

# EXTREMAL UNITARY LOCAL SYSTEMS ON $\mathbb{P}^1 - \{p_1, \dots, p_s\}$

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## INTRODUCTION

In [2] and [3] (also see [8]) the following problem was considered: Characterize  $s$  tuples  $\bar{A}_1, \dots, \bar{A}_s$  of conjugacy classes in  $SU(n)$  such that there are elements  $A_j \in SU(n)$  with conjugacy class of  $A_j$  in  $\bar{A}_j$  for  $j = 1, \dots, s$  and

$$A_1 A_2 \dots A_s = 1$$

This problem is in a natural way related to unitary representations of the fundamental group of  $\mathbb{P}^1 - \{p_1, \dots, p_s\}$  where  $\{p_1, \dots, p_s\}$  is a set of distinct points on  $\mathbb{P}^1$ . The characterization involves quantum cohomology of Grassmannians as we explain below.

Conjugacy classes in  $SU(n)$  are parameterized by points in the  $n - 1$  dimensional simplex:

$$\Delta(n) = \{(\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mid \delta_1 \geq \dots \geq \delta_n \geq \delta_1 - 1, \sum_{b=1}^n \delta_b = 0\}.$$

To an element  $(\delta_1, \dots, \delta_n) \in \Delta(n)$ , we associate the conjugacy class of the diagonal matrix with entries  $\exp(2\pi i \delta_b)$ ,  $b = 1, \dots, n$  on the diagonal.

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Now define  $\Gamma(n, s) \subseteq \Delta(n)^s$  be the set of  $(\Delta^1, \dots, \Delta^s) \in \Delta(n)^s$  such that there exist  $A^{(j)} \in \mathrm{SU}(n)$  in the conjugacy class  $\Delta^j$  for  $j = 1, \dots, s$  satisfying the equality  $A^{(1)}A^{(2)} \dots A^{(s)} = 1$ .

Recall the following result from [3] (also [2]).

**Theorem 0.1.** *Let  $(\Delta^1, \dots, \Delta^s) \in \Delta(n)^s$  with  $\Delta^j = (\delta_1^j, \dots, \delta_n^j)$  for  $j = 1, \dots, s$ . Then,  $(\Delta^1, \dots, \Delta^s) \in \Gamma(n, s)$  if and only if: For any integers  $r, d$  with  $0 < r < n$ ,  $d \geq 0$  and subsets  $I^1, \dots, I^s$  of  $\{1, \dots, n\}$  each of cardinality  $r$ , such that the Gromov-Witten number (see Sections 1.3, 1.5)*

$$(\dagger) \quad \langle \omega_{I^1}, \dots, \omega_{I^s} \rangle_d = 1,$$

the following inequality holds:

$$\sum_{j=1}^s \sum_{b \in I^j} \delta_b^j \leq d.$$

The natural question at this point is whether there are more such conditions and to find an irredundant (and sufficient) set of inequalities for this problem. To prove such a result one has to produce representations of (see Conventions 0.1)  $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_s\}, b) \rightarrow \mathrm{SU}(n)$  which are “in correspondence” with tuples  $(I^1, \dots, I^s, d)$  such that  $(\dagger)$  holds (with other conditions). In [14], Knutson, Tao and Woodward (also see [9] for a weaker result) prove an irredundancy result for the Lie algebra version of Theorem 0.1 (Klyachko [13] with improvements by Belkale [3]). We have however, not been able to understand the source of the extremal “representations” in [14] (or in [9]). The starting point for this work was the desire to construct these extremal representations directly (so as to apply to our situation).

The construction we give in this paper produces one such correspondence. Geometric Horn [6], which is in this context related to previous works of Witten [19] and Agnihotri [1], provides a way of producing a unitary representation of  $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_s\}, b)$  from a non vanishing Gromov-Witten number. This representation lands in the smaller unitary group  $\mathrm{SU}(r)$ . In a key inductive step we perform an operation called minimal extension and use it to obtain a representation  $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_s\}, b) \rightarrow \mathrm{SU}(n)$  with the desired extremal properties. As a consequence, the set of inequalities in Theorem 0.1 are necessary, sufficient and irredundant. We also get a new proof of the irredundancy result for the Lie algebra version of Theorem 0.1.

We produce the extremal representations by constructing their associated parabolic bundles. Perhaps, one can produce the associated local systems, in a “motivic” fashion (that is, as factors in a geometric variation of Hodge structure on  $\mathbb{P}^1 - \{p_1, \dots, p_s\}$ ). Using the results of N. Katz [11], it follows that this is a posteriori the case with our extremal local systems (which are direct sums of irreducible rigid unitary local systems).

The proof of a key theorem (Theorem 5.1) which relates the condition  $(\dagger)$  to the rigidity of certain unitary local systems on  $\mathbb{P}^1$ , will appear elsewhere. Instead, we sketch a proof of a weaker result which shows the importance of rigidity in questions on irredundance.

0.1. **Conventions.** All our varieties and schemes are defined over  $\mathbb{C}$ . We make the following notational conventions:

- (1) Fix a finite collection of points  $S = \{p_1, \dots, p_s\}$  on  $\mathbb{P}^1$ . Also fix a base point  $b \in \mathbb{P}^1 - S$ .
- (2) For a vector space  $V$ , denote by  $\text{Fl}(V)$  the space of complete flags

$$F_\bullet : 0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_{\text{rk}(V)} = V.$$

- (3) A vector bundle  $\mathcal{V}$  on  $\mathbb{P}^1$  is said to be a  $(d, r)$ -bundle if  $\deg(\mathcal{V}) = -d$  and  $\text{rk}(\mathcal{V}) = r$ . Note the negative sign in the expression for the degree (which is introduced for conformity with the notation used in quantum cohomology).

0.2. **Some ideas and methods in the proof.** Use notation from Conventions 0.1. Let  $\mathcal{W} = \mathcal{O}_{\mathbb{P}^1}^n$  and  $\mathcal{E} = \prod_{p \in S} E_p^\bullet \in \prod_{p \in S} \text{Fl}(\mathcal{W}_p)$  a generic point. It is a standard fact that maps from  $\mathbb{P}^1$  to the Grassmannian  $\text{Gr}(r, n)$  of degree  $d$  correspond to  $(d, r)$ -subbundles  $\mathcal{V}$  of  $\mathcal{W}$  (see Conventions 0.1). Therefore if given for each  $p \in S$  a subset  $I^p \subset \{1, \dots, n\}$  of cardinality  $r$ , the computation of the integer

$$(0.1) \quad \langle \omega_{I^{p_1}}, \dots, \omega_{I^{p_s}} \rangle_d$$

reduces to counting (if finite and 0 otherwise)  $(d, r)$ -subbundles  $\mathcal{V}$  of  $\mathcal{W}$  such that (see Section 1.3)  $\mathcal{V}_p \in \Omega_{I^p}^d(E_p^\bullet)$  for each  $p \in S$ .

Assume now that the Gromov-Witten invariant 0.1 is non zero, and  $\mathcal{V} \subseteq \mathcal{W}$  a subbundle as above. For any such  $\mathcal{V}$  we have an associated point  $\mathcal{E}(\mathcal{V})$  in  $\prod_{p \in S} \text{Fl}(\mathcal{V}_p)$  by intersecting the flag on  $\mathcal{W}_p$  with the subspace  $\mathcal{V}_p$  (for each  $p \in S$ ).

We obtain a parabolic bundle  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{E}(\mathcal{V}))$  on  $\mathbb{P}^1$  with parabolic structure at points of  $S$  where

$$w_a^p = \frac{n - r + a - i_a^p}{n - r}, \quad p \in S, \quad a = 1, \dots, r.$$

(See Section 2 for our definition of parabolic bundles). It follows from the proof of quantum generalization of Horn's conjecture [5] that  $\underline{\mathcal{V}}$  is semistable. We now extend this parabolic structure "in a minimal manner" to  $\mathcal{W}$  (with flags  $\mathcal{E}$ ) so that the induced structure on  $\mathcal{V}$  is the one above.

Our approach towards irredundancy originates in the naive hope that, under the assumption ( $\dagger$ ), this structure on  $\mathcal{W}$  will be give a situation where  $\mathcal{V}$  contradicts semistability and is the only subbundle contradicting semistability. This idea is essentially correct and provides the key induction step. Along the way, we examine and identify the induced parabolic structure on  $\mathcal{W}/\mathcal{V}$ . Also, there is some work to deal with extensions in the category of (semistable) parabolic bundles.

In the above discussion, for the purposes of induction, we want to consider  $(\mathcal{V}, \mathcal{E}(\mathcal{V}))$  instead of  $(\mathcal{W}, \mathcal{E})$ . But the degree of  $\mathcal{V}$  is not necessarily 0. This necessitates a generalization of Gromov-Witten invariants (Section 1.5). We use the properties (see Section 5) that the pair  $(\mathcal{V}, \mathcal{F})$  is "generic" and that  $\mathcal{V}$  is evenly split (see Section 1.1).

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## 1. SOME PRELIMINARIES

**1.1. Evenly split bundles on  $\mathbb{P}^1$ .** A  $(D, n)$ -vector bundle (see Conventions 0.1)  $\mathcal{W}$  on  $\mathbb{P}^1$  is said to be evenly split (ES) if  $\mathcal{W} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$  with  $|a_i - a_j| \leq 1$  for  $0 < i < j \leq n$ . It is easy to see that  $\mathcal{W}$  is ES if and only if  $H^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{W})) = 0$ .

Let  $D$  and  $n$  be integers with  $n > 0$ . It is easy to show there is a unique (upto isomorphism) ES -bundle of degree  $-D$  and rank  $n$  on  $\mathbb{P}^1$ . We denote this bundle by  $\mathcal{Z}_{D,n}$ .

Let  $\mathcal{W}$  be a bundle on  $\mathbb{P}^1$ . Define  $\mathfrak{Gr}(d, r, \mathcal{W})$  to be the moduli space of  $(d, r)$ -subbundles of  $\mathcal{W}$ . This can be obtained as an open subset of the Quot scheme of quotients of  $\mathcal{W}$  of degree  $d - D$  and rank  $n - r$ . In the notation of [17], we are looking at the open subset of  $\text{Hilb}^{n-r, d-D}(\mathcal{W})$  formed by points where the quotient is locally free.

For  $p \in \mathbb{P}^1$  define maps  $\pi_p : \mathfrak{Gr}(d, r, \mathcal{W}) \rightarrow \text{Gr}(r, \mathcal{W}_p)$  (fiber of the subbundle at  $p$ ). If  $D, d$  are integers and  $0 \leq r \leq n$ , define  $\mathfrak{Gr}(d, r, D, n) = \mathfrak{Gr}(d, r, \mathcal{Z}_{D,n})$ . We will also write the maps as  $\pi_p(d, r, \mathcal{W})$ .

**Proposition 1.1.** [6]  $\mathfrak{Gr}(d, r, D, n)$  is smooth and irreducible of dimension

$$d(n - r) - (D - d)r + r(n - r).$$

The subset of  $\mathfrak{Gr}(d, r, D, n)$  formed by ES-subbundles  $\mathcal{V} \subseteq \mathcal{Z}_{D,n}$  such that  $\mathcal{Z}_{D,n}/\mathcal{V}$  is also ES, is open and dense in  $\mathfrak{Gr}(d, r, D, n)$ .

