

- (1) (10 points) Show that every group of order 275 is solvable (you may use any theorem that we proved in class).

Let  $G$  be such a group. Note  $275 = 5^2 \times 11$

By Sylow's theorem

$n_{11} =$  number of 11-Sylow subgroups in  $G \equiv 1 \pmod{11}$

and divides  $25$

$$\Rightarrow n_{11} = 1, 5, 25$$

only one of these is  $1 \pmod{11}$

$$\Rightarrow n_{11} = 1.$$

$\Rightarrow$  the 11-Sylow subgroup is  $\mathbb{Q}$  unique, hence normal.

$\bullet$   $\mathbb{Q}$  itself is cyclic

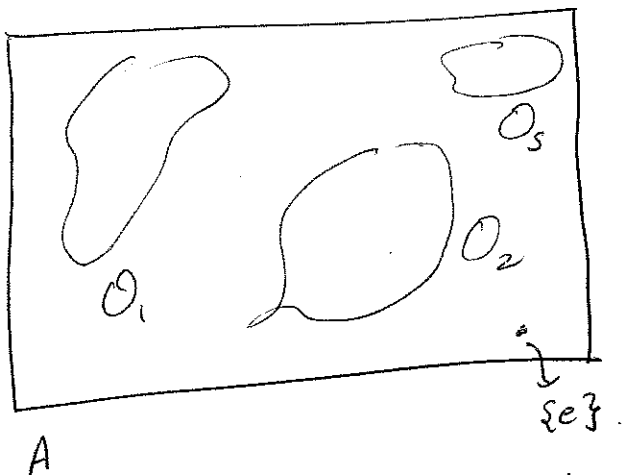
Now  $G/\mathbb{Q}$  is a group of order  $5^2$  hence abelian  
(groups of order  $p^2$  are abelian).

$\Rightarrow G \triangleright \mathbb{Q} \triangleright 1$  is a composition series for  $G$  with abelian factors.

$\Rightarrow G$  is solvable.

(2) (10 points) Let  $G$  be a  $p$ -group (i.e.  $|G| = p^k$ ,  $p$  a prime), and  $H \neq \{1\}$  be a normal subgroup of  $G$ . Show that  $H \cap Z(G) \neq \{1\}$  (Hint: Group action!).

Let  $A = H$ .  $G$  acts on  $A$  by conjugation  
 (note  $gag^{-1} \in gHg^{-1} \subseteq H = A$ ), let  $|A| = p^l$ .  
 ( ~~$|H|$~~   $|A|$ ).



any  $G$ -orbit in  $A$  has order dividing  $|G| = p^k$ . So it is either  $1, p, p^2, \dots$  or  $p^k$ . The orbit of  $\{1\}$  has one element

Suppose all <sup>other</sup> orbits had order  $\neq 1$   $\rightarrow$  here  $O_1, \dots, O_s, \{e\}$  are the orbits.

$$\Rightarrow 1 + \sum_{a=1}^s |O_a| = p^l (=|A|)$$

$$\Rightarrow \sum_{a=1}^s |O_a| = p^l - 1$$

But  $p$  divides the lhs and not the rhs  $\Rightarrow \Leftarrow$

so some  $|O_a| = 1 \Rightarrow O_a = \{a\} \quad a \neq e$

$$\Rightarrow gag^{-1} \in \{a\} \quad \forall g \in G$$

$$\Rightarrow \forall g \quad gag^{-1} = a \Rightarrow a \in Z(G)$$

- (3) (10 points) Using semidirect products, construct a non-abelian group of order 56 which has a normal 7-Sylow subgroup and a cyclic 2-Sylow subgroup.

$$\text{let } H = \mathbb{Z}/7\mathbb{Z}, \quad Q = \mathbb{Z}/8\mathbb{Z}.$$

First construct a non-trivial

$$Q \xrightarrow{\varphi} \text{Aut}(H). \quad [\text{no image has 1 or 2 elements}]$$

$|\text{Aut}(H)| = (\mathbb{Z}/7\mathbb{Z})^\times$  has 6 elements: Hence has an element of order 2.

Explicitly let  $\varphi(\bar{a}) = 6\bar{a}$  where  $\mathbb{Z}/7\mathbb{Z}$  is written additively

$$\varphi^2(\bar{a}) = 36\bar{a} = \bar{a}$$

if  $\mathbb{Z}/7\mathbb{Z}$  were written multiplicatively with generator  $x$ ,

$$\varphi(x) = x^6$$

with this  $\varphi$ ; let  $G = H \rtimes Q$   
In this group

$\downarrow \varphi$   
group generated by  $x$       generated by  $b$

$G = \langle (x, 1), (1, b) \rangle$  is generated by  $(x, 1)$  and  $(1, b)$ .

