defined by

$$\mathcal{H}_1 \equiv \{\psi \in L^2(\Omega) : \operatorname{div} \psi = 0, \psi|_{\partial\Omega} = 0\}, \quad \mathcal{H}_2 \equiv \{\psi \in L^2(\Omega) : \psi|_{\partial\Omega} = 0\}$$

The outline for the paper is as follows. The next Section 2 begins with a formulation of the problem and obtaining the corresponding energy equation. In Section 3, we state the maximum-principle for the convection–diffusion problem which is then employed to show stability for the basic state. We provide the variational form of the perturbation equations to obtain the sharpest estimates possible on the critical Rayleigh number below which we can establish stability.

2. PROBLEM FORMULATION

The physical setting for our problem is as follows. We consider the convection of a viscous fluid in a horizontal container of infinite length and height d (i.e. $0 \leq z \leq d$) and choose the origin of our co-ordinate system at the bottom wall. In this section, we discuss the central governing equations for the problem and obtain the perturbation equations about the basic state solution, which need to be analysed. The central equations for this problem are given by the well known Navier–Stokes and convection equations. The Navier–Stokes equations with the boundary conditions are given by

$$\left. \begin{aligned} \rho(\partial_t u + u \cdot \operatorname{grad} u) &= \operatorname{div}(\tau) + g\alpha T \\ u(0) = u(d) &= 0 \end{aligned} \right\} \quad (1)$$

where ρ is the density and τ is the well known stress tensor for the Newtonian fluid under consideration, given by

$$\tau = -pI + 2\mu(T)D(u)$$

α is the coefficient of thermal expansion, g is the gravity vector which we will take to point in the negative- z -direction. In the Newtonian constitutive model, p represents the pressure, I the identity tensor and $D(u)$ the symmetric part of the velocity gradient, i.e. $\frac{1}{2}(\text{grad } u + \text{grad}^T u)$. Also note that in our problem, the viscosity is dependent upon the temperature, T . In addition, we have the incompressibility equation

$$\text{div } u = 0 \quad (2)$$

and the convective equation,

$$\left. \begin{aligned} \partial_t T + u \cdot \text{grad } T &= \kappa \Delta T \\ T(0) = T_l, \quad T(d) &= T_u \end{aligned} \right\} \quad (3)$$

where κ is the thermal diffusivity coefficient and $T_l > T_u$. We will, in particular consider the quadratic viscosity-temperature model given by

$$\mu(T) = \mu_0(1 - \gamma(T - T_l)^2) \quad (4)$$

where $\mu_0 = \mu(T_l)$. The steady solution corresponding to Equations (1) and (3) are given by

$$\begin{aligned} u^{(0)} &\equiv 0, \quad T^{(0)}(z) \equiv T_l - \beta z \\ \beta &= \frac{T_l - T_u}{d} \end{aligned}$$

The basic state pressure field, denoted $p^{(0)}$, can be obtained by solving the linear momentum equation with $u = u^{(0)}$ and $T = T^{(0)}$. Next, we will setup the perturbation equations for the problem. For this purpose, we consider the following perturbations to the basic state:

$$\begin{aligned} u(x, y, z, t) &= u^{(0)} + w(x, y, z, t) \\ p(x, y, z, t) &= p^{(0)} + \pi(x, y, z, t) \\ T(x, y, z, t) &= T^{(0)} + \theta(x, y, z, t) \end{aligned}$$

and non-dimensionalize our governing equations with suitable choice of the variables [7]

$$\begin{aligned} V &= \frac{\mu_m}{d}, \quad u^* = \frac{u}{V}, \quad p^* = \frac{p}{\rho V^2}, \quad t^* = \frac{td^2}{\mu_m}, \quad \mu^* = \frac{\mu}{\mu_m}, \quad T^* = \frac{T - T_l}{T_u - T_l} \\ Pr &= \frac{\mu_m}{\kappa}, \quad Ra = R^2 = \frac{g\alpha\beta d^4}{\kappa\mu_m}, \quad \Gamma = \gamma\beta^2 d^2 \end{aligned}$$

where Ra is the Rayleigh number, Pr is the Prandtl number and μ_m is the average viscosity given by $\frac{1}{d} \int_0^d \mu(T) dz$. Substituting these into Equations (1)–(3), we get, upon simplification the non-dimensional version of the perturbation equations, which in the same notation of

Diaz and Straughan [7] is given by

$$\partial_t w + w \cdot \text{grad } w = -\text{grad } \pi + R\theta \mathbf{e}_3 + \text{div}([f(z) + hz\theta - \zeta\theta^2]D(w)) \quad (5)$$

$$\text{div } w = 0 \quad (6)$$

$$Pr(\partial_t \theta + w \cdot \text{grad } \theta) = \Delta \theta + R w \cdot \mathbf{e}_3 \quad (7)$$

where $\mathbf{e}_3 = (0, 0, 1)$ and we have finally discarded the superscript, *, for sake of notational convenience. Also,

$$f(z) = \frac{6(1 - \Gamma z^2)}{(3 - \Gamma)}, \quad h = \frac{12\Gamma Pr}{R(3 - \Gamma)}, \quad \zeta = \frac{6\Gamma Pr^2}{R^2(3 - \Gamma)}$$