**1.2. Partial Flag varieties.** Let  $\Lambda = (\Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_k = n)$  be given sequence of nonnegative integers and  $W$  a vector space of dimension  $n$ . Define the variety of partial flags (partial filtrations by vector subspaces)

$$X_\Lambda(W) = \{F_{\Lambda_1} \subset F_{\Lambda_2} \subset \dots \subset F_{\Lambda_k} = W \mid \text{rk}(F_{\Lambda_l}) = \Lambda_l \mid l = 1, \dots, k\}.$$

For notational convenience we set  $\Lambda_0 = 0$ . A simple dimension computation gives

**Lemma 1.2.**

$$\dim(X_\Lambda(W)) = \Lambda_1(n - \Lambda_1) + (\Lambda_2 - \Lambda_1)(n - \Lambda_2) + \dots + (\Lambda_k - \Lambda_{k-1})(n - \Lambda_k).$$

**1.3. Schubert cells in Grassmannians.** Let  $I \subseteq \{1, \dots, n\}$  be a subset of cardinality  $r$ . Let  $E_\bullet$  be a complete flag in an  $n$ -dimensional vector space  $W$ . Define the Schubert cell  $\Omega_I^o(E_\bullet) \subseteq \text{Gr}(r, W)$  by

$$\Omega_I^o(E_\bullet) = \{V \in \text{Gr}(r, W) \mid \text{rk}(V \cap E_u) = a \text{ for } i_a \leq u < i_{a+1}, a = 0, \dots, r\}$$

where  $i_0$  is defined to be 0 and  $i_{r+1} = n$ .  $\Omega_I^o(E_\bullet)$  is smooth and its closure is denoted by  $\Omega_I(E_\bullet)$ . For a fixed complete flag on  $W$ , every  $r$ -dimensional vector subspace belongs to a unique Schubert cell. The cohomology class of  $\Omega_I(E_\bullet)$  in  $H^*(\text{Gr}(r, n))$  is denoted by  $\omega_I$ .

1.4. **Complete Flags.** For a bundle  $\mathcal{W}$  on  $\mathbb{P}^1$ , define

$$\mathrm{Fl}_S(\mathcal{W}) = \prod_{p \in S} \mathrm{Fl}(\mathcal{W}_p).$$

If  $\mathcal{E} \in \mathrm{Fl}_S(\mathcal{W})$ , we will assume that it is written in the form  $\mathcal{E} = \prod_{p \in S} E_\bullet^p$ .

For a bundle  $\mathcal{W}$  on  $\mathbb{P}^1$ , a subbundle  $\mathcal{V} \subseteq \mathcal{W}$  and a collection of flags  $\mathcal{E} = \prod_p E_\bullet^p \in \mathrm{Fl}_S(\mathcal{W})$  we have associated induced complete flags on  $\mathcal{V}$  and on  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$  at points of  $S$ . We denote these by  $\mathcal{E}(\mathcal{V}) = \prod_p E_\bullet^p(\mathcal{V}) \in \mathrm{Fl}_S(\mathcal{V})$  and  $\mathcal{E}(\mathcal{Q}) = \prod_p E_\bullet^p(\mathcal{Q}) \in \mathrm{Fl}_S(\mathcal{Q})$ .

1.5. **Schubert states and Generalized Gromov-Witten numbers.** A Schubert state is a 5-tuple  $\mathcal{I} = (d, r, D, n, I)$  where  $d, D, r$  and  $n$  are integers,  $n \geq r \geq 0$  and  $I$  is an assignment to each  $p \in S$  a subset  $I^p$  of  $\{1, \dots, n\}$  of cardinality  $r$ . We will use the notation  $I^p = \{i_1^p < \dots < i_r^p\}$  for  $p \in S$ .

Let  $\mathcal{W} = \mathcal{Z}_{D,n}$ ,  $\mathcal{E} \in \mathrm{Fl}_S(\mathcal{W})$  a generic point and  $\mathcal{I} = (d, r, D, n, I)$  a Schubert state. Define  $\langle \mathcal{I} \rangle$  to be the number of points in the (if finite and 0 otherwise) intersection

$$(1.1) \quad \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E}) = \bigcap_{p \in S} \pi_p^{-1}[\Omega_{I^p}^o(E_\bullet^p)] \subseteq \mathfrak{Gr}(d, r, \mathcal{W}).$$

By a theorem of Kleiman (see [12]) the dimension of Intersection 1.1 is

$$(1.2) \quad \dim(\mathcal{I}) = \dim(\mathfrak{Gr}(d, r, D, n)) - \sum_{p \in S} \mathrm{codim}(\omega_{I^p}).$$

Notice that by definition, if  $\dim(\mathcal{I}) \neq 0$ , then  $\langle \mathcal{I} \rangle = 0$ . In the case  $D = 0$ , the numbers  $\langle \mathcal{I} \rangle$  correspond to the usual Gromov-Witten invariants  $\langle \omega_{I^{p_1}}, \dots, \omega_{I^{p_s}} \rangle_d$  which have an interpretation as enumerative numbers for maps  $\mathbb{P}^1 \rightarrow \mathrm{Gr}(r, n)$ , and are encoded as structure coefficients in the small quantum cohomology of  $\mathrm{Gr}(r, n)$  ([15], [10]). The generalized Gromov-Witten invariants can be recovered from the usual ones by making use of the shift operations (Section 3.1).

1.6. **Grassmann Duality.** Let  $W$  be a  $n$ -dimensional complex vector space. We denote the dual of a vector space  $V$  by  $V^*$ . We have a natural isomorphism

$$\mathrm{Gdual} : \mathrm{Gr}(r, W) \rightarrow \mathrm{Gr}(n - r, W^*)$$

obtained by taking  $V \subseteq W$  to  $(W/V)^* \subseteq W^*$ . A complete flag  $E_\bullet$  on  $W$  gives rise to a complete flag  $E_\bullet^*$  on  $W^*$ :  $E_{n-i}^* = \text{kernel of } W^* \rightarrow E_i^*$ . Suppose that  $I = \{i_1 < i_2 < \dots < i_r\} \subset \{1, \dots, n\}$ . Define

$$J = \mathrm{Gdual}(I, n) = \{a \in \{1, \dots, n\} \mid n + 1 - a \notin I\}.$$

Then,

**Lemma 1.3.**

$$\mathrm{Gdual}(\Omega_J^o(E_\bullet)) = \Omega_J^o(E_\bullet^*).$$

*Proof.* Let  $V \in \Omega_I^o(E_\bullet)$ , and  $T = \text{Gdual}(V)$ . We have,  $\dim(T \cap E_a^*) = \dim(\ker(W/V)^* \rightarrow E_{n-a}^*) = \dim(\text{coker}(E_{n-a} \rightarrow W/V)) = (n-r) - \dim(E_{n-a}/E_{n-a} \cap V)$  so  $\dim(T \cap E_a^*) \neq \dim(T \cap E_{a-1}^*)$  if and only if

$$n - a - \dim(E_{n-a} \cap V) \neq n - a + 1 - \dim(E_{n-a+1} \cap V)$$

that is  $\dim(E_{n-a+1} \cap V) - \dim(E_{n-a} \cap V) \neq 1$  or that  $n - a + 1 \notin I$ .  $\square$

It is easy to check that  $\mathcal{Z}_{D,n}^*$  is  $\mathcal{Z}_{-D,n}$ . Given a subbundle  $\mathcal{V}$  of  $\mathcal{W} = \mathcal{Z}_{D,n}$ , we get a subbundle  $(\mathcal{W}/\mathcal{V})^* \subset \mathcal{W}^*$  and  $\deg(\mathcal{W}/\mathcal{V})^* = -(\deg(\mathcal{W}) - \deg(\mathcal{V})) = D + \deg(\mathcal{V})$ . For  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$ . Denote by  $\mathcal{E}^*$  the induced point in  $\text{Fl}_S(\mathcal{W}^*)$ .

Consider a Schubert state of the form  $\mathcal{I} = (d, r, D, n, I)$ . Define  $\text{Gdual}(\mathcal{I})$  to be the Schubert state  $(d - D, n - r, -D, n, J)$  where  $J^p = \text{Gdual}(I^p, n)$  for each  $p \in S$ . If  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ , then the kernel  $\mathcal{T}$  of  $\mathcal{W}^* \rightarrow \mathcal{V}^*$  satisfies  $\mathcal{T} \in \Omega^o(\text{Gdual}(\mathcal{I}), \mathcal{W}^*, \mathcal{E}^*)$ . It is now easy to see that  $\langle \mathcal{I} \rangle = \langle \text{Gdual}(\mathcal{I}) \rangle$ .

## 2. PARABOLIC BUNDLES

A parabolic bundle  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$  on  $(\mathbb{P}^1, S)$  is a  $(D, n)$ -vector bundle  $\mathcal{W}$  on  $\mathbb{P}^1$ , a collection of complete flags  $\mathcal{E} = \prod_{p \in S} E_\bullet^p$  and a function (“weights”)

$$w : S \times \{1, \dots, n\} \rightarrow \mathbb{R}.$$

Such that, denoting  $w(p, a)$  by  $w_a^p$ , we for each  $p \in S$ ,

$$(2.1) \quad w_1^p \geq w_2^p \geq \dots \geq w_n^p \geq w_1^p - 1.$$

For a parabolic bundle  $\underline{\mathcal{W}}$  as above and a subbundle  $0 \subsetneq \mathcal{V} \subseteq \mathcal{W}$ , let  $\mathcal{I} = (d, r, D, n, I)$  be the Schubert state determined from  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ .

(1) Define the weight of  $\mathcal{V}$  by

$$\text{wt}(\mathcal{V}, \underline{\mathcal{W}}) = \sum_{p \in S} \sum_{a \in I^p} w_a^p$$

(2) The parabolic degree of  $\mathcal{V}$  is defined to be

$$\text{pardeg}(\mathcal{V}, \underline{\mathcal{W}}) = -d + \text{wt}(\mathcal{V}, \underline{\mathcal{W}}).$$

(3) The parabolic slope of  $\mathcal{V}$  is defined to be

$$\mu(\mathcal{V}, \underline{\mathcal{W}}) = \frac{\text{pardeg}(\mathcal{V}, \underline{\mathcal{W}})}{r}$$

The parabolic bundle  $\underline{\mathcal{W}}$  is said to be semistable (resp. stable) if  $\mu(\mathcal{V}, \underline{\mathcal{W}}) \leq \mu(\mathcal{W}, \underline{\mathcal{W}})$  (resp.  $\mu(\mathcal{V}, \underline{\mathcal{W}}) < \mu(\mathcal{W}, \underline{\mathcal{W}})$ ) for every subbundle  $0 \subsetneq \mathcal{V} \subsetneq \mathcal{W}$ . The following is standard:

**Lemma 2.1.** *Suppose  $\underline{\mathcal{W}}$  is a parabolic bundle as above and  $\mathcal{V} \subseteq \mathcal{W}$  a subbundle, and  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$  the quotient. These acquire parabolic structures. Call the resulting parabolic*

bundles  $\underline{\mathcal{V}}$  and  $\underline{\mathcal{Q}}$ . Let  $\mathcal{T} \subset \mathcal{W}$  be a subbundle. Denote by  $\tilde{\mathcal{T}}$  the saturation of the image of  $\mathcal{T}$  in  $\mathcal{Q}$ . We then have:

$$\mu(\mathcal{T}, \underline{\mathcal{W}}) \leq \frac{\mu(\mathcal{V} \cap \mathcal{T}, \underline{\mathcal{V}}) \operatorname{rk}(\mathcal{V} \cap \mathcal{T}) + \mu(\tilde{\mathcal{T}}, \underline{\mathcal{Q}}) \operatorname{rk}(\tilde{\mathcal{T}})}{\operatorname{rk}(\mathcal{T})}$$

**2.1. Morphisms between good parabolic bundles.** The system of weights  $w$  is said to be **good** if  $0 \leq w_1^p < 1$  for each  $p \in S$ . A parabolic bundle is said to be good if the weights on it are good. The parabolic bundles of [16] are the same as our good parabolic bundles.

A morphism between good parabolic bundles  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E}) \rightarrow \underline{\mathcal{T}} = (\mathcal{T}, \tilde{w}, \mathcal{F})$  is a morphism  $\phi : \mathcal{W} \rightarrow \mathcal{T}$  such that if  $b \geq 1$ ,  $a \geq 1$  are integers and  $p \in S$ ,  $\phi_p(E_a^p) \subset F_{b-1}^p$  whenever  $w_a^p < \tilde{w}_b^p$ .