$$\begin{aligned} \text{ad } (1, b)(x, 1)(1, b^{-1}) &= (\varphi(x), 1) \\ &= (x^6, 1) \\ &= (x, 1)^6 \end{aligned}$$

$$\begin{aligned} \text{writing } \alpha &= (x, 1) \\ \beta &= (1, b) \end{aligned}$$

$$G \text{ is generated by } \alpha, \beta \quad \begin{aligned} \alpha^7 &= 1 \\ \beta^8 &= 1 \end{aligned}$$

$$\text{ad } \beta \alpha \beta^{-1} = \alpha^6$$

- (4) (a) (12 points) Show that the commutator subgroup of the symmetric group  $S_n$  equals the alternating group  $A_n$  (Hint: the commutator  $[(123), (124)]$  equals  $(12)(34)$  and  $[(12), (23)] = (32)(21)$ . So argue that even products of transpositions are always generated by commutators)
- (b) Modify the above argument to show that the commutator subgroup of  $A_n$  is  $A_n$  for  $n \geq 5$ .
- (c) Show that the alternating group  $A_n$ ,  $n \geq 5$ , does not have any non-trivial one dimensional complex representations. Does  $A_4$  or  $A_3$  have non-trivial one dimensional complex representations?
- (d) Determine all one dimensional complex representations of the symmetric group  $S_n$ .

(a) Any even product of transposition is therefore a product of commutators.  $\Rightarrow A_n \subseteq [S_n, S_n]$   
 but commutators are always even  $\left[ \begin{matrix} \text{in} \\ [a, b] \\ \text{write } a, b \text{ as} \\ \text{product of transpositions} \\ a b a^{-1} b^{-1} \text{ has } \\ 2(r+s) \text{ transpositions} \\ \text{where } r = \# \text{ in } a \\ s = \# \text{ in } b \end{matrix} \right]$

$\Rightarrow [S_n, S_n] \cong A_n$   
 $\Rightarrow [S_n, S_n] = A_n$

(b) Note that  $(123)$  is even, so  $(12)(34) \in$  commutator of  $A_n$ .  
 but  $(12)$  and  $(34)$  are odd. Hence  $[(12), (23)] = (32)(21)$  does not show  $(32)(21)$  as  $\in$  commutator of  $A_n$ .

But  $(32)(21) = [(12)(45), (23)(45)]$   
 $(45)$  commutes with  $(12), (23)$   
 and  $(12)(45), (23)(45)$  are even!

we could add  $(45)$  because  $n \geq 5 \Rightarrow A_n = [A_n, A_n]$   
 $n \geq 5$

(c) if  $A_n \xrightarrow{\varphi} GL(V) = F^x$ , then  $\varphi([A_n, A_n]) = 1$   
 is a hom  $\downarrow$   $\downarrow$   $\downarrow$   
 idm  $\downarrow$   $\downarrow$   $\downarrow$   
 abeli  $\Rightarrow \varphi(A_n) = 1 \Rightarrow \varphi$  trivial

(d) if  $S_n \xrightarrow{\varphi} F^x$  then  $\varphi(A_n) = 1$  (with (c))  
 $\Rightarrow \varphi$  is induced from a homomorphism  $S_n/A_n \xrightarrow{\varphi} F^x \Rightarrow \varphi = \text{sign}$   
 "  $A_n \xrightarrow{\varphi} F^x \Rightarrow \varphi = 1$   
 $\mathbb{Z}/2$  (because  $\varphi(x)^2 = 1$   
 and  $\varphi(A_n) = 1$ ).

(5) (10 points) Let  $G$  be a finite group.

- (a) Let  $V$  be a finite dimensional complex representation of  $G$ . Define what it means for  $V$  to be irreducible.  
 (b) Let  $G$  be the cyclic group of order two and let  $x \in G$  be the generator of the group  $G$ . Let  $\rho: G \rightarrow GL(2, \mathbb{C})$  be the representation determined by

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Is  $\rho$  irreducible? Explain your answer.

(a)  $V$  is irreducible if there does not exist any subspace  $W$  of  $V$  such that  $0 \subsetneq W \subsetneq V$  and  $G$  keeps  $W$  invariant. (i.e.  $gW \subseteq W \forall g \in G$ )  
 not true

(b) NO! Let us look for invariant subspaces. They have to look like  $\mathbb{C}v$  for some  $v \in \mathbb{C}^2$ .

and

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$  should be a multiple of  $v$ .

$\Rightarrow$  we should look for eigenvalues of  $\rho(x)$  & eigenvectors

Eigenvalues  $\det \begin{pmatrix} x-0 & -1 \\ -1 & x-0 \end{pmatrix} = 0 \Rightarrow x^2 - 1 = 0$   
 $\Rightarrow x = +1$   
 $x = -1$

Positive eigenvalue. The eigenvector for eigenvalue 1 is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$   
 " -1 is  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Both give  $G$ -invariant subspaces of  $\mathbb{C}^2$ .