The non-dimensional boundary conditions representing no-slip and fixed temperature at the channel walls are

$$w(x, y, 0, t) = w(x, y, 1, t) = 0, \quad \theta(x, y, 0, t) = \theta(x, y, 1, t) = 0 \quad (8)$$

We now derive the energy equation corresponding to the perturbations w and θ which satisfy Equations (5)–(7) above. Multiplying (5) by w , Equation (7) by θ , integrating over Ω and applying Equations (8), we get

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 = R \int_{\Omega} \theta \mathbf{e}_3 \cdot w - \int_{\Omega} [f(z) + hz\theta - \zeta\theta^2] D(w) : D(w) \quad (9)$$

Similarly, the perturbation equation satisfies the equation

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 = R \int_{\Omega} \theta \mathbf{e}_3 \cdot w - \int_{\Omega} |\text{grad } \theta|^2$$

Hence, combining all the terms, we obtain the energy equation

$$\frac{d\mathcal{E}(t)}{dt} = R(1 + \lambda) \langle \theta, w \rangle - \langle f(z) |D(w)|^2 \rangle - \langle hz\theta |D(w)|^2 \rangle + \langle \zeta\theta^2 |D(w)|^2 \rangle - \lambda \|\text{grad } \theta\|^2 \quad (10)$$

where λ is a positive constant, $\mathcal{E}(t) = \frac{1}{2}(\|w\|^2 + \lambda Pr \|\theta\|^2)$ and the symbol $\langle \cdot, \cdot \rangle$ represents the integral over Ω . In the section below we shall derive a condition for universal stability of the steady solutions $(u^{(0)}, p^{(0)}, T^{(0)})$.

3. NON-LINEAR STABILITY

In this section we employ a non-linear stability argument to find conditions for stability of the basic state. First, we state a maximum principle lemma for Equation (7) [12,13]. This lemma proves to be very useful in certain key estimates that we need to make for the main theorem.

Lemma 1

Let $\Theta(\mathbf{x}, t)$ satisfy Equation (3). Then, if

$$\Theta(\mathbf{x}, 0) \leq \bar{\Theta}_0$$

for constant $\bar{\Theta}_0 > 0$, it follows that

$$\Theta(\mathbf{x}, t) \leq \bar{\Theta}_0 + \max |\Theta(\mathbf{x}, t)|_{\partial\Omega}$$

for all $\mathbf{x} \in \Omega$ and for all times, t .

Proof

For proof of this theorem see References [12,13]. □

Remark 1 (Consequences of Lemma 1)

Two immediate consequences follow from Lemma 1. We can easily see from the non-dimensionalization that since $0 \leq T^{(0)} \leq 1$, then defining $\theta_0 \equiv \theta(x, y, z, 0)$ such that $|\theta_0| < \max |\theta_0|$ (for fixed $\max |\theta_0|$), we have

$$T(x, y, z, 0) = T^{(0)} + \theta(x, y, z, 0) < 1 + \max |\theta_0|$$

Now applying Lemma 1, it is easily seen that

$$T^{(0)} + \theta(x, y, z, t) < 2 + \max |\theta_0| \Rightarrow |\theta(x, y, z, t)| < 2 + \max |\theta_0| \quad (11)$$

where, recall that θ_0 as defined above is the initial thermal disturbance. A second result that follows from Lemma 1 and the positivity of the viscosity function is that $\max |\theta_0| < \sqrt{1/\gamma} - 2$, implying that $0 < \gamma < \frac{1}{4}$, which in turn suggests that $0 < \Gamma < \frac{1}{4}$.

Theorem 1

Let $\mu(T)$ represent the temperature dependent viscosity function as defined in Equation (4). Then for velocity and temperature perturbations $(w, \theta) \in \mathcal{H}_1 \times \mathcal{H}_2$, satisfying Equations (5)–(7), there exists a positive constant, ε depending only upon Ra , Pr , and Γ , such that for $|\theta(x, y, z, 0)| < C$ (C constant) and $0 < \Gamma < \min\{\frac{1}{4}, R^2/(R - 2Pr)^2\}$,

$$\frac{d}{dt} \mathcal{E}(t) \leq -\varepsilon \mathcal{E}(t) \quad (12)$$

where $\mathcal{E}(t) = (\|w\|^2 + \lambda Pr \|\theta\|^2)$. Furthermore, there exists a $Ra_c > 0$, such that system (5)–(7) is non-linearly stable provided $Ra < Ra_c$.