**Lemma 2.2.** ([16]) *The category of good semistable vector bundles on  $(\mathbb{P}^1, S)$  of a given slope is abelian.*

*Remark 2.3.* Without the assumption of goodness of the weights, Lemma 2.2 fails to hold. One could also take for the definition of goodness the condition  $0 < w_1^p \leq 1$  for each  $p \in S$ . This point comes up while taking (twisted) duals of good weights.

Let  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$  be a parabolic bundle with good weights. The **twisted dual** of  $\underline{\mathcal{W}}$  is the parabolic bundle  $(\mathcal{W}^*, \tilde{w}, \tilde{\mathcal{E}})$  where  $\tilde{\mathcal{E}}$  is the point in  $\operatorname{Fl}_S(\mathcal{W}^*)$  induced from  $\mathcal{E} \in \operatorname{Fl}_S(\mathcal{W})$  and  $\tilde{w}_a^p = 1 - w_{\operatorname{rk}(\mathcal{W})+1-a}^p$  for  $p \in S$  and  $a = 1, \dots, \operatorname{rk}(\mathcal{W})$ . The system of weights  $\tilde{w}$  may not be good but see Remark 2.3. It is easy to see that (twisted) duality preserves stability and semistability.

*Remark 2.4.* The parabolic dual of  $\underline{\mathcal{W}}$  in a categorical sense (see [20]) is different from the twisted dual above. For example suppose our weights satisfied  $w_a^p \in (0, 1)$  for  $p \in S$  and  $a = 1, \dots, \operatorname{rk}(\mathcal{W})$ . Then the categorical dual in the situation above is

$$\mathcal{W}^* \otimes \mathcal{O}(-p_1) \otimes \dots \otimes \mathcal{O}(-p_s)$$

with the collection of flags induced from  $\mathcal{E}^*$  and weights  $\tilde{w}$ . If some  $w_a^p$  is 0 a suitable shift (Section 3) has to be made to the parabolic bundle above.

Consider a parabolic bundle  $(\mathcal{W}, w, \mathcal{E})$ . The member  $E_a^p$  of the flag at  $p$  such that  $a < \operatorname{rk}(\mathcal{W})$  and  $w_a^p = w_{a+1}^p$  does not play any role in the notion of a morphism of parabolic bundles  $\underline{\mathcal{W}} \rightarrow \underline{\mathcal{T}}$ . Indeed, in their paper [16], Mehta and Seshadri work with partial flag varieties at each parabolic point. We will sometimes define a parabolic bundle by giving a partial flag at each parabolic point and assign weights to each member of the partial flag (which weakly increase under specialization and pairwise differences of absolute value  $\leq 1$ ).

The existence of direct sums of good parabolic bundles is proved in [16].

**2.2. Moduli Spaces.** Let  $w : S \times \{1, \dots, n\} \rightarrow \mathbb{R}$  be a good system of weights,  $D \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{>0}$ . By results in [16], there is an irreducible and normal moduli space “parametrizing” semistable bundles  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$  where  $\mathcal{W}$  is a  $(D, n)$ -bundle on  $\mathbb{P}^1$ . Two semistable parabolic bundles  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$  and  $\underline{\mathcal{T}} = (\mathcal{T}, w, \mathcal{F})$  give the same point of this moduli space if they are “Jordan-Hölder equivalent”. Recall that the notion of a morphism of parabolic bundles ignores steps in the parabolic flags where the weights do not increase (from the previous step).

A semistable parabolic bundle  $\underline{\mathcal{W}}$  with good weights is said to be **rigid** if the corresponding moduli space is a point. If the parabolic degree of  $\underline{\mathcal{W}}$  is zero, this is equivalent to: Let  $\mathcal{L}$  be the unitary local system on  $\mathbb{P}^1 - S$  corresponding to  $\underline{\mathcal{W}}$  by the theorem of Mehta and Seshadri ([16]). Then, any unitary local system on  $\mathbb{P}^1 - S$  with the same local monodromies as  $\mathcal{L}$  is isomorphic to  $\mathcal{L}$ .

If  $\underline{\mathcal{W}}$  is stable, then rigidity of  $\underline{\mathcal{W}}$  implies that any other semistable parabolic bundle with the same weights, degree and rank as  $\underline{\mathcal{W}}$  is isomorphic to it (as parabolic bundles). Also, in this case, if the parabolic degree of  $\underline{\mathcal{W}}$  is zero, rigidity is the same as infinitesimal rigidity of the corresponding local system (with deformations preserving the local monodromies at points of  $S$ ).

### 3. SHIFT OPERATIONS

**3.1. Shift operations on Gromov-Witten invariants.** Suppose  $\mathcal{W}$  is a  $(D, n)$ -bundle of  $\mathbb{P}^1$  and  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  and  $p \in \mathbb{P}^1$ . Let  $t_p$  be a uniformising parameter at  $p$  (the construction does not depend upon the choice of  $t_p$ ). Define  $\tilde{\mathcal{W}} \supset \mathcal{W}$  as follows:  $\tilde{\mathcal{W}}$  coincides with  $\mathcal{W}$  on  $\mathbb{P}^1 - \{p\}$  and sections of  $\tilde{\mathcal{W}}$  over a small open subset containing  $p$  are meromorphic sections  $s$  of  $\mathcal{W}$  such that  $t_p s$  is a holomorphic section of  $\mathcal{W}$  with fiber at  $p$  in  $E_1^p$ . It is easy to see that  $\mathcal{E}$  induces a  $\tilde{\mathcal{E}} \in \text{Fl}_S(\tilde{\mathcal{W}})$  and that if  $\mathcal{W}$  is ES and  $\mathcal{E}$  generic then  $\tilde{\mathcal{W}}$  is ES as well.

**Lemma 3.1.** *There is 1 – 1 correspondence between subbundles of  $\mathcal{W}$  and of  $\tilde{\mathcal{W}}$  (by taking saturations). If  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$  and  $\tilde{\mathcal{V}} \subset \tilde{\mathcal{W}}$  the corresponding subbundle of  $\tilde{\mathcal{W}}$ , then  $\tilde{\mathcal{V}} \in \Omega^o(\mathfrak{S}_p(\mathcal{I}), \tilde{\mathcal{V}}, \tilde{\mathcal{E}})$  where  $\mathfrak{S}_p(\mathcal{I}) = (\tilde{d}, r, D - 1, n, J)$  is defined as follows:  $\tilde{d}$  and  $J$  are given by:  $J^q = I^q$  if  $q \in S - \{p\}$ , and*

- (1) If  $i_1^p > 1$ , let  $J^p = \{i_1^p - 1 < \dots < i_r^p - 1\}$  and  $\tilde{d} = d$ .
- (2) If  $i_1^p = 1$ , let  $J^p = \{i_2^p - 1 < \dots < i_r^p - 1 < n\}$  and  $\tilde{d} = d - 1$ .

Moreover  $\langle \mathcal{I} \rangle = \langle \mathfrak{S}_p(\mathcal{I}) \rangle$ .

**3.2. Shift operations on parabolic bundles.** Let  $\underline{\mathcal{W}} = (\mathcal{W}, \tilde{w}, \mathcal{E})$  be a parabolic bundle and  $p \in S$ . Let  $p \in S$  and construct a vector bundle  $\tilde{\mathcal{W}}$  and the point  $\tilde{\mathcal{E}} \in \text{Fl}_S(\tilde{\mathcal{W}})$  as in Section 3.1. Define a new system of weights  $\tilde{w}$  which is the same as  $w$  for points different from  $p$  and  $\tilde{w}_a^p = w_{a+1}^p$  for  $a = 1, \dots, \text{rank}(\mathcal{W}) - 1$  and  $\tilde{w}_{\text{rk}(\mathcal{W})}^p = w_1^p - 1$ . The parabolic bundle  $(\tilde{\mathcal{W}}, \tilde{w}, \tilde{\mathcal{E}})$  is denoted by  $\mathfrak{S}_p(\underline{\mathcal{W}})$

If  $\mathcal{V}$  is a subbundle of  $\mathcal{W}$  and  $\tilde{\mathcal{V}}$  the corresponding subbundle of  $\tilde{\mathcal{W}}$ , then it is easy to see that ([3], Appendix)

$$\mu(\mathcal{V}, \underline{\mathcal{W}}) = \mu(\tilde{\mathcal{V}}, \mathfrak{S}_p(\underline{\mathcal{W}})).$$

#### 4. IRREDUNDANCY, FIRST STEPS

Consider a Schubert state of the form  $\mathcal{I} = (d, r, D, n, I)$ . Let  $\mathcal{W} = \mathcal{Z}_{d,n}$  and  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  a generic point.  $\mathcal{I}$  is said to be **irredundant for eigenvalue problems** if there is a choice of weights  $w$  (not necessarily good, but satisfying inequalities 2.1) for which the only contradictions to semistability (among all proper subbundles) of  $(\mathcal{W}, w, \mathcal{E})$  are subbundles  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . It follows from [3] that  $\langle \mathcal{I} \rangle = 1$  if  $\mathcal{I}$  is irredundant.

In [3], we worked with a different notion of weights. In that paper the requirement was that the weights satisfied

$$(4.1) \quad w_1^p \geq w_2^p \geq \cdots \geq w_n^p \geq w_1^p - 1 \text{ and } \sum_{a=1}^n w_a^p = 0$$

for each  $p \in S$ . It is easy to see that if  $w$  as above shows the irredundancy of  $\mathcal{I}$ , then we can find a choice of weights of the form (4.1) for which the corresponding irredundancy statement holds: Replace  $w_a^p$  by  $w_a^p - \frac{1}{n} \sum_{b=1}^n w_b^p$ .

The main theorem of this paper is

**Theorem 4.1.** *If  $\langle \mathcal{I} \rangle = 1$  then  $\mathcal{I}$  is irredundant.*

The following is immediate from the existence of shift operations (Section 3):

**Lemma 4.2.** *For  $p \in S$ ,  $\mathcal{I}$  is irredundant iff  $\mathfrak{S}_p(\mathcal{I})$  is irredundant.*

Using Lemma 4.2, one can assume without loss of generality that  $i_r^p = n$  for each  $p \in S$ . We also make an observation of boundedness. Given a  $(D, n)$ -vector bundle  $\mathcal{W}$  on  $\mathbb{P}^1$ , there are only finitely many Schubert states of the form  $\mathcal{I} = (d, r, D, n, I)$  such that there exists  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  satisfying

- (1)  $\Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E}) \neq \emptyset$ .
- (2) There exists a choice of weights  $w$  such that any  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$  contradicts the semistability of  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$ .

The following is a “preparation lemma” for showing irredundancy:

**Lemma 4.3.** *The inequality corresponding to a Schubert state  $\mathcal{I}$  is irredundant if and only if there is a parabolic bundle  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$  where  $\mathcal{W} = \mathcal{Z}_{D,n}$  and  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  is a generic point, such that if  $\{\mathcal{V}\} = \Omega^o(\mathcal{I}, \mathcal{V}, \mathcal{E})$ ,*

- (1) *The induced parabolic structures on  $\mathcal{V}$  and  $\mathcal{W}/\mathcal{V}$  are semistable of the same slope ( $(\mathcal{W}, w, \mathcal{E})$  is therefore semistable, but not stable).*
- (2) *The weights for the subbundle  $\mathcal{V}$  are **separated away** from the weights of the quotient  $\mathcal{W}/\mathcal{V}$ . That is, for  $p \in S$ ,  $i \in I^p, j \in \{1, \dots, n\} - I^p$ ,*

$$0 < |w_i^p - w_j^p| < 1.$$

*Proof.* It is clear that if  $\mathcal{I}$  is irredundant, we can find a point on the face determined by it which is in the interior of the weight space: That is  $0 < |w_i^p - w_j^p| < 1$  if  $i \neq j$ .

Let us look at the “issues” with the other direction: We have weights  $w$  that make  $\mathcal{W}$  semistable,  $\mathcal{V}$  is a contradictor to stability and the weights of  $\mathcal{V}$  are separated from those of  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$ . We want weights for which the subbundle  $\mathcal{V}$  is the sole contradiction to semistability. We do this in as follows: Perturb (using induction) the given weights on  $\mathcal{V}$  (i.e  $w_a^p$  for  $a \in I^p$ ) and on  $\mathcal{Q}$  (i.e  $w_a^p$  for  $a \notin I^p$ ) so that both become stable (there is space to do this and induce these weights to  $\mathcal{W}$ ) whilst keeping the slopes fixed. Therefore all subbundles of  $\mathcal{W}$  other than  $\mathcal{V}$  (and other than  $\mathcal{Q}$  if it splits off as a parabolic subbundle) have slope less than  $\mu(\mathcal{V}) - \delta$  for some  $\delta > 0$ . Add a small constant  $\epsilon \ll \delta$  to the weights of  $\mathcal{V}$  (that is, increase the values  $w_a^p$  for  $a \in I^p$  by  $\epsilon$ ). It is easy to see now that  $\mathcal{V}$  is the only contradiction to semistability (see Lemma 2.1).  $\square$

## 5. GEOMETRY OF HORN’S CONJECTURE

Let  $\mathcal{I} = (d, r, D, n, I)$  be a non null Schubert state with  $\langle \mathcal{I} \rangle \neq 0$ . Let  $\mathcal{W} = \mathcal{Z}_{D,n}$ ,  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  a generic point and  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . Let  $\mathcal{F} = \mathcal{E}(\mathcal{V})$ . It is known that (see [6]), that

- (1)  $\mathcal{V}$  is ES. This follows from Kleiman’s transversality theorem applied to the intersection 1.1 (from Section 1.3) and Proposition 1.1 (which tells us that the subset of  $\mathfrak{Gr}(d, r, \mathcal{W})$  consisting of ES-subbundles is nonempty and dense).
- (2) If to  $F_a^p (= E_a^p(\mathcal{V}) \subseteq E_{i_a^p}^p)$  we assign the weight  $w_a^p = \frac{n-r+a-i_a^p}{n-r}$ , then the parabolic bundle  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{F})$  is semistable of slope  $1 + \frac{d-D}{n-r}$ . This parabolic bundle will be called the **W-bundle** corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ .
- (3) If  $\mathcal{S} \subset \mathcal{V}$ , let  $\mathcal{S} \in \Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{E}(\mathcal{V}))$  and  $\mathcal{S} \in \Omega^o(\mathcal{L}, \mathcal{W}, \mathcal{E})$  for Schubert state  $\mathcal{K}$  and  $\mathcal{L}$ . Then

$$\mu(\mathcal{S}, \underline{\mathcal{V}}) = \mu(\mathcal{V}, \underline{\mathcal{V}}) + \frac{\dim(\mathcal{K}) - \dim(\mathcal{L})}{\text{rk}(\mathcal{S})(n-r)}$$

(see Equation 1.2 for the definition of  $\dim(\mathcal{K})$  and  $\dim(\mathcal{L})$ ). Therefore  $\mathcal{S}$  contradicts stability of  $\underline{\mathcal{V}}$  if and only if

$$(5.1) \quad \dim(\mathcal{L}) = \dim(\mathcal{K}).$$

The semistability of  $\underline{\mathcal{W}}$  just says the “geometrically obvious” fact that  $\dim(\mathcal{K}) \leq \dim(\mathcal{L})$ .

- (4) The induced point  $\mathcal{E}(\mathcal{V}) \in \text{Fl}_S(\mathcal{V})$  is “generic”. More precisely, if we have a nonempty open subset  $U$  of  $\text{Fl}_S(\mathcal{Z}_{d,r})$  which is preserved by the automorphisms of  $\mathcal{Z}_{d,r}$ , then the induced point can be assumed to satisfy  $\mathcal{E}(\mathcal{V}) \in U$ .

It is unfortunate that this system of weights may not be good (which happens exactly when  $i_1^p = 1$  for some  $p \in S$ ).

**5.1. Intersection Rigidity.** Let  $\mathcal{I} = (d, r, D, n, I)$  be a non null Schubert state with  $\langle \mathcal{I} \rangle \neq 0$ . Let  $\mathcal{W} = \mathcal{Z}_{D,n}$ ,  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  a generic point and  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . Let  $\mathcal{F} = \mathcal{E}(\mathcal{V})$  and  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{F})$  the  $\mathcal{W}$ -bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ . Assume that the system of weights  $w$  is good (the results of this section will eventually be applied to the Grassmann dual of  $\mathcal{I}$ ). That is, for each  $p \in S$ ,  $i_1^p \neq 1$ . We can then form a moduli space  $\mathcal{M}(\mathcal{I})$  of parabolic bundles “of the type of  $\underline{\mathcal{V}}$ ” with weights  $w$ . It follows from the work of Witten and Agnihotri that  $\langle \mathcal{I} \rangle = H^0(\mathcal{M}(\mathcal{I}), \Theta(\mathcal{I}))$  for a natural **ample** line bundle  $\Theta(\mathcal{I})$  on  $\mathcal{M}(\mathcal{I})$ . The restriction  $i_1^p \neq 1$  is innocuous in view of Lemmas 3.1 and 4.2. The proof of the following theorem will appear elsewhere:

**Theorem 5.1.**  $\langle \mathcal{I} \rangle = 1$  if and only if  $\mathcal{M}(\mathcal{I}) = a$  point.

The classical part of this theorem (when  $d = D = 0$ ) is essentially equivalent to a conjecture of Fulton which was proved by Knutson, Tao and Woodward [14] by using the Honeycomb theory.

Instead of a proof (which uses ideas from the authors proof of the quantum generalization of Horn’s conjecture [5]), we sketch an argument here which shows that if  $\mathcal{M}(\mathcal{I})$  is positive dimensional then  $\mathcal{I}$  is not irredundant (i.e redundant!) for Eigenvalue problems.

First we define the scaling of a Schubert state: For a Schubert state  $\mathcal{I} = (d, r, D, n, I)$  and a positive integer  $N$ , define

$$N\mathcal{I} = (d, r, N(D - d) + d, N(n - r) + r, J)$$

where  $j_a^p = N(i_a^p - a) + a$  for  $p \in S$  and  $a = 1, \dots, r$ .

Suppose that  $\mathcal{W}$  is a  $(D, n)$ -bundle on  $\mathbb{P}^1$ ,  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  and  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . Consider the vector bundle

$$\mathcal{W}(N) = \mathcal{V} \oplus \bigoplus_{j=1}^N (\mathcal{W}/\mathcal{V}).$$

There is a natural point  $\mathcal{E}(N) \in \text{Fl}_S(\mathcal{W}(N))$  induced from  $\mathcal{E}$  and also a natural inclusion  $\mathcal{V} \subseteq \mathcal{W}(N)$ . The geometry of the scaling is clear from the following lemma which is a very easy computation.

**Lemma 5.2.** (1)  $\mathcal{V} \in \Omega^o(N\mathcal{I}, \mathcal{W}(N), \mathcal{E}(N))$ .

(2) The tangent space to  $\mathfrak{Gr}(d, r, \mathcal{W}(N))$  (see Section 1.1) at  $\mathcal{V}$  is direct sum of  $N$  copies of the tangent space to  $\mathfrak{Gr}(d, r, \mathcal{W})$  at  $\mathcal{V}$ . If  $\mathcal{W}$  is ES, then  $\mathfrak{Gr}(d, r, \mathcal{W}(N))$  is smooth at  $\mathcal{V}$  (with the local dimension the expected one).

(3) The tangent space at  $\mathcal{V}$  to the scheme  $\Omega^o(N\mathcal{I}, \mathcal{W}(N), \mathcal{E}(N))$  is a direct sum of  $N$  copies of the tangent space at  $\mathcal{V}$  to  $\Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ .

*Proof.* The first statement is proved by a simple computation. For the second, the tangent space to  $\mathfrak{Gr}(d, r, \mathcal{W}(N))$  is  $\text{Hom}(\mathcal{V}, \mathcal{W}(N)/\mathcal{V})$  which is  $N$  copies of  $\text{Hom}(\mathcal{V}, \mathcal{W}/\mathcal{V})$ . The smoothness statement follows from Grothendieck’s theory.

For the second, we use the description of tangent spaces to Schubert varieties ([18], Section 2.7).  $\square$

We also check that

**Lemma 5.3.** (1)  $\dim(N\mathcal{I}) = N \dim(\mathcal{I})$ . (see Equation 1.2).  
 (2) If  $\dim(\mathcal{I}) = 0$ , then  $\mathcal{M}(\mathcal{I}) = \mathcal{M}(N\mathcal{I})$  and  $\Theta(N\mathcal{I}) = \Theta(\mathcal{I})^N$ .

Now suppose that  $\mathcal{I}$  is irredundant. Therefore  $\langle \mathcal{I} \rangle = 1$  and there exists a parabolic bundle  $\underline{\mathcal{W}} = (\mathcal{W}, w, \mathcal{E})$  such that  $\mathcal{W}$  is an ES  $(D, n)$ -bundle,  $\mathcal{E}$  a generic point of  $\text{Fl}_S(\mathcal{W})$  and such that the only contradictor of semistability of  $\underline{\mathcal{W}}$  is a point  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . It is easy to see that  $\mathcal{W}(N)$  has a natural parabolic structure and  $\mathcal{V}$  is still the Harder-Narasimhan maximal contradictor of semistability.

Also assume now (by way of contradiction) that  $\mathcal{M}(\mathcal{I})$  is positive dimensional. Since  $\Theta(\mathcal{I})$  is ample, this will give us a  $N$  such that

$$\langle N\mathcal{I} \rangle = h^0(\mathcal{M}(N\mathcal{I}), \Theta(N\mathcal{I})) = h^0(\mathcal{M}(\mathcal{I}), \Theta(\mathcal{I})^N) > 1.$$

Now we attempt to evaluate the intersection number  $\langle N\mathcal{I} \rangle$  using the pair  $(\mathcal{W}(N), \mathcal{E})$ . The point  $\mathcal{V} \in \Omega^o(N\mathcal{I}, \mathcal{W}(N), \mathcal{E}(N))$  corresponds to a transverse point of intersection because of Lemmas 5.2, 5.3 and the assumed transversality at  $\mathcal{V}$  of the intersection  $\Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . Since  $\langle N\mathcal{I} \rangle > 1$ , there should be “one other point of intersection”:

Consider a family  $(\mathcal{T}_t, \mathcal{E}_t)$  of  $(d + N(D - d), N(n - r) + r)$ -vector bundles  $\mathcal{T}_t$  along with collections of flags  $\mathcal{E}_t \in \text{Fl}_S(\mathcal{T}_t)$  parameterized by points  $t$  in a disc such that at  $t = 0$ , we have  $(\mathcal{W}(N), \mathcal{E}(N))$  and for  $t \neq 0$ , the pair  $(\mathcal{T}_t, \mathcal{E}_t)$  is generic, that is,  $\mathcal{T}_t$  is ES and  $\mathcal{E}_t$  is generic for intersection theory. For each  $t$  we find two points  $\mathcal{V}_t$  and  $\tilde{\mathcal{V}}_t$  in  $\Omega(N\mathcal{I}, \mathcal{T}_t, \mathcal{E}_t)$  such that as  $t \rightarrow 0$ ,  $\mathcal{V}_t \rightarrow \mathcal{V}$ , but  $\tilde{\mathcal{V}}_t$  stays away from  $\mathcal{V}$ . Here we use the fact that  $\mathcal{V}$  is a transverse point of the intersection  $\Omega^o(N\mathcal{I}, \mathcal{W}(N), \mathcal{E}(N))$  (the quot scheme of subbundles of  $\mathcal{W}(N)$  is smooth at  $\mathcal{V}$ ).