Proof

Consider the energy equation for the velocity and thermal perturbations as derived in Equation (10), namely

$$\frac{d\mathcal{E}(t)}{dt} \leq R(1 + \lambda) \langle \theta, w \rangle - \langle f(z) |D(w)|^2 \rangle - \langle hz\theta |D(w)|^2 \rangle + \langle \zeta\theta^2 |D(w)|^2 \rangle - \lambda \|\text{grad } \theta\|^2 \quad (13)$$

If we now define

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} (\|w\|^2 + \lambda Pr \|\theta\|^2) \\ I(t) &= R(1 + \lambda) \langle \theta, w \rangle \\ D(t) &= \langle f(z) |D(w)|^2 \rangle + \lambda \|\text{grad } \theta\|^2 \end{aligned} \quad (14)$$

Then it follows from applying the Holder's inequality, the Poincare's inequality and the results of Remark 1, that

$$\begin{aligned}
 \frac{d\mathcal{E}(t)}{dt} &\leq -D(t) \left(1 - \frac{I(t)}{D(t)}\right) - \langle (hz\theta - \zeta\theta^2)|D(w)|^2 \rangle \\
 &\leq -D(t) \left(1 - \frac{1}{R_E}\right) + (h(\max |\theta|) + \zeta(\max |\theta|)^2)\|D(w)\|^2 \\
 &\leq -D(t) \left(1 - \frac{1}{R_E}\right) + (h\beta + \zeta\beta^2)\|D(w)\|^2 \\
 &\leq -\left(1 - \frac{1}{R_E}\right) (\bar{\gamma}\|D(w)\|^2 + \lambda\|\text{grad } \theta\|^2) \\
 &\leq -\left(1 - \frac{1}{R_E}\right) \min\left(8\bar{\gamma}\pi^2, \frac{2\pi^2}{Pr}\right) \mathcal{E}(t)
 \end{aligned} \tag{15}$$

where $f(z) > f_0 = 6(1 - \Gamma)/(3 - \Gamma)$ and $\beta = 2 + \max |\theta_0|$, $\bar{\gamma} = f_0 - h\beta - \zeta\beta^2$ and $1/R_E = \max(I(t)/D(t))$. In order that $\bar{\gamma} > 0$, we choose $\max |\theta_0|$ such that

$$\beta_0 \equiv \frac{h - \sqrt{h^2 + 4\zeta f_0}}{2\zeta} - 2 \leq \max |\theta_0| \leq \frac{h + \sqrt{h^2 + 4\zeta f_0}}{2\zeta} - 2 \equiv \beta_1 \tag{16}$$

$$\Rightarrow 0 < \Gamma < \frac{R^2}{(R - 2Pr)^2} \tag{17}$$

Therefore, as long as $R_E > 1$ and Equation (17) is satisfied, then, upon some manipulation, we have

$$\frac{d\mathcal{E}(t)}{dt} \leq -\varepsilon\mathcal{E}(t), \quad \varepsilon = \left(1 - \frac{1}{R_E}\right) \min\left(8\bar{\gamma}\pi^2, \frac{2\pi^2}{Pr}\right)$$

therefore implying non-linear stability. □

In order to obtain the sharpest estimate on the critical Rayleigh number, it is best to analyse the corresponding variational problem. As a result, we obtain the Euler-Lagrange equations with the constraint $R_E = 1$

$$\left. \begin{aligned}
 R(1 + \lambda)\theta\mathbf{e}_3 + 2 \operatorname{div}(f(z)D(w)) &= -\operatorname{grad } \pi \\
 \operatorname{div } w &= 0 \\
 R(1 + \lambda)w \cdot \mathbf{e}_3 + 2\lambda\Delta\theta &= 0
 \end{aligned} \right\} \tag{18}$$

The Euler–Lagrange equations corresponding to the linearized problem are

$$\left. \begin{aligned} R\theta\mathbf{e}_3 + \operatorname{div}(f(z)D(w)) &= -\operatorname{grad} \pi \\ \operatorname{div} w &= 0 \\ R w \cdot \mathbf{e}_3 + \Delta\theta &= 0 \end{aligned} \right\} \quad (19)$$

Upon comparison, we see that problems (18) and (19) are the same if we set $\lambda=1$ in Equations (18). Problem (19), we note is the very same problem solved in Reference [7]. Furthermore, these authors have obtained sharp critical estimates on the critical Rayleigh number which applies equally to our problem. Their calculation reveals that for $0 < \Gamma < 1$ and λ nearly 1, the critical Rayleigh number lies between $1040.56 \leq Ra_L \leq 1707.76$. In our work, the restriction upon the parameter, Γ is $0 < \Gamma < \min\{\frac{1}{4}, R^2/(R - 2Pr)^2\}$. Therefore, it is easily seen by comparison with the results of previous work that the critical Rayleigh number must lie in the range: $1621.16 \leq Ra_c \leq 1707.76$. Hence, in summary, for $Ra < Ra_c$ and $0 < \Gamma < \min\{\frac{1}{4}, R^2/(R - 2Pr)^2\}$, it follows that the states $(u^{(0)}, T^{(0)})$ are asymptotically stable.

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