Now we can take the limit of  $\tilde{\mathcal{V}}_t$  in an appropriate quot scheme and take the saturation of this coherent subsheaf of  $\mathcal{W}(N)$ . The saturation picks up at least as much parabolic slope as  $\mathcal{V}$  (we consider the induced parabolic structure on  $\mathcal{W}(N)$ ) by arguments well known in the theory of parabolic bundles. This is contradiction to the fact that  $\mathcal{V}$  is the maximal (Harder-Narasimhan) contradictor of semistability of the parabolic structure on  $\mathcal{W}(N)$ .

**5.2. Rigidity bundle.** Let  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{F})$  be a parabolic vector bundle with a good system (see Section 2) of weights  $w$ . Suppose that  $\mathcal{V}$  is a  $(d, r)$ -bundle on  $\mathbb{P}^1$ .

For each point  $p \in S$ , let  $\Lambda^p = (0 < \Lambda_1^p < \dots < \Lambda_{l(p)}^p = r)$  be defined as follows

$$\Lambda^p = \{a \in \{1, \dots, r\} \mid a = r \text{ or } w_{a+1}^p < w_a^p\}$$

$\Lambda^p$  corresponds to elements of the flag  $F_{\bullet}^p$  where the weight jumps. We calculate the dimensions of moduli spaces when they contain a stable point.

**Lemma 5.4.** *Assume that the moduli  $\mathcal{M}$  of  $\underline{\mathcal{V}}$  contains at least one stable point. Then the dimension of this moduli space is (see Section 1.2 for the definition of the partial flag varieties  $X_{\Lambda^p}(\mathcal{V}_p)$ )*

$$(1 - r^2) + \sum_{p \in S} \dim(X_{\Lambda^p}(\mathcal{V}_p)).$$

*Proof.* Assume that  $\mathcal{V}$  is ES, and that  $\underline{\mathcal{V}}$  is a stable point. The space

$$Y = \prod_{p \in S} X_{\Lambda^p}(\mathcal{V}_p)$$

is a local universal family of parabolic bundles. The fibers of  $Y \rightarrow \mathcal{M}$  over the point in  $\mathcal{M}$  corresponding to  $\underline{\mathcal{V}}$  are stable bundles in  $Y$  which are isomorphic to  $\underline{\mathcal{V}}$ . The fibers have dimension  $\dim(\text{Aut}(\mathcal{V})) - 1 = r^2 - 1$  (Lemma 9.1). We make use of the fact that  $\text{Aut}(\mathcal{V})$  has a natural action on  $Y$  and the stabilizer of the point corresponding to  $\underline{\mathcal{V}}$  is 1 dimensional i.e. scalars.  $\square$

We make the following definition (see Claim 6.1 and Section 0.2):

**Definition 5.5.** Define the rigidity bundle  $\text{Rig}(\underline{\mathcal{V}})$  of  $\underline{\mathcal{V}}$  to be  $(\mathcal{V}, \text{rig}, \mathcal{F})$  where to  $F_{\Lambda_k^p}$  we assign the weight  $\text{rig}_k^p = \frac{r - \Lambda_k^p}{r}$  (these weights obviously lie in  $[0, 1)$ ).

It is easy to see that  $\text{Rig}(\bigoplus_{j=1}^N \underline{\mathcal{V}}) = \bigoplus_{j=1}^N \text{Rig}(\underline{\mathcal{V}})$ . The following ‘‘centroid’’ proposition is proved in Section 9:

**Proposition 5.6.** Assume that the moduli of  $\underline{\mathcal{V}}$  (as in Section 2.2) is 0 dimensional and consists of a stable point (it is always connected). Then  $(\mathcal{V}, \text{rig}, \mathcal{F})$  is a rigid, stable parabolic bundle of slope  $\mu(\mathcal{V}, \text{Rig}(\underline{\mathcal{V}})) = 1 - \frac{d}{r} - \frac{1}{r^2}$ . If  $0 \subsetneq \mathcal{S} \subsetneq \mathcal{V}$  is a subbundle of  $\mathcal{V}$  then,

$$(5.2) \quad \mu(\mathcal{S}, \text{Rig}(\underline{\mathcal{V}})) \leq 1 - \frac{d}{r} - \frac{1}{\text{rk}(\mathcal{S})r}$$

## 6. EXTREMAL WEIGHTS

We start with basic remarks. Let  $\mathcal{W} = \mathcal{Z}_{D,n}$  and  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  a generic point. The space of weights  $w$  on  $(\mathcal{W}, \mathcal{E})$  is clearly convex (if  $w_1$  and  $w_2$  are weights, then so is  $tw_1 + (1-t)w_2$  for any real number  $t \in [0, 1]$ ). It is known that if  $s \geq 3$  there are weights  $w$  of any given slope such that  $(\mathcal{W}, w, \mathcal{E})$  is stable. This is because then there exist irreducible unitary representations  $\pi_1(\mathbb{P}^1 - S, \text{any base point}) \rightarrow \text{SU}(n)$  for any  $n$ .

**6.1. Minimal Extension.** Consider a Schubert state  $\mathcal{I} = (d, r, D, n, I)$  with  $\langle \mathcal{I} \rangle = 1$ . Assume that for each  $p \in S$ ,  $i_r^p = n$ . This implies that the  $W$ -bundle coming from the Grassmann dual of  $\mathcal{I}$  has good weights (see Section 1.6). Let  $\mathcal{W} = \mathcal{Z}_{D,n}$  and  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  a generic point. Let  $\Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E}) = \{\mathcal{V}\}$ .

In the situation above, consider the  $W$ -bundle  $(\mathcal{V}, w, \mathcal{E}(\mathcal{V}))$  corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ . We may try to induce the weights  $w_a^p$  to weights  $u$  on  $(\mathcal{W}, \mathcal{E})$  so that the inclusion  $(\mathcal{V}, w, \mathcal{E}(\mathcal{V})) \subset (\mathcal{W}, u, \mathcal{E})$  is an inclusion of parabolic bundles. So, we are looking for a system of weights  $u$  such that  $u_{i_a^p}^p = w_a^p$  for  $p \in S$  and  $a = 1, \dots, r$ . Define a ‘‘minimal’’ extension of weights  $u$  as follows. Let  $p \in S$ .

- (1)  $u_{i_a^p}^p = w_a^p$  for  $p \in S$  and  $a = 1, \dots, r$ .
- (2) If  $i_a^p < b < i_{a+1}^p$  where  $0 < a < r - 1$ , then  $u_b^p = w_{a+1}^p$ .

(3) If  $0 < b < i_1^p$ , then  $u_b^p = w_1^p$ .

$(\mathcal{W}, w, \mathcal{E})$  is said to be **minimal extension** of the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ . The crucial claim is that the induced parabolic structure on  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$  can also be obtained as follows:

- (1) Form the W-bundle on  $\mathcal{Q}^*$  corresponding to  $(\mathcal{Q}^*, \mathcal{W}^*, \text{Gdual}(\mathcal{I}), \mathcal{E}^*)$  ( $\mathcal{E}^* \in \text{Fl}_S(\mathcal{W}^*)$  is the point induced from  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$ ). Take the (twisted) dual of this structure and get one on  $\mathcal{Q}$ .
- (2) Form the rigidity bundle (see Definition 5.5) of this parabolic structure on  $\mathcal{Q}$ .

**Claim 6.1.** *The Rigidity structure on  $\mathcal{Q}$  (Section 5.2) in (2) above is the same as the weights induced on  $\mathcal{Q}$  by the minimal extension  $(\mathcal{W}, w, \mathcal{E})$  of the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ .*

*Proof.* This is a verification by brute force. Let us first write down the weights obtained from (1) on  $\mathcal{Q}$ . Clearly this is a pointwise statement and fix a  $p \in S$ :  $\{1, \dots, n\} \setminus I^p = \{\alpha_1^p < \dots < \alpha_{n-r}^p\}$ .

Let  $J^p = \text{Gdual}(n, I^p) = \{a \in \{1, \dots, n\} \mid n+1-a \notin I^p\}$ , and let  $J = \{j_1^p < \dots < j_{n-r}^p\}$ .

The W-bundle on  $\mathcal{Q}^*$  (corresponding to  $(\mathcal{Q}^*, \mathcal{W}^*, \text{Gdual}(\mathcal{I}), \mathcal{E}^*)$ ) assigns to  $E_a^{*,p}(\mathcal{Q}^*)$  the weight  $\frac{r+a-j_a^p}{r}$ . Therefore for the induced twisted dual structure on  $\mathcal{Q}$ , the weight assigned to  $E_b^p(\mathcal{Q})$  is  $1 - \frac{r+a-j_a^p}{r}$  where  $a = n - r + 1 - b$ . This simplifies to  $1 - \frac{r+n-r+1-b-(n+1-\alpha_b^p)}{r} = \frac{r+b-\alpha_b^p}{r}$ . Therefore the weight assigned to  $E_b^p(\mathcal{Q})$  is different from the weight assigned to  $E_{b+1}^p(\mathcal{Q})$ , then  $\alpha_{b+1}^p > \alpha_b^p + 1$ . That is,  $\alpha_b^p + 1 = i_k^p$  for some  $k$ . This shows  $k = \alpha_b^p - b + 1$  and  $b = i_k^p - k$ .

So suppose that

$$i_{k-1}^p = \alpha_{b-t}^p - 1 < \alpha_{b-t}^p < \alpha_{b-t+1}^p < \dots < \alpha_{b-1}^p < \alpha_b^p < i_k^p = \alpha_b^p + 1$$

(where  $i_0^p = 0$  and note that  $i_r^p = n$ ). Then the rigidity weight attached to  $E_{b-l}^p(\mathcal{Q})$  for  $l = 0, \dots, t$  is  $\frac{n-r-(i_k^p-k)}{n-r}$ . This is the same as the weight assigned to each  $E_{\alpha_{b-l}^p}^p$  for  $l = 0, \dots, t$  in the minimal extension  $(\mathcal{W}, w, \mathcal{E})$  of the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ .  $\square$

Although we chose the minimal extension so as to minimize the slope of the quotient, we will need to know the maximum slope the quotient can acquire so that the subbundle has the weights as in the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ .

Suppose that the W-bundle corresponding to  $(\mathcal{Q}^*, \mathcal{W}^*, \text{Gdual}(\mathcal{I}), \mathcal{E}^*)$  is the direct sum of  $N$  copies of the same parabolic bundle  $\underline{\mathcal{S}}$ . It follows then from Claim 6.1 that  $N$  divides each of the numbers  $i_{a+1}^p - i_a^p - 1$  (and  $i_1^p - 1$ ).

If  $p \in S$  and  $i_a^p < b < i_{a+1}^p$  where  $0 < a < r - 1$ , then the weights in the minimal extension is  $u_b^p = w_{a+1}^p$ . But we can assign for such  $p$  and  $b$  the weight  $w_a^p$ . The difference between the two is  $\frac{i_{a+1}^p - i_a^p - 1}{n-r}$  which is atleast  $\frac{1}{\text{rk}(\underline{\mathcal{S}})}$ .

Similarly if  $1 \leq b < i_1^p$ , then the weights in the minimal extension is  $u_b^p = w_1^p$ . But we can assign for such  $p$  and  $b$  the weight 1. The difference between the two is  $\frac{i_1-1}{n-r}$  which is once again atleast  $\frac{1}{\text{rk}(\mathcal{S})}$ .

If the number of points in  $S$  is  $s$ , then we can increase the slope of (the quotient structure on)  $\mathcal{Q}$  by atleast  $\frac{s}{\text{rk}(\mathcal{S})} - \delta$  for any  $\delta > 0$ . We will always add the same constant to all of the weights corresponding to  $\mathcal{Q}$ .

**6.2. The key inductive step.** Assume  $\langle \mathcal{I} \rangle = 1$  and therefore we have  $\langle \text{Gdual}(\mathcal{I}) \rangle = 1$ . According to Theorem 5.1, and the results in Section 2.2, the W-bundle corresponding to  $(\mathcal{Q}^*, \mathcal{W}^*, \text{Gdual}(\mathcal{I}), \mathcal{E}^*)$  is rigid. In this section we consider the case when it has exactly one simple factor. In this case it is a direct sum of copies of this factor (see Lemma 8.1). We urge the reader to first consider the case when the number of factors is 1.

Denote the twisted dual of this bundle by  $\underline{\mathcal{Q}}_1$  (the underlying vector bundle is  $\mathcal{Q}$ ). By Lemma 8.1,  $\underline{\mathcal{Q}}_1 = \bigoplus_{j=1}^N \underline{\mathcal{S}}$  for a rigid parabolic bundle. Hence  $N \text{rk}(\mathcal{S}) = n - r$ .

Now  $\text{Rig}(\underline{\mathcal{Q}}_1) = \bigoplus_{j=1}^N \text{Rig} \underline{\mathcal{S}}$ . By Proposition 5.6,  $\text{Rig}(\underline{\mathcal{S}})$  is **stable** of slope

$$(6.1) \quad 1 - \frac{D-d}{n-r} - \frac{1}{\text{rk}(\mathcal{S})^2}.$$

Now consider the minimal extension  $(\mathcal{W}, w, \mathcal{E})$  of the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ . By Section 6.1, this induces on  $\mathcal{Q}$  the parabolic bundle  $\text{Rig}(\underline{\mathcal{Q}}_1)$  which is semistable.

The subbundle  $\mathcal{V}$  has slope (by Section 5)  $1 - \frac{D-d}{n-r}$  which is greater than Expression 6.1.

We can clearly a suitable constant (and the same constant)  $\epsilon$  to each of the weights corresponding to  $\mathcal{Q}$  in the structure  $(\mathcal{W}, w, \mathcal{E})$  so that  $\mathcal{V}$  becomes a contradictor of stability and does not contradict semistability. The constant is  $\epsilon = \frac{1}{s \text{rk}(\mathcal{S})^2} < \frac{s}{\text{rk}(\mathcal{S})}$  (since  $s \geq 2$ , see Section 6.1). We therefore have a system of weights which satisfies the requirements of the preparation lemma (Lemma 4.3). The irredundancy is therefore proved in this case.

**6.3. Conclusion of the proof.** If  $\mathcal{V}, \mathcal{I}, \mathcal{W}$  and  $\mathcal{E}$  are as above with  $\langle \mathcal{I} \rangle = 1$ , we have the following proposition which will be proved in Section 8.

**Proposition 6.2.** *There is a Schubert state  $\mathcal{J} = (d_1, r_1, D, n, J)$  with  $r_1 \neq n$  such that  $\langle \mathcal{J} \rangle = 1$  and if  $\{\mathcal{T}\} = \Omega^o(\mathcal{J}, \mathcal{W}, \mathcal{E})$ , the following properties are satisfied:*

- (1)  $\mathcal{V} \subseteq \mathcal{T} \subsetneq \mathcal{W}$ .
- (2) *The W-bundle corresponding to  $((\mathcal{W}/\mathcal{T})^*, \mathcal{W}^*, \text{Gdual}(\mathcal{J}), \mathcal{E}^*)$  has exactly one simple factor.*

If  $\mathcal{J}$  and  $\mathcal{T}$  are as above and  $\mathcal{K}$  be defined from  $\mathcal{V} \in \Omega^o(\mathcal{K}, \mathcal{T}, \mathcal{E}(\mathcal{T}))$ . Clearly  $\langle \mathcal{K} \rangle = 1$  (because the induced point  $\mathcal{E}(\mathcal{T})$  is “generic”). In this section we consider the case  $\mathcal{T} \neq \mathcal{V}$ .

Consider the minimal extension  $(\mathcal{W}, w, \mathcal{E})$  of the W-bundle corresponding to  $(\mathcal{T}, \mathcal{J}, \mathcal{W}, \mathcal{E})$  and add a suitable constant (as in Section 6.2) to the weights corresponding to  $\mathcal{W}/\mathcal{T}$  in

the above structure so that slope of  $\mathcal{T}$  equals the slope of  $\mathcal{W}$ . Call the resulting parabolic bundle  $\underline{\mathcal{W}}$ . By Section 5,  $\mu(\mathcal{V}, \underline{\mathcal{W}}) = \mu(\mathcal{T}, \underline{\mathcal{W}}) = \mu(\mathcal{W}, \underline{\mathcal{W}})$ . We cannot apply the preparation lemma 4.3, because the weights of  $\mathcal{V}$  may not be separated from those of  $\mathcal{W}/\mathcal{V}$ . Indeed, the problem is only with the weights of  $\mathcal{T}/\mathcal{V}$ . This can be solved by perturbation (and induction) using  $\langle \mathcal{K} \rangle = 1$ . That is, perturb the weights for the restriction of the parabolic structure to  $\mathcal{T}$ , so that  $\mathcal{V}$  is the only contradictor to stability (and not to semistability) of the restriction of the parabolic structure to  $\mathcal{T}$ , the slopes remaining the same. We can now lift this structure up to  $\mathcal{W}$  keeping the weights on  $\mathcal{W}/\mathcal{T}$  the same as before.

## 7. EXAMPLES

All the examples below are from the classical part. One could use the shift operations to get “quantum” examples - but we expect that there are quantum examples that do not arise from the classical ones by application of the shift operation.

In each of these examples we are given an intersection in a  $\text{Gr}(r, n)$  corresponding to a Schubert state  $\mathcal{I} = (d, r, D, n, I)$  with  $s = 3$ ,  $d = D = 0$  and  $\langle \mathcal{I} \rangle = 1$ . We write  $I^j = I^{p_j}$  for  $j = 1, \dots, 3$ .

Having chosen generic flags  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  where  $\mathcal{W} = \mathcal{O}^n$ , we will find a subspace  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$  of rank  $r$ . Let  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$  and  $\mathcal{Q}^* \subset \mathcal{W}^*$ . By Grassmann duality  $\mathcal{Q}^* \in \Omega^o(\text{Gdual}(\mathcal{I}), \mathcal{W}^*, \mathcal{E}^*)$  where  $\mathcal{E}^*$  is the induced point in  $\text{Fl}_S(\mathcal{W}^*)$ . Let  $\text{Gdual}(\mathcal{I}) = (0, n - r, 0, n, J)$  and write  $J^j = J^{p_j}$  for  $j = 1, \dots, 3$ .

**First Example:** For  $n = 8, r = 5, d = D = 0, I^1 = \{3, 4, 5, 7, 8\}, I^2 = I^3 = \{2, 3, 5, 6, 8\}$ . The Grassmann dual is happening in  $\text{Gr}(3, 8)$ , with  $J^1 = \{3, 7, 8\}$  and  $J^2 = J^3 = \{2, 5, 8\}$ .

The W-bundle corresponding to  $(\mathcal{Q}^*, \text{Gdual}(\mathcal{I}), \mathcal{W}^*, \mathcal{E}^*)$  has weights  $(\frac{3}{5}, 0, 0), (\frac{4}{5}, \frac{2}{5}, 0)$ , and  $(\frac{4}{5}, \frac{2}{5}, 0)$ . This induces weights on  $\mathcal{Q}$  (we take the negatives and add 1 and rearrange in descending order):  $(1, 1, \frac{2}{5}), (1, \frac{3}{5}, \frac{1}{5})$  and  $(1, \frac{3}{5}, \frac{1}{5})$ . The associated rigidity weights of this structure on  $\mathcal{Q}$  is  $(\frac{1}{3}, \frac{1}{3}, 0), (\frac{2}{3}, \frac{1}{3}, 0)$  and  $(\frac{2}{3}, \frac{1}{3}, 0)$ .

The W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$  has weights  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ , and  $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$  at  $p_2$  and  $p_3$ . We now consider the minimal extension  $(\mathcal{W}, w, \mathcal{E})$  of the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$ . The weights are  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$ , and  $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$  at  $p_2$  and  $p_3$ . And the quotient  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$  has (induced) weights  $(\frac{1}{3}, \frac{1}{3}, 0), (\frac{2}{3}, \frac{1}{3}, 0), (\frac{2}{3}, \frac{1}{3}, 0)$  of slope  $1 - \frac{1}{9}$ . This structure on  $\mathcal{Q}$  is the same as the rigidity structure of the previous paragraph.

From the theory developed in the paper,  $\mathcal{V}$  is the Harder-Narasimhan maximal contradictor of stability for  $(\mathcal{W}, w, \mathcal{E})$ . But there are other contradictors of stability because the W-bundle corresponding to  $(\mathcal{V}, \mathcal{I}, \mathcal{W}, \mathcal{E})$  is not stable. Example: the line  $\mathcal{S} \subset \mathcal{V}$  corresponding to the Schubert state  $(0, 1, 0, 5, L)$  in with  $l_1^1 = 5, l_1^2 = 4$  and  $l_1^3 = 2$ . This picks up weights:  $0, \frac{1}{3}, \frac{2}{3}$ , and hence has the same slope as that of  $\mathcal{V}$  ( $= 1$ ).

So how does the paper make  $\mathcal{V}$  the only contradictor of semistability? First we add weights  $\epsilon$  to all the weights of  $(\mathcal{W}, w, \mathcal{E})$  which correspond to the quotient  $\mathcal{Q}$  so that

the slope of  $\mathcal{Q}$  becomes equal to that of  $\mathcal{V}$ . In this case, the constant addition  $\epsilon$  is given by  $9\epsilon = \frac{1}{3}$ . So now the weights on  $\mathcal{W}$  become  $(\frac{1}{3} + \epsilon, \frac{1}{3} + \epsilon, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \epsilon, 0, 0)$ , and  $(\frac{2}{3} + \epsilon, \frac{2}{3}, \frac{2}{3}, \frac{1}{3} + \epsilon, \frac{1}{3}, \frac{1}{3}, \epsilon, 0)$  at  $p_2$  and  $p_3$ . One checks in general that there is “space” for this addition (so that the order of the weights remains unchanged). The weights of  $\mathcal{V}$  and of  $\mathcal{W}/\mathcal{V}$  are now separated (the absolute value of the difference of any weight from  $\mathcal{V}$  and one from  $\mathcal{T}/\mathcal{V}$  is greater than 0 and less than 1). This new structure on  $\mathcal{T}$  is semistable and we are done by the preparation lemma 4.3.

**Second Example:** We give an example where W-bundle corresponding to the tuple  $(\mathcal{Q}^*, \text{Gdual}(\mathcal{I}), \mathcal{W}^*, \mathcal{E}^*)$  is semistable but not stable. Here  $r = 2$  and  $n = 5$  and  $I^1 = I^2 = I^3 = \{2, 5\}$ . The Schubert state  $\text{Gdual}(\mathcal{I})$  is in  $Gr(3, 5)$  and corresponds to  $(J^1 = J^2 = J^3 = \{2, 3, 5\})$ . The W-bundle corresponding to  $(\mathcal{Q}^*, \text{Gdual}(\mathcal{I}), \mathcal{W}^*, \mathcal{E}^*)$  is unstable - there is a line which contradicts semistability : The line

$$E^*(\mathcal{Q}^*)_2^{p_2} \cap E^*(\mathcal{Q}^*)_2^{p_3}.$$

This gives a subspace of dimension 2 of  $\mathcal{Q}$  in Schubert state  $\{2, 3\}, \{1, 3\}, \{1, 3\}$ . This in turn gives a 4 dimensional subspace  $\mathcal{T}$  of  $\mathcal{W}$  in Schubert state

$$\mathcal{J} = (0, 4, 0, 5, \{\{2, 3, 4, 5\}, \{1, 2, 4, 5\}, \{1, 2, 4, 5\}\}).$$

This hyperplane contains  $\mathcal{V}$  and is unique in its Schubert state. Now consider the W-bundle corresponding to  $(\mathcal{T}, \mathcal{J}, \mathcal{W}, \mathcal{E})$ : the weights are  $(0, 0, 0, 0), (1, 1, 0, 0), (1, 1, 0, 0)$ . The minimal extension of these weights to  $\mathcal{W}$  is  $(0, 0, 0, 0, 0), (1, 1, 0, 0, 0), (1, 1, 0, 0, 0)$ . One checks that  $\mathcal{T}$  is the Harder-Narasimhan bundle of  $\mathcal{W}$  with the above weights, and  $\mathcal{V}$  has the same slope of  $\mathcal{W}$  ( $= 1$ ). There are two issues here: The first to make  $\mathcal{V}$  the Harder-Narasimhan bundle, and also to produce weights for which  $\mathcal{V}$  is the sole contradiction to semistability.

We can first add a constant to the weights of  $\mathcal{W}/\mathcal{T}$  so that  $\mathcal{W}$  becomes semistable: the constant is  $3\epsilon = 1$ , and the weights on  $\mathcal{W}$  are now

$$(\epsilon, 0, 0, 0, 0), (1, 1, \epsilon, 0, 0), (1, 1, \epsilon, 0, 0).$$

With these weights  $\mathcal{W}$  is semistable and  $\mathcal{T}, \mathcal{V}$  (among others) contradict stability. The weights on  $\mathcal{T}$  and on  $\mathcal{T}/\mathcal{W}$  are separated.

It is easy to see that  $\mathcal{V} \in \Omega^o(\mathcal{K}, \mathcal{T}, \mathcal{E}(\mathcal{T}))$  where  $\mathcal{K} = (0, 2, 0, 4, K)$  where  $K^{p_1} = \{1, 4\}$ ,  $K^{p_2} = \{2, 4\}$  and  $K^{p_3} = \{2, 4\}$ . By induction there are (semistable) weights  $w'$  on  $(\mathcal{T}, \mathcal{E}(\mathcal{T}))$  for which  $\mathcal{V}$  is a contradiction to stability, the weights of  $\mathcal{V}$  and  $\mathcal{T}/\mathcal{V}$  are separated and such that the slope of  $\mathcal{T}$  is still 1. We can perturb the weights on  $\mathcal{T}$  from the previous paragraph in the direction of  $w'$  and induce these weights upto  $\mathcal{W}$  to obtain conditions under which the preparation lemma 4.3 can be applied on  $(\mathcal{V}, \mathcal{I}, \mathcal{W})$ .

## 8. PROOF OF PROPOSITION 6.2

We begin with the following lemma on direct sum of rigid stable bundles.

**Lemma 8.1.** *Let  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{F})$  be a stable parabolic bundle with good weights which is rigid. Then, any semistable parabolic bundle with the same degree, rank and weights as  $\bigoplus_{j=1}^N \underline{\mathcal{V}}$  is isomorphic to it.*

*Proof.* Let  $\mathcal{W} = \bigoplus \mathcal{V}$  which is ES. Let  $X$  be the product of partial flag varieties corresponding to  $\bigoplus_{j=1}^N \underline{\mathcal{V}}$ . This is  $N^2$  times the dimension of the corresponding variety  $Y$  for  $\underline{\mathcal{V}}$ . But  $\dim(Y) = \dim(\text{Aut}(\mathcal{V})) - 1$  because of the rigidity and stability of  $\mathcal{V}$ .

$X$  has a distinguished point corresponding to  $\bigoplus \underline{\mathcal{V}}$ . The stabilizer of this point in  $\text{Aut}(\mathcal{W})$  is  $N^2$  dimensional. Therefore,  $\text{Aut}(\mathcal{W})$  has an orbit in  $X$  of dimension

$$N^2 \dim(\text{Aut}(\mathcal{V})) - N^2 = N^2 \dim(Y) = \dim(X).$$

Hence  $\text{Aut}(\mathcal{W})$  has a dense orbit on  $X$ .

In a universal family of semistable vector bundles of the same type as  $\bigoplus_{j=1}^N \underline{\mathcal{V}}$ , the subbundles which are isomorphic to  $\bigoplus_{j=1}^N \underline{\mathcal{V}}$  form a closed subset (because we know that the simple factors are all  $\underline{\mathcal{V}}$ 's). The lemma is therefore proved.  $\square$

**Theorem 8.2.** *Let  $\mathcal{V}$  be an ES-bundle,  $\mathcal{F} \in \text{Fl}_S(\mathcal{V})$  a generic point and  $w$  a good system of weights. Assume that  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{F})$  is a semistable parabolic bundle which is rigid (corresponding Mehta-Seshadri moduli space is a point) with at least two non isomorphic simple JH-factors. There then exists a subbundle  $\mathcal{S} \subsetneq \mathcal{V}$  of  $\mathcal{V}$  which satisfies*

- (1)  $\mu(\mathcal{S}, \underline{\mathcal{V}}) = \mu(\mathcal{V}, \underline{\mathcal{V}})$ .
- (2) Determine  $\mathcal{K}$  by the requirement  $\mathcal{S} \in \Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{F})$ , then  $\Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{F})$  consists of a single reduced point. This implies that  $\langle \mathcal{K} \rangle = 1$ .

*Proof.* Let  $\mathcal{A}$  be the abelian category of slope semistable parabolic bundles of a given parabolic slope.

In this situation by classical Jordan-Hölder theory, there exists a filtration

$$\underline{\mathcal{V}} = \underline{\mathcal{L}}_0 \supset \underline{\mathcal{L}}_1 \supset \cdots \supset \underline{\mathcal{L}}_k \supset \underline{\mathcal{L}}_{k+1} = 0$$

with the graded quotients

$$\frac{\underline{\mathcal{L}}_l}{\underline{\mathcal{L}}_{l+1}}$$

simple (i.e stable) and rigid (because of our assumption) for  $l = 0, \dots, k$ . Such a filtration is not unique but the unordered list of graded quotients is unique. Let  $\underline{\mathcal{L}} = \underline{\mathcal{L}}_k$ .

Consider a maximal element  $\underline{\mathcal{S}}$  of the set

$$\{\underline{\mathcal{S}} \subseteq \underline{\mathcal{V}} \mid \text{each graded quotient in the JH filtration of } \underline{\mathcal{S}} \text{ is isomorphic to } \underline{\mathcal{L}}\}.$$

Claim: This choice of  $\mathcal{S}$  “works”.

Proof of the claim: Let  $\mathcal{K}$  be as in (2). If  $\Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{F})$  is positive dimensional at  $\mathcal{S}$  then we will find a map (tangent spaces)  $\mathcal{S} \rightarrow \mathcal{V}/\mathcal{S}$  which preserves parabolic structures. The image  $\mathcal{T}$  of this map has all factors isomorphic to  $\underline{\mathcal{L}}$ . The inverse image of  $\mathcal{T}$  in  $\mathcal{V}$  is larger than  $\mathcal{S}$  and has all simple factors isomorphic to  $\underline{\mathcal{L}}$ . This leads to a contradiction.

The genericity of  $\mathcal{F}$  now implies that  $\langle \mathcal{K} \rangle > 0$  and the local multiplicity of  $\Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{F})$  at  $\mathcal{S}$  is 1. If  $\mathcal{S}'$  is any other point of  $\Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{F})$ , the induced flags on  $\mathcal{S}'$  is generic and

the induced parabolic structure on  $\mathcal{S}'$  is the same as that on  $\mathcal{S}$  (Lemma 8.1). The natural map  $\psi : \mathcal{S}' \rightarrow \mathcal{V}/\mathcal{S}$  is not 0 and preserves parabolic structures. We could take the inverse image of  $\text{im}(\psi)$  in  $\mathcal{V}$  and obtain a subbundle bigger than  $\mathcal{S}$  all of whose simple factors are isomorphic to  $\underline{\mathcal{L}}$ . This is a contradiction. Hence  $\langle \mathcal{K} \rangle = 1$ .  $\square$

**Lemma 8.3.** *Let  $\mathcal{W} = \mathcal{Z}_{D,n}$ ,  $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$  a generic point,  $0 \subset \mathcal{S} \subset \mathcal{V} \subset \mathcal{W}$  a sequence of subbundles. Consider Schubert states  $\mathcal{I}, \mathcal{K}, \mathcal{L}$  which are given by the requirements*

- (1)  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ ,
- (2)  $\mathcal{S} \in \Omega^o(\mathcal{K}, \mathcal{V}, \mathcal{E}(\mathcal{V}))$  and
- (3)  $\mathcal{S} \in \Omega^o(\mathcal{L}, \mathcal{W}, \mathcal{E})$ .

*Suppose that (see Expression 1.2)  $\dim(\mathcal{L}) = 0$  and  $\langle \mathcal{K} \rangle = \langle \mathcal{I} \rangle = 1$ . Then,  $\langle \mathcal{L} \rangle = 1$ .*

*Proof.* Suppose that  $\langle \mathcal{L} \rangle > 1$ . Since the flags on  $\mathcal{W}$  are generic and  $\dim(\mathcal{L}) = 0$ , we will find (finitely many) points other than  $\mathcal{S}$  in  $\Omega^o(\mathcal{L}, \mathcal{W}, \mathcal{E})$  which are however not subbundles of  $\mathcal{V}$  (the induced flags on  $\mathcal{V}$  are generic so can be used to compute  $\langle \mathcal{K} \rangle = 1$ ).

But monodromically, (the parameter space is an open subset of  $\text{Fl}_S(\mathcal{W})$ ) all the (finitely many) points in  $\Omega^o(\mathcal{L}, \mathcal{W}, \mathcal{E})$  look the same (the universal family is connected and irreducible). But  $\mathcal{S}$  is special in the sense that it is contained in the unique  $\mathcal{V} \in \Omega^o(\mathcal{I}, \mathcal{W}, \mathcal{E})$ . Other points of  $\Omega^o(\mathcal{L}, \mathcal{W}, \mathcal{E})$  do not have this property. This is a contradiction.  $\square$

We now prove Proposition 6.2. Use notation from that proposition. Let  $\mathcal{Q} = \mathcal{W}/\mathcal{V}$ . Apply Lemma 8.2 to the  $\mathcal{W}$ -structure corresponding to  $(\mathcal{Q}^*, \text{Gdual}(\mathcal{I}), \mathcal{W}^*, \mathcal{E}^*)$  (which we assume has more than one simple factor, otherwise there will be nothing to prove) to produce an inclusion  $0 \subsetneq \mathcal{S} \subsetneq \mathcal{Q}^*$  such that if  $\mathcal{K}$  is defined from  $\mathcal{S} \in \Omega^o(\mathcal{K}, \mathcal{Q}^*, \mathcal{E}^*(\mathcal{Q}^*))$  and  $\mathcal{L}$  from  $\mathcal{S} \in \Omega^o(\mathcal{L}, \mathcal{W}^*, \mathcal{E}^*)$  we have

- (1)  $\langle \mathcal{K} \rangle = 1$ .
- (2)  $\mathcal{S}$  contradicts the semistability (and not stability) of the  $\mathcal{W}$ -bundle corresponding to  $(\mathcal{Q}^*, \mathcal{W}^*, \text{Gdual}(\mathcal{I}), \mathcal{E}^*)$ . The equality 5.1 reads as  $\dim(\mathcal{K}) = \dim(\mathcal{L})$  in this situation. But  $\dim(\mathcal{K}) = 0$  from (1), therefore  $\dim(\mathcal{L}) = 0$ .

Now consider  $\mathcal{T}$  be the kernel of the composite  $\mathcal{W} \rightarrow \mathcal{Q} \rightarrow \mathcal{S}^*$ . Clearly  $\mathcal{T}$  is a subbundle of  $\mathcal{W}$  which contains  $\mathcal{V}$ . By Lemma 8.3  $\langle \mathcal{L} \rangle = 1$  and hence by Grassmann duality,  $\langle \text{Gdual}(\mathcal{L}) \rangle = 1$ . Clearly,  $\mathcal{T} \in \Omega^o(\text{Gdual}(\mathcal{L}), \mathcal{W}, \mathcal{E})$ . Iterating this procedure (with  $\mathcal{T}$  replacing  $\mathcal{V}$ ) we obtain a sequence of subbundles:

$$\mathcal{V} = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_h \subsetneq \mathcal{W}$$

such that

- (1) For  $i = 0, \dots, h$  if  $\mathcal{K}_i$  is defined by  $\mathcal{V}_i \in \Omega^o(\mathcal{K}_i, \mathcal{W}, \mathcal{E})$ , then  $\langle \mathcal{K}_i \rangle = 1$ .
- (2) The  $\mathcal{W}$ -bundle corresponding to  $((\mathcal{W}/\mathcal{V}_h)^*, \mathcal{W}^*, \text{Gdual}(\mathcal{K}_h), \mathcal{E}^*)$  has exactly one simple factor.

Clearly  $\mathcal{T} = \mathcal{V}_h$  satisfies the conclusion of Proposition 6.2.

## 9. PROOF OF PROPOSITION 5.6

We need the following result on automorphisms of ES-bundles:

**Lemma 9.1.** *The dimension of the group of automorphisms of  $\mathcal{Z}_{D,n}$  is  $n^2$ .*

*Proof.* Let  $\mathcal{W} = \mathcal{Z}_{D,n}$ . Clearly, the degree of  $\mathcal{E}nd(\mathcal{W})$  is 0, so by Riemann-Roch,  $\chi(\mathcal{E}nd(\mathcal{W})) = n^2$ . But if  $\mathcal{W}$  is generic, the summands of  $\mathcal{E}nd(\mathcal{W})$  are either  $\mathcal{O}(-1)$ ,  $\mathcal{O}$  or  $\mathcal{O}(1)$  all of which have  $H^1 = 0$ . So  $\dim(\mathcal{E}nd(\mathcal{W})) = n^2$ . The identity is an automorphism and the set of automorphisms is an open subset of the ring of endomorphisms (see Lemma 9.2), so the group of automorphisms of  $\mathcal{W}$  is  $n^2$  dimensional.  $\square$

**Lemma 9.2.** *Let  $X$  be a projective algebraic variety,  $\mathcal{W}$  a vector bundle and  $x$  a point. Consider the natural map  $\phi_x : \mathcal{E}nd(\mathcal{W}) \rightarrow \mathcal{E}nd(\mathcal{W}_x)$ .*

- (1) *For  $s \in \mathcal{E}nd(\mathcal{W})$ , the characteristic polynomial of  $\phi_x(s)$  does not depend on  $x$ .*
- (2) *Given  $s \in \mathcal{E}nd(\mathcal{W})$ , it is an automorphism if and only if  $\phi_x(s)$  is so.*

*Proof.* The coefficients in characteristic polynomial are global regular functions, hence constant. In particular the determinant is a constant.  $\square$

Now suppose that  $\underline{\mathcal{V}}$  is as in Proposition 5.6.

$$\begin{aligned} \text{wt}(\mathcal{V}, \text{Rig}(\underline{\mathcal{V}})) &= \frac{1}{r} \left( \sum_{p \in S} \sum_{a=1}^{l(p)} (\Lambda_a^p - \Lambda_{a-1}^p)(r - \Lambda_a^p) \right) \\ &= \frac{1}{r} \left( \sum_{p \in S} \dim(X_{\Lambda^p}(\mathcal{V}_p)) \right) = \frac{1}{r}(r^2 - 1) \end{aligned}$$

where in the last equality we used the assumed stability and rigidity of  $\mathcal{V}$ , and Lemma 5.4. Therefore,  $\mu(\mathcal{V}, \text{Rig}(\underline{\mathcal{V}})) = 1 - \frac{d}{r} - \frac{1}{r^2}$  and the slope assertion is verified.

We first show that  $\text{Rig}(\underline{\mathcal{V}})$  is semistable. Assume the contrary and let  $\mathcal{S}$  be the Harder-Narasimhan maximal contradictor of semistability of degree  $-\tilde{d}$  and rank  $\tilde{r}$ .

Suppose that  $F_{\Lambda_a^p} \cap \mathcal{S}_p$  is  $u_a^p$  dimensional for  $p \in S$  and  $a \in \{1, \dots, l(p)\}$ . The uniqueness of the Harder-Narasimhan element gives (the flags on  $\mathcal{V}$  are generic)

$$(9.1) \quad \dim(\mathfrak{Gr}(\tilde{d}, \tilde{r}, \mathcal{V})) - \sum_{p \in S} \text{codim}(Y_p) = [\tilde{r}(r - \tilde{r}) + \tilde{d}\tilde{r} - d\tilde{r}] - \sum_{p \in S} \text{codim}(Y_p) = 0$$

where

$$Y_p = \{S \in \text{Gr}(\tilde{r}, \mathcal{V}_p) \mid \dim(F_{\Lambda_a^p}(\mathcal{V}_p)) \cap S \geq u_a^p, a = 1, \dots, l(p)\}.$$

The codimension of  $Y_p$  is

$$\tilde{r}(r - \tilde{r}) - \sum_{a=1}^{l(p)} (u_a^p - u_{a-1}^p)(\Lambda_a^p - u_a^p)$$

(with the usual understanding that  $u_0^p = 0$ )

$$\begin{aligned} & \tilde{r}(r - \tilde{r}) - r\tilde{r} + \sum_{a=1}^{l(p)} (u_a^p - u_{a-1}^p)(r - \Lambda_a^p) - \sum_{a=1}^{l(p)} (u_a^p - u_{a-1}^p)(-u_a^p) \\ &= r(\text{wt}_p(\mathcal{S})) - r\tilde{r} + \tilde{r}(r - \tilde{r}) - \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(-u_a^p) \end{aligned}$$

(where  $\text{wt}_p$  is the contribution to  $\text{wt}(-, \text{Rig}(\underline{\mathcal{V}}))$  from  $p$ )

$$= r(\text{wt}_p(\mathcal{S})) - \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(\tilde{r} - u_a^p)$$

Using Equation 9.1, we get

$$-r(\text{wt}(\mathcal{S}, \text{Rig}(\underline{\mathcal{V}}))) + \sum_{p \in S} \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(\tilde{r} - u_a^p) + \tilde{d}r - d\tilde{r} + \tilde{r}(r - \tilde{r}) = 0$$

or that

$$(\text{wt}(\mathcal{S}, \text{Rig}(\underline{\mathcal{V}})) - \tilde{d})r = \tilde{r}(r - \tilde{r}) - d\tilde{r} + \sum_{p \in S} \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(\tilde{r} - u_a^p).$$

We have  $(\text{wt}(\mathcal{S}, \text{Rig}(\underline{\mathcal{V}})) - d)r > \mu(\mathcal{V}, \text{Rig}(\underline{\mathcal{V}}))r\tilde{r}$  by assumption, so

$$\tilde{r}(r - \tilde{r}) - d\tilde{r} + \sum_{p \in S} \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(\tilde{r} - u_a^p) > r\tilde{r} - d\tilde{r} - \frac{\tilde{r}}{r}$$

So,

$$\sum_{p \in S} \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(\tilde{r} - u_a^p) - \tilde{r}^2 > -\frac{\tilde{r}}{r}$$

But we claim

**Claim 9.3.**

$$(9.2) \quad \sum_{p \in S} \sum_{a=1}^{a=l(p)} (u_a^p - u_{a-1}^p)(\tilde{r} - u_a^p) - \tilde{r}^2 + 1 \leq 0.$$

*Proof.* (Of the Claim) Note that  $\mathcal{S}$  is ES (see section 5) and gets a partial flag in each of its fibers  $\mathcal{S}_p$  for  $p \in S$  made from  $F_{\Lambda_a^p}(\mathcal{W}_p) \cap \mathcal{S}_p$ . Let this corresponding flag be in  $X_{\Delta^p}(\mathcal{S}_p)$  where  $\Delta^p = \Delta_1^p < \dots < \Delta_{b^p}^p$  is the set  $\{u_a^p \mid a = 1, \dots, l(p)\}$ . Inequality 9.2 can now be interpreted as

$$(9.3) \quad \sum_{p \in S} \dim(X_{\Delta^p})(\mathcal{S}_p) - \dim(\text{Aut}(\mathcal{S})) + 1 \leq 0$$

Intuitively, Equality 9.3 follows from the “rigidity” of  $\mathcal{S}$  with its induced partial flag structure (as will be explained below). The rigidity property of  $\mathcal{S}$  comes from being the Harder-Narasimhan subbundle of a “rigid”  $(\mathcal{V}, \text{partial flags})$ . We give a more formal argument below.

It suffices to show that  $\text{Aut}(\mathcal{S})$  has a dense orbit on

$$FL_{\Delta}(\mathcal{S}) = \prod_{p \in S} X_{\Delta^p}(\mathcal{S}_p)$$

because that would imply Inequality 9.3.

Consider the scheme

$$X = \{G^p \in \prod_{p \in S} X_{\Lambda^p}(\mathcal{V}_p) \mid \text{rk}(G_{\Lambda_a^p} \cap \mathcal{S}_p) = u_a^p \text{ for } p \in S, a \in \{1, \dots, l(p)\}\}.$$

The point  $\underline{\mathcal{V}} = (\mathcal{V}, w, \mathcal{F})$  gives a point in  $X$  (by ignoring steps in the flags  $F_i^p$ ).

It is easy to see that there is a morphism  $X \rightarrow FL_{\Delta}(\mathcal{S})$  which is **dominant**. Given two generic points  $f_0$  and  $f_1$  in  $X$  we get two stable parabolic bundles if we consider the original system of weights  $w$  on  $\mathcal{V}$ . Since the dimension of the moduli of  $\underline{\mathcal{V}}$  is 0, we find that there is an automorphism of  $\mathcal{V}$  which takes  $f_0$  to  $f_1$ . This automorphism has to necessarily preserve the subbundle  $\mathcal{S}$  (because  $\mathcal{S}$  being the Harder-Narasimhan element for the rigidity weights, is unique in its Schubert state). We deduce that  $\text{Aut}(\mathcal{S})$  has a dense orbit on  $FL_{\Delta}(\mathcal{S})$  and hence the claim is proved.  $\square$

Assuming the claim, the semistability of  $\text{Rig}(\underline{\mathcal{V}})$  follows. We now show Inequality 5.2 which will prove the stability of  $(\mathcal{V}, \text{rig}, \mathcal{F})$  as well. If  $\mathcal{T} \subsetneq \mathcal{V}$  is a  $(\tilde{d}, \tilde{r})$ -subbundle, then the semistability result gives  $\mu(\mathcal{T}, \text{Rig}(\underline{\mathcal{V}})) \leq \mu(\mathcal{V}, \text{Rig}(\underline{\mathcal{V}}))$ . Multiplying the equation by  $r\tilde{r}$ , we get

$$\mu(\mathcal{T}, \text{Rig}(\underline{\mathcal{V}}))\tilde{r}r \leq \tilde{r}r - d\tilde{r} - \frac{\tilde{r}}{r}.$$

But  $\mu(\mathcal{T}, \text{Rig}(\underline{\mathcal{V}}))r\tilde{r} \in \mathbb{Z}$ , so the above inequality is strict and in fact

$$\mu(\mathcal{T}, \text{Rig}(\underline{\mathcal{V}}))\tilde{r}r \leq \tilde{r}r - d\tilde{r} - 1.$$

which gives Inequality 5.2.

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